Teaching mathematics with high cognitive activation:

instructional formats and connection-making interactions in high-level Nordic lessons

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Cognitive activation is a dimension of teaching quality which considers to what extent the teacher addresses the educational goal of student understanding, such as through successful implementation of demanding tasks. This study aimed to enrich empirical understandings of instructional formats and teacher-student interactions in cognitively activating lessons. Eight lessons were purposefully selected from a Nordic video database containing 125 lessons. The interactions in the lessons were analysed using reflexive thematic analysis and instructional formats using content analysis. Whole-class discussions and group work were the dominant formats. Four types of connection-making interactions were observed, connecting both within mathematics and to non-mathematical experience.

For any educational system with ambitions for positive student outcomes, it is rational to strive for teaching to be of high quality. Educational researchers generally agree that teaching quality is a multi-dimensional construct, which means it is defined, conceptualized, and measured in multiple sub-constructs, as opposed to one single construct or metric (Croninger et al., 2012). Some dimensions can be viewed as generic and independent of subject matter, such as classroom management. Other dimensions can be viewed as subject-specific and closer to the heart of the content being taught. What constitutes teaching quality also depends on which perspective of teaching is taken. Possible connections to student outcomes may be sensitive to the social context in which the research takes place (Franke et al., 2007; Hiebert & Grouws, 2007).

From a constructivist perspective, a key objective of mathematics teaching is offering students opportunities to build understanding of concepts and their relations (Glasersfeld, 1995; Skemp, 1976). Cognitive

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activation, i.e. teaching for understanding, has been conceptualized as a subject-specific dimension of teaching quality (Praetorius et al., 2018). Research on cognitive activation has operationalized the dimension in different ways. Indicators of cognitive activation involve the cognitive challenge of tasks and the level of student engagement in classroom discourse. Cognitive activation in mathematics has been measured both by directly rating the tasks that students are set to solve, such as on class tests (e.g. Kunter & Voss, 2013), and by rating lesson segments using observation frameworks (e.g. Pianta & Hamre, 2009). Some studies have found cognitive activation to be positively linked with student outcomes in mathematics, such as enjoyment and achievement gains (Cantley et al., 2017; Klieme et al., 2001; Krauss et al., 2020; Kunter et al., 2013; Lipowsky et al., 2009). However, research is scarce for developing empirical understandings of how teachers successfully facilitate cognitive activation strategies through interactions in lessons, particularly in lower secondary mathematics. In this lesson observation study, cognitive activation is understood to constitute how teachers address the goal of student understanding through both intellectually rigorous tasks and, particularly, how teachers implement tasks during lessons through extended mathematics related talk and interaction in the classroom (Grossman, 2019). To develop empirical understandings of what exemplifies high cognitive activation in mathematics lessons, it is critical to study teacher-student interactions in such lessons and the lesson context in which they transpire.

Literature review

While teaching, teachers must both attend to various educational goals and maintain quality in their teaching in a constant flow of interactions with students. What constitutes teaching quality is fundamentally based on the goals of education. Diederich and Tenorth (1997) identified three generic goals for teaching and learning: student attentiveness, student motivation, and student understanding. The degree to which teachers address these three goals is described in the Three Basic Dimensions framework of teaching quality, respectively: classroom management, student support, and cognitive activation (Klieme et al., 2006; Praetorius et al., 2018; Vieluf & Klieme, 2023). In this sense, cognitive activation refers to the dimension of teaching that addresses student understanding.

Cognitive activation has been conceptualized and measured in a variety of frameworks for teaching quality. Examples of such frameworks are Teaching for Robust Understanding (Schoenfeld, 2017), Protocol for Language Arts Teaching Observations (PLATO; Grossman et al., 2013), and the Classroom Assessment Scoring System (Pianta, La Paro, & Hamre, 2008). In an analysis of 12 observation frameworks, Praetorius and Charalambous (2018) defined cognitive activation in terms of three teaching practices: (1) selection of appropriately challenging tasks and use of mathematically rich practices, (2) facilitation of cognitive activity, and (3) support of meta-cognitive learning from cognitively activating tasks. The theoretical foundation of the concept lies in both the application of cognitive science to educational situations and (socio)constructivist theories of learning. The primary theoretical assumptions are that to activate students cognitively, i.e. teach for understanding, the teacher must: (1) in the constructivist view, engage students in cognitive conflicts through challenging problems and questions, and (2) in the socio-constructivist view, invite students to participate in classroom discourse and communicate their ideas to develop conceptual understanding (Praetorius et al., 2018; Smith & Stein, 2011). The definition of cognitive activation used for selection of lessons in this study builds on these two assumptions.

