Norwegian student teachers' perspectives on linear equations

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The topic of linear equations is part of school mathematics all over the world. Hence, it is important that teachers are proficient in teaching equations, explaining not only the arithmetic operations in the solving process, but also the objectives of the solution, the nature of an unknown and the balance property of an equation. This study employs a set of 16 low-inference codes to analyse 146 Norwegian student teachers' explanations of the solution of a linear equation. One important result is that a majority of the students explained additive operations rather by "swap sides swap signs" than by "do the same to both sides".

This paper reports from a twin study to the study published in the paper "Swedish primary teacher education students' perspectives on linear equations" (Andrews, 2020). Both papers originate from a collaboration between the authors, and data were collected the same way for both studies. The participants in Andrews' study were student teachers for primary school at a Swedish university, while the participants in this study were student teachers at a Norwegian university, following either the programme for teachers in grades 1–7 or the programme for teachers in grades 5–10, in compulsory school in Norway. For the Swedish student teachers and the Norwegian group for grades 1–7 mathematics was mandatory, while the Norwegian group for grades 5–10 voluntarily had chosen mathematics. The data collected were also used in a study published in a conference paper (Larson & Larsson, 2021). The paper presented here draws on the same categorisation of data that was made for the conference paper. The conference paper highlighted differences between the two countries, while this paper further elaborates on features in the Norwegian data.

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The topic of equations is a part of school mathematics all over the world (Houang et al., 2004), and plays an important role in the development of other parts of the subject of mathematics (Cai et al., 2010). Furthermore, knowledge of equations, and algebra in general, is a ”gatekeeper” to further studies (Ladson-Billings, 1997; Moses & Cobb Jr, 2001). Consequently, it is not surprising that the topic of equations emerges in several grades in the Norwegian curriculum for compulsory school (Norwegian Directorate for Education and Training, 2020). In turn, this entails that mathematics teachers in Norway will treat equations regularly, which implies that proficiency in explaining how to solve equations is essential for future mathematics teachers, as the participants in this study.

The aim of this study is to explore Norwegian student teachers' understanding of linear equations. This was implemented by inviting the student teachers to explain the solution of the equation $x + 5 = 4x - 1$. One matter was if they preferred to explain the additive operations by ”do the same to both sides” or ”swap sides swap signs”. The study is framed by the following research questions.

How do Norwegian student teachers explain the solution of a linear equation?

How do different parts of their explanation interact?

Two important teacher competencies are knowledge about the topic and knowledge about the teaching and learning of the topic (cf. Ball et al., 2008). Hence, enhanced knowledge about how student teachers understand linear equations and how they explain how to solve them, is useful for teacher educators. Thus, the outcomes of this study might underpin teacher educators' choice of what to emphasise in mathematics education about equation solving.

Background

A linear equation in one unknown is any equation that can be written on the form $ax + c = 0$, where a and c are constants. However, in compulsory school mathematics, it is common that an equation takes either of the forms $ax + c = d$ or $ax + c = bx + d$, where, in addition, the constants a, b, c, d are integers. The former has the unknown on one side of the equals sign only, while the latter has unknowns on both sides of the equals sign. The two types are fundamentally different from a didactical point of view (cf. Andrews, 2020), since the former can always be solved by inverse arithmetic operations or by informal techniques as ”cover the unknown”, while for the latter it is ”necessary to operate on what is represented" (Filloy & Rojano, 1989, p. 20). Thus, this partitioning enlightens the "didactic cut" (Filloy & Rojano, 1989, p. 20) between equations with the unknown on one side only, and equations with unknowns on both sides.

This paper will utilise a linear equation with unknowns on both sides of the equals sign, namely $x + 5 = 4x - 1$. There are two main strategies for solving this type of linear equation, do the same to both sides (DSBS) and swap sides swap signs (SSSS) (de Lima & Tall, 2008; Tall, 2017), which historically go back to the Eulerian and Viète model respectively (Filloy & Rojano, 1989). An initial step by DSBS might be "to get rid of x on the left side, you subtract both sides by x", and by SSSS "you move x to the right side, and when you move over the equals sign you must remember to swap signs such that it becomes $-x$".