While cognitive activation is defined as the dimension of teaching which addresses student understanding, it is generally not specified what kind of understanding. In Skemp's seminal paper (1976), he distinguishes between two types of understanding. The first type, instrumental understanding is also known as ”rules without reasons”: the possession of a rule and the ability to use it. The second type is relational understanding: knowing both what to do and why. The taken-as-shared meaning of understanding here is that of relational understanding. This is consistent with operationalisations of cognitive activation in conceptual frameworks (e.g. Grossman, 2019; Kunter & Voss, 2013).

Studies in different educational contexts have found a positive connection between cognitive activation and student outcomes. Large-scale studies in a German context have found cognitive activation to be positively related to student achievement in mathematics (Baumert et al., 2013; Klieme et al., 2001; Lipowsky et al., 2009). A study on mathematics teaching quality in Norway, building on TIMSS data, found the same positive association (Bergem, Nilsen & Scherer, 2016). Further, a positive connection between cognitive activation and student interest and enjoyment has been found in both mathematics and science in different educational contexts (Cantley et al., 2017; Ekatushabe et al., 2021).

Studies have also indicated a connection between cognitive activation and teacher-related variables. A Nordic study of science teachers found a connection between teachers' perceived time-constraints and low cognitive activation, indicating that teachers may need specific support in lesson planning to implement cognitive activation strategies (Teig et al., 2019). Moreover, cognitive activation measures have been found to be significantly predicted by teachers' pedagogical content knowledge (Krauss et al., 2020; Kunter et al., 2013).

The notion of selection and use of tasks is represented in the "offer-use-model" of teaching and learning (Helmke, 2015; Weingartner, 2021). The "offer side" refers to two aspects of teaching practice: the objective cognitive activation potential and the implemented cognitive activation potential. In mathematics, the objective potential is typically found in tasks as they appear in textbooks or other teaching materials. The implemented potential refers to the way the teacher intends for students to engage with the task. This involves how the task is presented or potentially adapted in teacher-student interactions during the solution process, or whether the solution process is intended to be carried out individually or in interaction with other students.

Teachers can implement tasks in a wide variety of ways. One is to have students use and evaluate multiple solution methods, or what has been called flexibility in mathematics problem solving (Star et al., 2015). This competency has been phrased as adaptive expertise, in contrast to routine expertise (Hatano & Inagaki, 1986). For students to develop such expertise and understanding, they need to make connections between multiple representations or solution strategies, an endeavour which has been associated with achievement gains (Brenner et al., 1997). In a more recent framework, explicit connections in conjunction with productive struggle is argued to be a key feature of instruction that facilitates deep understanding, makes students "see the structure" of a domain and transfer knowledge to real-life situations (Fries et al., 2021; Hiebert & Grouws, 2007).

Yet, to directly connect mathematics to reality is not as straightforward as it may seem. Within his model of learning milieus, Skovsmose (2001) distinguished between real-life references and references to a "semi-reality" – a reality which is imagined by the author of a task, commonly ignoring many constraints and complexities of reality. The second dimension of his model contrasts the exercise paradigm (or "traditional mathematics education") with landscapes of investigation, a paradigm in which students search for explanations in a process of exploration and project work. Skovsmose argued that a challenge to the school mathematics tradition in Denmark was presented by the learning milieu exemplified by a landscape of investigation with real-life references. To complicate matters further, a paradox has been observed in features of teaching and to what extent students find mathematics relevant for their lives. Clarke's "expanded relevance paradox" (2006) describes application-oriented teaching in Sweden being associated with students finding mathematics irrelevant, while a pure-mathematics oriented teaching in Hong Kong was associated with students there finding mathematics highly important and relevant.

School mathematics in the Nordic countries has largely been described as within the exercise paradigm and procedurally oriented. Studies of lower secondary mathematics lessons in Norway have indicated a large majority of lesson time to be used for either individual seatwork or whole-class instruction (Bergem & Pepin, 2013) and that considerable lesson time is used to give feedback focusing on procedural skills (Stovner & Klette, 2022). Classroom practices in 197 mathematics lessons in Sweden were found to be mostly focused on developing competency in carrying out procedures, even after national competence reform (Boesen et al., 2014). A more recent study in Sweden showed similar findings as in Norway in terms of lesson time and instructional formats in mathematics; around half of observed lesson segments focused on students' individual work and 30 % on whole-class instruction (Tengberg et al., 2021). A similar proportion has been reported in Iceland, both through observation and teacher reporting: most lesson time in lower secondary mathematics was used for students' individual work in textbooks while the teacher walked between desks interacting with students (Gunnarsdóttir & Pálsdóttir, 2015; Sigurjónsson, 2023; Þórðardóttir & Hermannsson, 2012). In a small-scale video study of mathematics teachers in Iceland and Finland, the only previously published classroom study comparing mathematics teaching in Iceland to teaching in another country, a major contrast was found in the independent student learning in the Iceland lessons compared to the emphasis on whole-class interaction in Finland's lessons (Savola, 2010). Although by no means encapsulating any "national patterns" of teaching, these studies do give indications and insights into what constitutes mathematics teaching in Nordic schools. The current study aimed to contribute to these previous efforts by addressing the research question: What characterises instructional formats and teacher-student interactions in lower secondary mathematics lessons considered high in cognitive activation in a Nordic context?

Method

The study is based on a qualitative analysis of instructional formats and interactions in eight specially selected video-recorded mathematics lessons from four Nordic countries. The study draws from a Nordic video database built by the Quality in Nordic Teaching (QUINT) research centre (Klette, 2022). QUINT's vision is to systematically research teaching quality in the Nordic countries using classroom video recordings and accompanied background data. In mathematics, QUINT's video database comprises 125 lessons in 40 lower secondary classrooms, ten classrooms from each of the four countries considered for this study: Iceland, Sweden, Norway, and Denmark. Students in these lower secondary classrooms are generally aged 13-14. The database contains scores for the lessons according to the mathematics-adapted Protocol for Language Arts Teaching Observations (PLATO; see Grossman et al., 2013; Grossman, 2019). In PLATO, each 15-minute segment of a lesson receives a score for each element of teaching defined in the protocol. Raters in each country are certified by passing a course taught by experts in the protocol from Stanford University. Inter-rater reliability within countries was periodically checked with double scoring, and between countries with joint video workshops organised by the QUINT centre. In figure 1, the relevant data generation process is outlined. A more detailed description of the entire QUINT database is available (see Klette, 2022).

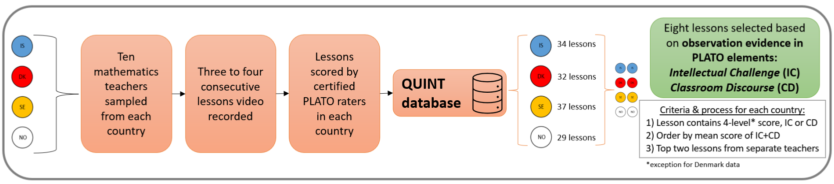

Figure 1. *Data generation process in the mathematics part of the QUINT database and its use for this study*

Selection of lessons

For this study, a three-fold selection criteria was constructed to identify specific lessons where the teacher offered students rich opportunities for cognitive activation. This was achieved by considering PLATO scores in the database for intellectual challenge (IC) and classroom discourse (CD). In table 1, a shortened version of the four-level rubric for IC and CD is portrayed.

The first criterion was that a lesson contained a score at the 4-level in either IC or CD. This filtered out over 90% of lessons in the database. The second criterion was to prioritise the lessons with the highest sum of mean scores in IC and CD. The third criterion was to select two lessons from two separate teacher in each country. This process yielded a total of eight cognitively activating lessons from eight separate classrooms[2]. Figure 2 shows a scatterplot of all lessons in the database by mean scores in IC and CD with the selected lessons for the study highlighted with a circle.

Table 1. *The four levels (shortened) of the PLATO elements IC and CD (Grossman, 2019)*

	Intellectual Challenge (IC): Teacher provides activities or assignments that are…	Classroom Discourse (CD): Teacher or students…
1-level	almost entirely rote or recall.	rarely if ever respond to students' ideas about mathematical content. Few to no opportunities for mathematics related student talk
2-level	largely rote or recall, a portion promotes analyses, interpretation, inferencing, or idea generation.	respond briefly to student ideas. Talk is tightly teacher-directed. Occasional opportunities for mathematics related student talk
3-level	a mix: most promote analysis, interpretation, inferencing, or idea generation; a few are focused on recall or rote tasks.	show multiple instances where student ideas are specifically addressed. There are opportunities for mathematics related student talk but may be substantial teacher direction.
4-level	rigorous and largely promote sophisticated or high-level analytic and inferential thinking, including synthesising and evaluating information and/or justifying. or defending their answers or positions	consistently engage in high-level uptake of student ideas. Most students participate by speaking or actively listening and students respond to each other. Open-ended questions, a clear focus and on-track conversation.

The chosen lessons from Sweden, Norway, and Denmark were transcribed and translated by colleagues within the QUINT network. The colleagues also provided contextual information as well as confirming mutual understandings of the data.

Data analysis

The instructional format of the lessons was analysed using content analysis (see Hsieh & Shannon, 2005; Braun & Clarke, 2013). The video-recordings and transcripts were analysed on a minute-by-minute basis using the categories: individual work, group work, whole-class discussion, and direct instruction. Lesson time used for administrative tasks (e.g. taking attendance) or other non-math related matters (e.g. discussing Eurovision) was categorized as downtime. The content analysis also involved identifying tasks used in the lessons to visually map what students were assigned to do in any given format throughout the lessons.