Although it is fair to assume that mathematics students have individual preferences, there also exist cultural differences in what method teachers promote. While the DSBS method is more popular in many western countries, SSSS is preferred in some Asian countries (Ngu et al., 2015). The motives for which method is the most successful vary. Arguments against SSSS are that the action of swapping signs in SSSS is considered as a "mystical operation" (de Lima & Tall, 2008, p. 8) and a "piece of magic" (p. 14), and that using SSSS entails "you probably do not understand really what you are doing" (Andrews & Öhman, 2017, p. 5). Furthermore, SSSS promotes rote learning that does not support conceptual learning (Star & Seifert, 2006), which may cause difficulties in future mathematics studies (Capraro & Joffrion, 2006), while DSBS rather supports the learning of algebraic structures (Wasserman, 2014). On the other hand, there are arguments that DSBS is "too complicated, error prone and inefficient" (Ngu et al., 2015, p. 271), and experimental studies where students who used SSSS outperformed those who used DSBS (Ngu & Phan, 2016a). Moreover, it is not unlikely that the SSSS method is inferred from DSBS, so some students might perceive SSSS as just a less complicated version of DSBS, rather than consider one of them as more sophisticated than the other.

It is customary to use models or analogies when teaching linear equations (Araya et al., 2010). The balance scale is common (Araya et al., 2010; Filloy & Rojano, 1989; Linchevski & Herscovics, 1996; Vlassis, 2002), although it has its shortcomings for equations including subtraction of terms or negative values of the unknown, as well as the complication that today's children are not familiar with balance scales (Pirie & Martin, 1997). Other models that stress the equality between both sides are a geometrical model employing rectangle areas (Filloy & Rojano, 1989), and a model with boxes and beans (e.g. Araya et al., 2010; Rystedt et al., 2016).

Most models build on the balance between the two sides of the equation, stressing the relational aspect of the equals sign. The understanding of the equals sign as relational, and not just as operational, is crucial for solving equations, in particular equations with unknowns on both sides of the equals sign (Herscovics & Linchevski, 1994; Kieran, 1981; Knuth et al., 2006; Stephens et al., 2013).

Further, it is likely to be important that teachers emphasise the role of the variable, in equations often called the unknown, and that the objective of the solving process is to find which value of the unknown that makes the equality true. This is regardless of whether the students' understanding of the concept of unknown is poor (MacGregor & Stacey, 2007; Stacey & MacGregor, 1997) or adequate (Knuth et al., 2005).

The task and the analytic tool

In this section the task used for data collection is presented, as well as results from earlier studies on the same task (Andrews, 2020; Andrews & Öhman, 2017; Andrews & Xenofontos, 2017; Larson & Larsson, 2021; Xenofontos & Andrews, 2017). During these studies an analytic tool was gradually developed (Andrews & Larson, 2019). That tool was utilised to analyse the data the current study is based on, and it will be presented in table 1.

In the data collection for these studies, the participants were given a correct but short solution to a linear equation, with no annotations that showed the fictional solver's thinking. The equation and the solution given were

$$x + 5 = 4x - 1$$
$$5 = 3x - 1$$
$$6 = 3x$$
$$2 = x$$

It was explicated to the participants of these studies that they should suppose they had a friend, who was absent when the teacher introduced the topic. They were invited to explain to their friend how this equation was solved. The solution of the equation and some instructions were presented on an A4-sheet, and the participants were instructed to give their written reply below on the same page, which gave them approximately half a blank A4-page for their reply.

The equation $x + 5 = 4x - 1$ is a linear equation with unknowns on both sides of the equals sign. This equation was chosen for several reasons. First, since the equation has unknowns on both sides of the equals sign, it cannot easily be solved by inverse arithmetic operations, but requires a more strategic approach. Second, there are no brackets or fractions, that could be an algebraic challenge not directly connected to the solving strategies, which would risk that the participants elaborated on algebraic simplifications rather than the operations required to solve equations. Third, for the same reason, the coefficients and constants are small, natural numbers, that will not cause difficult calculations. Fourth, the solution ends with the unknown on the right-hand side, which highlights the balance property of the equation and provides a possibility to explain that the answer usually is given the other way round.

The first study on this task was made in Cyprus with Greek-Cypriot and Greek student teachers for primary school. Constant comparison analysis (Fram, 2013) was applied to develop the first analytic tool connected to this task (Andrews & Xenofontos, 2017; see also Xenofontos & Andrews, 2017). The analysis of the student teachers' replies yielded seven codes. A vast majority of the student teachers mentioned the concepts of known and unknown. A majority of the Cypriot student teachers explained that the objective of the solution was to separate the unknowns from the numbers, which was much less frequent among the Greek student teachers. However, only one of the 33 student teachers mentioned that the objective was to find the value of x. Almost all student teachers provided explanations of the additive steps of the solution (first to second line, and second to third line in the solution above). Interestingly, all these student teachers referred to ”swap sides swap signs” (SSSS), and no student teacher mentioned ”do the same to both sides” (DSBS). The DSBS model appeared, however, in some student teachers' explanation of the multiplicative step from the third to the fourth line. These student teachers clarified that you must divide $3x$ by 3, and hence you divide 6 by 3. Other student teachers were more unclear, and said that you divide 6 by 3 without referring to the coefficient of the unknown term.

Later, the same task was used for data collection in Sweden and Norway. In contrast to the Cypriot study (Andrews & Xenofontos, 2017), explanations by DSBS were frequent among Swedish student teachers for primary school (Andrews, 2020). The Swedish student teachers' use of DSBS mainly concerned the additive steps, while the multiplicative step to a greater extent was explained more vaguely like ”you divide 6 by 3”, without mentioning that this division stemmed from the coefficient in $3x$. Moreover, students in Swedish upper secondary school referred to SSSS as a process where you do not understand what you are doing, and that their teacher did not let them use SSSS (Andrews & Öhman, 2017). The results from these studies indicate DSBS is more established in the Swedish tradition, than in the Cypriot and Greek.

Since the initial analysis of the Scandinavian data showed these student teachers provided both SSSS and DSBS explanations (e.g. Andrews, 2020), we concluded it was favourable to develop the analytic tool. Through an iterative process originating from the data, codes were developed or changed, and new codes were included. This gave an analytic tool with 13 codes (Andrews & Larson, 2019), from which a slightly modified version was used in a report on the Swedish data (Andrews, 2020). Among these 13 codes, five treated DSBS, namely ”DSBS general”, ”DSBS general additive”, ”DSBS particular additive”, ”DSBS general division”, and ”DSBS particular division”, while just two treated SSSS, ”SSSS general” and ”SSSS particular additive”. ”Particular” meant an explanation explicitly referred to the equation in the task, while ”general” was a description of the principle without reference to this equation. For example, writing ”you can do anything in an equation as long as you do it on both sides” would be ”DSBS general”, since is does not refer to a specific operation. Further ”you can add or subtract any number from both sides of the equation” is ”DSBS general additive”, since it refers to an additive operation without referring to the equation under scrutiny. Finally, ”we take +1 on both sides of the equation” refers to an additive operation in the equation under scrutiny, and is thus ”DSBS particular additive”. The corresponding codes for the multiplicative operations and the SSSS approach are interpreted in a similar way (see also table 1).

One reason for including just two codes of SSSS was that when student teachers invoked SSSS, they almost always referred to the additive processes. However, that does not imply the remaining three codes for SSSS are redundant. Hence, these three codes were added to the set, which, in addition, gave a ”symmetric” tool with the same codes for DSBS and SSSS. This also enabled the possibility to code the few examples where SSSS referred to the multiplicative step, as well as potential replies comprising the fourth or fifth SSSS-code. Consequently, the current analytic tool (also used in Larson & Larsson, 2021) includes 16 codes (see table 1). The codes are claimed to be of low inference (Andrews, 2007), which implies the codes are distinct as well as culture independent.