The teacher-student interactions were analysed with reflexive thematic analysis (Braun & Clarke, 2022). Both the video-recordings and transcripts were used in this analysis. In the first phase, all lesson videos were watched from start to finish with the translated transcripts on a second screen to become familiar with the lessons. In this phase, I wrote

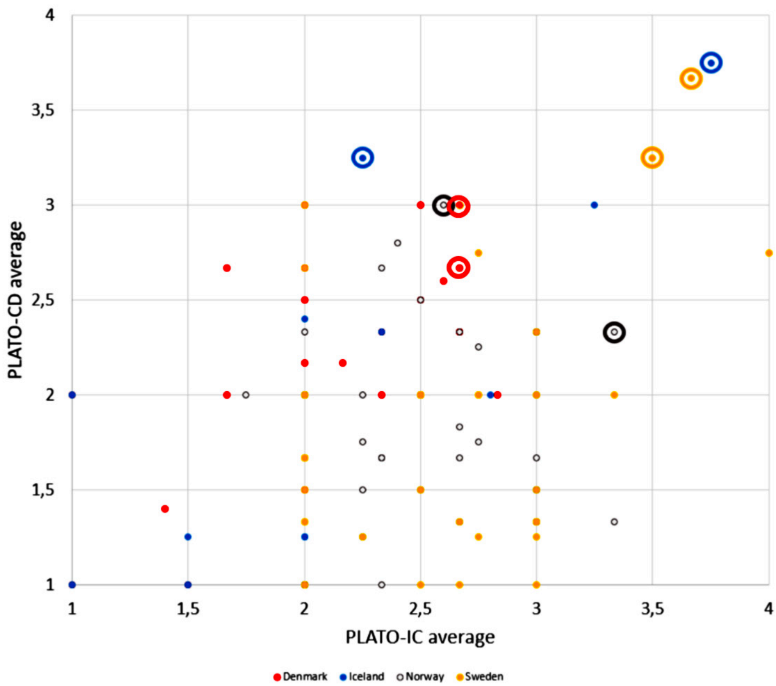

Figure 2. *Mean scores in IC and CD across all mathematics lessons in the QUINT dataset with selected lessons highlighted. Denmark (red), Iceland (blue), Norway (white), Sweden (yellow)*

notes in a separate document on the content and structure of the lessons and marked each interaction event. To distinguish between where one interaction ended and another began, one out of two criteria had to be met: 1) the teacher turned the interaction toward another person, or group of persons, without including the persons previously addressed, or 2) the teacher changed the topic of the interaction. For contextualization purposes in the transcript, each interaction was also marked regarding if the teacher addressed the entire class or a subset of the class. In the coding phase, the interaction-marked transcripts were used to code each interaction. The coding was open with no a priori codes. Clusters of codes were formed during the coding phase. In the theme-generation phase, these clusters were used to generate themes with a meaningful contribution toward addressing the research question. Around the time of the theme development phase, conversations at research seminars and feedback after conference presentations helped to refine, define, and name the themes. A second round of coding the lessons, in a reverse order, tested the applicability of the candidate themes and further refined them. The purpose of the themes was to further understand the nature of teacher-student interactions in these highly cognitively activating lessons without attempting to capture everything about the interactions. In this paper, I report findings on interactions related to one theme: connection-making within mathematics and to non-mathematical experience. Table 2 shows a full list of themes developed and their associated codes as inductively generated from the data material of teacher-student interactions[3]. The reason for bringing the focus specifically to connection-making is to be able to provide concrete examples and depth to a specific part of the findings that are a particularly relevant contribution to the field.

Table 2. *List of themes with examples of their associated codes from the teacher-student interactions*

Theme	Connections within mathematics and to non-mathematical experience	Frequent shifts in types of interactions	Use of formative feedback and explicit student roles
Associated codes	Connects to past experience/activity. Connects math to daily life . Connects one method to another. Connects one concept to another.	Prompts student to explain ”how”. Prompts student to explain what they are doing. Gives a hint. Explains a game/ group activity. Explains a method.	Gives feedback. Formative feedback. Assigns a role to student(s). Check student progress.

I adopted a critical realist perspective, which means the focus of the analysis was on the qualitative nature of interactions and instructional formats as objects mediated and shaped by a social and cultural context (Leung & Chung, 2019; Willig, 2013). As such, exact counts of minutes were not considered of importance in the analysis, but rather the relations to the outcome; a lesson considered cognitively activating. Numerically, it suffices to report that around 550 teacher-student interactions were coded.