Method

The purpose of this paper is to explore Norwegian student teachers’ understanding of linear equations, and how they explain how to solve them, for example if and how they employ the strategies and notions discussed above (see also table 1).

The 146 participants in this study were student teachers at a Norwegian university, who followed a programme for teachers in compulsory school. There were 83 student teachers from the programme for grades 1–7, with a median age of 20 years, and 63 student teachers from the programme for grades 5–10, with median age 21. Only 24 student teachers (14 + 10) were more than 23 years, which means most of them finished upper secondary school a few years ago. Since the topic of equations is recurrently treated in compulsory school and upper secondary school mathematics, also in the previous syllabus (Norwegian Directorate for Education and Training, 2013), it is fair to assume that the participants had studied linear equations on numerous occasions. For both groups, data were collected in the second semester of their education, in the

Table 1. *The analytic tool with 16 codes, including frequencies from the scripts*

Code	Description	Freq.	%
Discusses the nature of x	x is a variable or an unknown and may represent any number.	22	15
Equality of both sides	The balance property and/or both sides of the equals sign being equal.	24	16
Conceptual objective	To find the value of x. (Addresses the purpose of equation solving.)	65	45
Procedural objective	To get unknowns on one side and numbers on the other side of the equals sign. (Addresses the process of equation solving.)	128	88
SSSS general	The SSSS movement of objects with no reference to specific operations.	59	40
SSSS general additive	The SSSS additive movement of objects with no reference to the equation under scrutiny.	22	15
SSSS particular additive	The SSSS additive movement of objects of the equation under scrutiny.	94	64
SSSS general multiplicative	The SSSS multiplicative movement of objects with no reference to the equation under scrutiny.	0	0
SSSS particular multiplicative	The SSSS multiplicative movement of objects of the equation under scrutiny.	1	1
DSBS general	The principle of solving equations by doing the same operation on both sides.	24	16
DSBS general additive	Solving equations by adding or subtracting the same object to both sides with no reference to the equation under scrutiny.	4	3
DSBS particular additive	Solving equations by adding or subtracting the same object to both sides of the equation under scrutiny.	31	21
DSBS general multiplicative	Solving equations by multiplying or dividing both sides by the same object with no reference to the equation under scrutiny.	23	16
DSBS particular multiplicative	Solving equations by multiplying or dividing both sides of the equation under scrutiny by the same object.	111	76
Unspecified operation on the coefficient	A multiplicative operation on the equation based on the value of the coefficient with no reference to either SSSS or DSBS.	11	8
Checks solution	Checking or mentions the possibility of checking the solution.	13	9

beginning of their first mathematics course and before the topic of equations was treated. The courses included both subject content and mathematics didactics. It was mandatory to study mathematics for student teachers for grades 1–7, while the student teachers for 5–10 had chosen mathematics as one of the subjects to be included in their education. In the first semester, both groups had studied another school subject, e.g. English or science, and pedagogy. All student teachers were informed