Table 3 provides an overview of the lessons, including grade level of lessons within each national school system, lesson length, mathematical topic, and a pseudonym given to each teacher where, for ease of reference, the first letter of each pseudonym represents the first letter of the country in which the lesson was recorded. The lessons were differently placed in the learning trajectory. For instance, in Ída's lesson, students had their first introduction to algebraic expressions, as an extension from numerical expressions, with a focus on order of operations. However, Nadia's lesson aimed at revising division competency that students had already developed in previous grade levels. Other lessons appeared in the middle, neither introduction nor revision. The graphical presentation of instructional formats and tasks in the lessons was inspired by Mok & Lopez-Real (2006).

Table 3. *Overview of lessons: grade level within each school system, teacher pseudonym, length in minutes and main mathematical topic*

Country	Grade level	Teacher pseudonym	Lesson length (minutes)	Mathematical topic
Iceland	Grade 8	Ída	60	Algebra
		Íris	55	Probability
Sweden	Grade 7	Sabrine	63	Algebra
		Sandra	50	Fractions
Norway	Grade 8	Nadia	52	Division
		Nils	70	Fractions
Denmark	Grade 7	Daniel	44	Algebra
		Doris	45	Percentages

Results

I begin by presenting a visual timeline of the lessons in figure 3, showing tasks selected and assigned by teachers in the lessons, and explaining the variety of instructional formats found both within and between lessons as a result of the content analysis. What follows is a detailed account of the connections made by teachers in interactions with students that were developed from the thematic analysis. Examples of different types of connection-making interactions further illuminate how teachers contributed to the high cognitive activation in the lessons in the implementation of tasks. Instructional formats varied within and between the different lessons

The instructional format of the lessons varied both within and between lessons

As seen in figure 3, group work (green color) or whole-class discussion (blue color) was observed in every lesson – some had lesson time prioritized for group work while others prioritized whole-class discussion. Most lessons had some individual work (yellow color), but frequently in short bursts before moving to either group work or whole-class discussion. For example, in Sabrine's lesson students were to think about the first task (on candy-sharing) alone for a minute before talking with their group. Continuous individual work in the lessons never exceeded

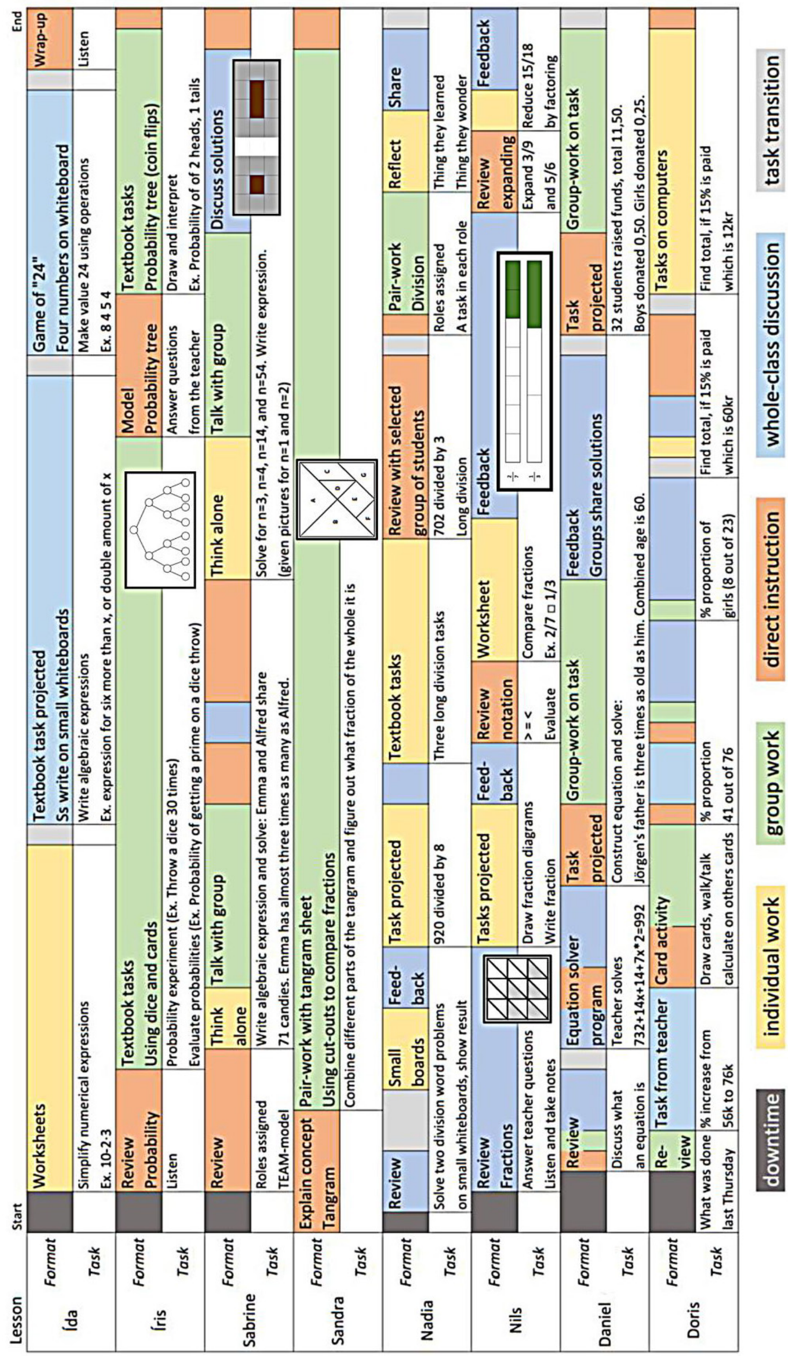

Figure 3. *Timeline of each lesson with instructional formats and tasks*

15 minutes before switching to a different format. All lessons included some direct instruction (red color), often in the beginning with review, or toward the end for wrapping up. Direct instruction was also found interspersed between other lesson formats, and commonly included explanation of the next lesson activity, a brief review of concepts or explanation of content that the students were engaging with. For instance, in Nadia's lesson she had a few students come to the blackboard to review division with her as the other students continued their textbook exercises. As such, the lessons were not devoid of the exercise paradigm or ”traditional mathematics teaching” – parts of some lessons had procedural exercises and rules or examples on a blackboard to follow, or sections in which the students were mostly to listen (not actively engaging with content). However, these sections did not extend a large portion of the lessons. The extended sections tended to be either group-work or whole-class discussions, sometimes in a game-like format (e.g. game of 24 in Ída's lesson) or with students assigned explicit roles (e.g. Sabrine's and Nadia's lessons – see figure 3). As many tasks were procedural in nature, it was in these implementations of the tasks that the cognitive activation potential was heightened, rather than in the tasks themselves. Further, none of the lessons followed a particular ”script” – all shifted between different formats, some very frequently (see e.g. Nils' and Doris' lessons in figure 3).

Interactions for making connections within mathematics and to non-mathematical experience

Every teacher was observed to engage in connection-making interactions with students, but in different ways. Four different types of connection-making were observed and are reported in these findings; two types making connections to non-mathematical experience and two types making connections within mathematics.

In every lesson, teachers connected the mathematical content to students' previous experience or past class activity. Sandra and Daniel were special cases in having exclusively connection-making interactions of this type, and no other types. For example, in Sandra's assistance to students with the fractional tangram task, she said: ”Do you remember how to expand and reduce fractions? [...] So think, use what you have learned, with expanding and reducing fractions, because you can only add fractions with the same denominator”. On the other hand, in the beginning of Daniel's lesson, he connected to past class activity: ”We are going back a while, we are going more than a week back.” These interactions can be interpreted as more targeted toward generally connecting to prior experience than to directly establish conceptual mathematical connections to support students' knowledge transfer and understanding.

Table 4. *Interactions that connected one method to another*

Teacher	Task	Setting of connection-making interaction	Interaction example
Sabrine	Write algebraic expression and solve: Emma and Alfred share 71 candies. Emma has almost three times as many as Alfred.	A student group is presented the option to use x for an unknown variable to search for an integer solution, but encouraged to pursue their own method.	
		Student groups invited to present at the whiteboard to explain to the whole class their different solution methods.	"Could we take a similar solution? Because [group of students] has a similar one. And then maybe you can explain it in another way?"
	Given pictures for n=1 and n=2, solve for n=3, n=4, n=14 and n=54. Write an expression.	Student groups invited to present at the whiteboard to explain to the whole class their different solution methods.	"Now you can see here that in [group] they have calculated it in a special way. Is there someone who has used another strategy? For example drawn a picture?"
Ída	Construct expression with value 24 from the numbers: 8, 7, 9, 1	Teacher writes on the whiteboard two different solutions, as explained by students from their seats.	
Íris	Expected frequency of a particular result from 30 dice rolls	Student groups are presented a way of seeing expected frequency as expanding the fraction of the outcome probability.	"And how many nodes gave me this result? It was one out of eight which I could calculate with this method [multiplying together three terms of one-half] or by creating a probability tree and count."
	Probability of getting the same result in three coin flips	Teacher discusses two solution methods with the whole class.	
Doris	Calculate the whole from a given part (60) and percentage (15%)	Teacher invites a different explanation from students after hearing 100 divided by 15, multiplied by 60.	"Has anyone done it in another way? […] Yes, that is the same. 60 divided by 15, multiplied by 100, is 400 kroner."