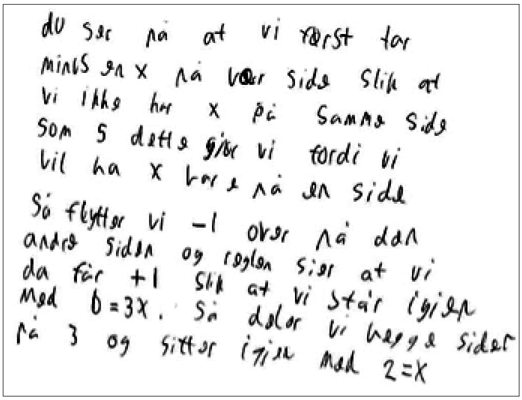

Figure 1. O*ne participant's script*

Table 2. *The codes present in the script presented in figure 1*

Norwegian	English	Assigned code
du ser på at vi først tar minus en x på hver side	you see that first we take minus one x on each side	DSBS particular additive (subtracting from both sides of the equation in the task)
slik at vi ikke har x på samme side som 5	such that we do not have x on the same side as 5	Procedural objective (we want to separate unknowns from knowns)
dette gjør vi fordi vi vil ha x bare på én side	we do that since we want to have x on one side only	Procedural objective (just clarifying what just has been said)
Så flytter vi -1 over på den andre siden og regelen sier at vi da får +1	Then we move -1 over to the other side and the rule says we then get +1	SSSS particular additive (moving and changing signs of a term in the equation in the task)
slik at vi står igjen med $6 = 3x$	so that we have $6 = 3x$ left	No code (this is just the result of an operation)
Så deler vi begge sider på 3	Then we divide both sides by 3	DSBS particular multiplicative (dividing both sides of the equation in the task)
og sitter igjen med $2 = x$	and have $2 = x$ left	No code (just the result of an operation)

that participation was voluntary and anonymous, and that they were free to choose their own pseudonym.

Each script of these Norwegian data was analysed by the author of this paper as part of another study (Larson & Larsson, 2021), using the set of codes in table 1. For that study, Larsson recoded the Swedish scripts used by Andrews (2020). After this coding, each researcher recoded a sample of 20 scripts chosen at random from those coded by the other. The two researchers' coding of these 40 scripts were compared. Cohen's kappa was found to be 0.82, which shows the consistency of their coding to be of high quality. In fact, aggregating the two codes "SSSS general" and "SSSS general additive", which were difficult to distinguish and can be aggregated without loss of vital information, gave Cohen's kappa 0.89, which is excellent.

The set of codes is presented in table 1, where also each code is briefly explained. In addition, table 1 shows the frequencies of each code in the Norwegian scripts (this is to provide the frequencies of all codes without repeating the whole table later). The codes are independent, which means every code can be combined with any other for a script. If the same code appeared repeatedly in the same script, it was registered just once.

One example of how the analytic tool was employed is provided in figure 1 and table 2. See Andrews (2020) for further examples based on the corresponding data from Sweden.

Results

In this section, the presence of codes obtained from the analysis of the scripts and connections between different codes are presented. Comparisons between student teachers following the programme for grades 1–7 and student teachers on the programme for 5–10 are also provided.

The analysis showed the maximum number of codes present in a script was eight (4 scripts), the mode was four codes (44 scripts), and the mean was 4.3 codes (std.dev. 1.5). The mean for the 83 student teachers aiming for grades 1–7 was almost 0.4 lower than for the 63 student teachers in the group for 5–10. Two scripts in the group for 1–7 were evaluated as having no code present. One of these scripts was blank, while in the other, the student teacher explained that they were sorry, but they didn't understand more than the absent friend.

Codes about the objective of solving an equation

The first four codes in the set (see table 1) connect to the overall understanding of solving equations. Only 12 student teachers in the group for 1–7 and 10 student teachers in the group for 5–10 mentioned anything connected to the nature of x, while the balance property of the equation was mentioned by 8 and 16 student teachers respectively. This means the proportion between the groups was equal for the nature of x, but the balance property was more frequently stressed in the group for 5–10. Anyway, a vast majority of the scripts included neither of these two codes.

The objectives, describing the purpose (conceptual objective) and the procedure (procedural objective) of the solving process, were more frequently mentioned. Since a script may include both objectives, it is also interesting to see how the appearances of objectives interact, which is shown in table 3.

Table 3 shows that only 13 student teachers omitted both objectives. Further, only 5 student teachers mentioned just a conceptual objective, and all these came from the group for 1–7. It was, however, common to mention a procedural objective only, or to mention both objectives. This shows almost all student teachers mentioning a conceptual objective, also included a procedural objective. However, among all student teachers mentioning a procedural objective, just under half of them included a conceptual objective too. One difference between the two groups was that it was less common among the group for 1–7 to include a conceptual objective than among the group for 5–10 (39 % vs. 52 %).

Codes about the operational steps in the solution

Regarding the operational steps, table 1 showed the most common explanation to the additive steps was made by SSSS. Table 4 shows how this code, ”SSSS particular additive”, interacts with two other codes connected to the operational steps.

In total, there were 94 scripts (59 + 35 in the groups for 1–7 and 5–10 respectively) coded as ”SSSS particular additive”, that is the student teachers explained one or both additive steps in the current equation by SSSS. Among these, 60 scripts (41 + 19) also mentioned some general aspect of the SSSS principle. That is, they mentioned the principle of SSSS independently from the equation under scrutiny. It was less common to invoke DSBS for the additive steps, 31 scripts (10 + 21) were coded as ”DSBS particular additive”, and only one script invoked both SSSS and DSBS with reference to the equation in the task.

Table 4 shows it was more frequent among the group for grades 1–7 to invoke SSSS for the additive steps than among the group for 5–10, although the SSSS additive strategy dominated also in the group for 5–10. However, one third of the group for 5–10 invoked DSBS for the additive steps, while the corresponding proportion in the group for 1–7 was just one eighth.

Table 3. *Frequencies of objectives in the scripts.*

Student teachers for grades 1–7

		Procedural objective		
		Absent	Present	Totals
Conceptual objective	Absent	8	43	51
	Present	5	27	32
	Totals	13	70	83

Student teachers for grades 5–10

		Procedural objective		
		Absent	Present	Totals
Conceptual objective	Absent	5	25	30
	Present	0	33	33
	Totals	5	58	63

Table 4. *Cross-tabulation originating from "SSSS particular additive"* *

Student teachers for grades 1–7

		SSSS general		DSBS particular additive		
		Absent	Present	Absent	Present	Totals
SSSS particular additive	Absent	13	11	14	10	24
	Present	18	41	59	0	59
	Totals	31	52	73	10	83

Student teachers for grades 5–10

		SSSS general		DSBS particular additive		
		Absent	Present	Absent	Present	Totals
SSSS particular additive	Absent	23	5	8	20	28
	Present	16	19	34	1	35
	Totals	39	24	42	21	63

* *Note*. In this table, SSSS general means either "SSSS general" or "SSSS general additive" was present

Although SSSS was the dominant strategy for the additive steps, DSBS dominated even more regarding the final division, as we see in the columns of table 5. Since the results from the two groups (1–7 and 5–10) were rather similar here, the groups are not split in this table. However, one

Table 5. *Cross-tabulations of the additive and multiplicative strategies*

		DSBS particular multiplicative		Unspecified operation on the coefficient		
		Absent	Present	Absent	Present	Totals
SSSS particular additive	Absent	20	32	49	3	52
	Present	15	79	86	8	94
DSBS particular additive	Absent	30	85	106	9	115
	Present	5	26	29	2	31
	Totals	35	111	135	11	146

important difference between the groups is presented in the following paragraph.

There were 111 scripts coded as ”DSBS particular multiplicative”, only 11 scripts coded as giving an unclear explanation of the final step, while the remaining 24 scripts were not identified with any code connected to the final division. Of the 111 scripts that invoked DSBS in the multiplicative step, 58 came from the group for grades 1–7, and 53 from the group for 5–10. This means the proportion of student teachers invoking DSBS was smaller in the group for 1–7 (70 % vs. 84 %).

In table 5, it is difficult to identify any interactions between the additive and multiplicative strategies, like if one additive strategy predicts a certain multiplicative strategy. One reason might be that so few scripts were coded as ”DSBS particular additive” and ”unspecified operation on the coefficient”.

This subsection is closed by a few notes about scripts with no operational codes present. Except for the two scripts evaluated with zero codes present, two more scripts (both from the group for grades 1–7) were coded as giving no explanation at all connected to SSSS or DSBS. One of these just said ”Don't know. Get the x:es on one hand side and the numbers on the other”, which is a ”procedural objective” only. The other explained that since it was given that $x = 2$, we can see that $2 + 5 = 4 \cdot 2 - 1$, which gives ”checks solution” as the only code present.