The other interaction type relating to non-mathematical experience was to connect the mathematical content to daily life. For instance, Nadia knew students would use lunch breaks to buy a kebab and made a connection to dividing a payment for a kebab. Similarly, Ída motivated the need for using letters in algebraic expressions by suggesting its application in calculating revenue for the local ski resort, with the amount of guests paying the entrance fee being an unknown quantity. Most teachers made such connections between mathematics and daily life. The exception to that was Sabrine, who never directly connected mathematics to daily life outside of the semi-reality of the tasks. However, in multiple interactions she made connections within mathematics.

Two interaction types involved making connections within mathematics. Table 4 shows interactions of the type Sabrine engaged in: connecting one method with another. She made these interactions on three occasions within the lesson, two of which were in the setting of student groups presenting their solutions. Other teachers who connected one method with another were Ída, Íris and Doris. In Ída's case, the task itself had multiple solutions, so she invited students to contribute more than one expression with the value 24 using the given numbers. Both Íris and Doris highlighted two different solution methods in a whole-class discussion.

The last type of connection-making interaction was to connect one mathematical concept to another. Nils was the only teacher observed to engage in this type of interaction. In the beginning of the lesson, he motivated the need for fractions by connecting them to decimals. He started to write out the infinite decimal for one-seventh (0.142857...) in comparison to the fraction 1/7 and then stated that "when you become proficient with fractions, you will be able to do things [...] in math, that you almost didn't even think was possible, it is very useful, to be able to do calculations with fractions, and, when you understand it ... have practiced and understood it... it is not that difficult either." Later, in expanding on a visual model proposed by a student to compare the fractions 1/3 and 2/7, Nils also emphasised multiple solution paths and to make connections for understanding: "There is no right or wrong way. The important thing is that you can explain it mathematically."

What exemplified these interactions was, on the one hand, to connect the mathematical content to student's previous experiences or daily lives, and on the other hand, to draw connections within mathematics, such as between different concepts or between different methods. One can interpret that the aim of these interactions was to move students toward relational understanding of mathematical concepts, and as exemplified in Nils' words, that a central goal was for students to be able to explain their mathematical thinking.

Discussion and implications

The study explored what characterises instructional formats and teacher-student interactions in cognitively activating lessons. The findings can be summarised in five key points.

First: there seems no one characterisation, no one "recipe", for a cognitively activating lesson. The instructional formats were characterised by variety rather than the dominance of one specific format. From the perspective of systematic observations, this empirical finding provides an argument against the notion that observation systems privilege "uniform solutions" to teaching. On the contrary, the findings within these lessons illustrate a wide range of teaching repertoires. What may be called "traditional" mathematics teaching, with individual seat work or exercises (e.g. Skovsmose, 2001), can occur in brief intervals – but in general as part of a wider variety of formats and activity structures to facilitate construction of knowledge (Glasersfeld, 1995).

Second, the interactions where connections were made within mathematics contributed to the cognitive activation potential in the lessons. These interactions were frequently made within a setting where students or student groups explained their solutions or thought processes, either to the teacher directly, or to the whole-class in a dialogue moderated by the teacher. In the lessons observed, connecting one method to another was more common than connecting one concept to another, as teachers highlighted different methods of approaching the tasks, seemingly to encourage flexibility and adaptive expertise (Hatano & Inagaki, 1986;

Star et al., 2015). I argue that explicitly connecting within mathematics supported students' relational understanding by highlighting both the what and the why (Skemp, 1976; Fries et al., 2021).

Third, the highlighting of multiple solution methods is attributed to the teachers' implementation of the tasks in interaction with the students, rather than the tasks themselves. As can be seen on figure 3, many of the tasks selected were procedural in nature and as such within the exercise paradigm (Skovsmose, 2001). From a practical perspective, this finding highlights the crucial importance of teacher implementation of tasks for cognitive activation. From a methodological perspective, it highlights potential shortcomings of solely relying on task analysis to determine levels of cognitive activation. If the tasks alone had been analysed at face-value as indicators of cognitive activation in these lessons, many are unlikely to have been considered high-level. As previous findings also suggest, teachers' pedagogical content knowledge is decisive for the implementation of tasks and thereby the quality of teaching (Baumert & Kunter, 2013; Krauss et al., 2020; Kunter et al., 2013).

Fourth, as individual work was far from the most common instructional format, the lessons selected may be considered outside the dominant paradigm in Nordic mathematics teaching (Gunnarsdóttir & Pálsdóttir, 2015; Stovner & Klette, 2022). While it is important to support students as independent learners (e.g. Savola, 2010), it is also of paramount importance to offer students time and space to think together about the same task. A shift to more collaboration between students in thinking about the same task can be one pathway to enhance cognitive activation in mathematics in the Nordic context, where the individual exercise paradigm still seems to dominate (e.g. Sigurjónsson, 2023; Tengberg et al., 2021).

Lastly, the interactions connecting to non-mathematical experience seemed to be more geared toward student motivation than student relational understanding. Considering Skovsmose's (2001) notion of references to a semi-reality and Clarke's expanded relevance paradox (2006), these connection-making ambitions of the teachers raise some questions. To what aim do teachers make connections to daily life – and is there a possibility that such ambitions are misguided? Few would reject the usefulness of a good analogy or a wider narrative to comprehend ideas. But are students more motivated to learn algebra if teachers point out its application to model company revenue? Do students gain more mathematical understanding if regularly reminded of its relevance to society and real-life application? Clarke's paradox suggests that the more likely answer is no. Yet, to competently transfer mathematical knowledge to real-life situations and contexts is a core educational goal (Fries et al., 2021). The interactions connecting to non-mathematical experience in these findings more generally made connections to prior experience. With no conclusive claims whether for "better or worse", an interpretive observation is that connections to "real-life" were more directed at students' motivation, i.e. to see a purpose for the mathematical topics at hand, than to develop students' understanding or direct transfer of knowledge to real-life situations.

The study's results have both limitations and implications. The lack of data on student outcomes sets limits to conclusions of to what extent the educational goal of student understanding was truly reached. Furthermore, PLATO is here used for the selection process and adapted, as a protocol originally made for teaching observations in language arts, to mathematics. The lessons were selected based on their high scores in PLATO's elements relating to cognitive activation (Grossman, 2019). Being high-level in that sense does not mean the lessons were perfect by any means – scores in other dimensions are not discussed. However, teachers or pre-service teachers who wish to develop cognitive activation in their lessons can look to some exemplary lessons and connection-making practices as ways to create opportunities for further addressing the educational goal of student understanding.

One implication of these results is what can be termed professional flexibility. This can be defined as teachers' readiness to shift their approach by being observant of the needs of students in the moment. In Hatano and Inagaki's terms, teaching requires expertise which is adaptive rather than routine (1986; Star et al., 2015). Providing students with opportunities to engage with tasks through different types of interactions that gauge their current level of understanding, explain their thinking to the teacher or each other with clear participatory roles, and make connections between mathematical concepts or methods alike – these appear as key exemplars of interactions in these cognitively activating Nordic mathematics lessons. Critically, this conclusion is limited to the situated and interpreted reality of these eight specifically chosen lesson from a Nordic social context. Empirical inquiries into lessons considered cognitively activating based on different observation frameworks, or in a different cultural or social context, may further illuminate the sensitivity of social context to measures of teaching quality. Furthermore, richer empirical understandings should inform further theoretical development.

Connection-making is not explicitly a component of cognitive activation as a theoretical construct (Klieme et al., 2001; Praetorius et al., 2018). The study does raise questions of the association between teacher-student interactions that involve making explicit connections and cognitive activation as a dimension of teaching quality. The interaction types presented, specifically the two types making connections within mathe-matics, are, I argue, aimed at student relational understanding. As such, drawing these connections contributes to the educational goal of student understanding, and therefore to the level of cognitive activation in the lessons (Skemp, 1976; Vieluf & Klieme, 2023).

This study may have implications for teacher education or professional development programs in a Nordic context. To support teachers in preparing cognitively activating lessons, their skills in successfully facilitating students' productive struggle during group or pair work should be cultivated, and their inclination to assigning individual seat work should be reflected upon (Hiebert & Grouws, 2007; Teig et al., 2019). Support for high cognitive activation should also include a careful selection of tasks for such planning, attending to the way a teacher implements them in interaction with students (Smith & Stein, 2011). This does not mean that lessons with the aim to practice and develop procedural fluency should vanish. However, such lessons must be placed within a sequence of other lessons with the aim to develop relational understanding alongside. Building on findings from the lessons observed in this study, I suggest that a possible direction for teachers and prospective teachers to develop their cognitive activation skills is in connection-making interactions with students. They can also gain from reflecting on the types of interactions they make with students in lessons and what instructional formats they prioritise. By exploring these directions, teachers and prospective teachers can develop a teaching repertoire of outstanding cognitive activation to invite more students to experience joy and success in their mathematics learning.

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Notes

1 In the time since this study was conducted, the QUINT database has also made data from Finland available.

2 In the case of Denmark, there were no lessons with 4-level scores in IC or CD. The first criterion was then modified for a 3-level score before considering the mean score.

3 Findings related to the other themes can be found in the author's doctoral dissertation (Sigurjónsson, 2023).

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