Interactions between codes for the objectives and operational strategies

Finally, possible interactions between the objectives present and the operational strategies chosen will be explored. Andrews' method (2020, pp. 40–41) for exploring these connections was adopted, that is to compare the ratios between the procedural and conceptual objectives for scripts invoking different operational strategies. Table 6 shows the frequencies

Table 6. *Cross-tabulations of the objectives and the additive strategies*

		SSSS particular additive		DSBS particular additive		
		Absent	Present	Absent	Present	Totals
Conceptual objective	Absent	26	55	70	11	81
	Present	26	*39*	45	*20*	*65*
Procedural objective	Absent	7	11	14	4	18
	Present	45	*83*	101	*27*	*128*
	Totals	52	94	115	31	146

Note. Numbers used in the calculations are in italics

for objectives filtered by the SSSS and DSBS additive aspects. Because no clear differences between the two groups (1–7 and 5–10) were identified regarding the interaction between the codes, all student teachers are treated as one group in this subsection.

To the right, we see the ratio between the procedural and conceptual objectives among all scripts is 128 : 65. The corresponding ratio for scripts including "SSSS particular additive" is 83 : 39, which is approximately equivalent to 128 : 65. For "DSBS particular additive" the ratio is 27 : 20, a clear difference from 128 : 65. A conclusion is that student teachers who included a conceptual objective were more likely to invoke DSBS for the additive operations. The corresponding ratios for the multiplicative operation are found in table 7.

The ratios between procedural and conceptual objective are 103 : 56 for scripts invoking "DSBS particular multiplicative" and 8 : 1 for scripts coded as "unspecified operation on the coefficient". The former is approximately equal to 128 : 65, while the latter clearly deviates, however based on a small number of scripts. A possible conclusion is that

Table 7. *Cross-tabulations of the objectives and the multiplicative strategies*

		DSBS particular multiplicative		Unspecified operation on the coefficient		
		Absent	Present	Absent	Present	Totals
Conceptual objective	Absent	26	55	71	10	91
	Present	9	*56*	64	*1*	*65*
Procedural objective	Absent	10	8	15	3	18
	Present	25	*103*	120	*8*	*128*
	Totals	35	111	135	11	146

Note. Numbers used in the calculations are in italics student teachers who included a conceptual objective avoided unclear explanations of the final division.

Discussion

The purpose of this paper was to explore Norwegian student teachers' understanding of linear equations, and particularly how they explained the solution of the equation $x + 5 = 4x - 1$. Supported by an analytic tool of 16 low-inference codes, it was possible to address the research questions "How do Norwegian student teachers explain the solution of a linear equation?" and "How do different parts of their explanation interact?"

The maybe most important result was the student teachers' preference of choosing SSSS as the explanation for the two additive steps in the solution. The code "SSSS particular additive" was identified in 64 % of the scripts, while "DSBS particular additive" was identified in 21 % of the scripts. This result is noteworthy also because western country tradition often promotes DSBS rather than SSSS (Ngu et al., 2015). Even though the proportion of student teachers invoking DSBS was much larger than in the Cypriot study (Andrews & Xenofontos, 2017), it was clearly smaller than in the Swedish study on primary student teachers (Andrews, 2020; Larson & Larsson, 2021). It might be surprising that student teachers from two culturally similar neighbour countries treat a standard step in a basic topic as linear equations rather differently. Although there were differences in the group settings, such as the Swedish student teachers were older, and the Norwegian group consisted of student teachers for both primary and secondary school, this observation is interesting. In addition, this difference becomes even larger if the Swedish group, which consisted of student teachers for primary school only, is compared only to the Norwegian group for grades 1–7. On the other hand, both approaches are well established and there are broad variations also within cultures (e.g. Andrews & Larson, 2017), so it is not unexpected that an SSSS preference evolves also in a western country (cf. Ngu et al., 2015). Since culture influences the ways in which mathematics is presented in classrooms, and traditions tend to be preserved, the result is not that remarkable.

Another result that at first glance might be surprising, is that so many Norwegian student teachers changed approaches when they came to the final division. A vast majority (111 of 146) invoked DSBS to explain the division by 3 in $6 = 3x$, and among the 94 student teachers who explained the additive steps by SSSS, 79 changed to DSBS for the division. This shift was evident in both groups, although it was larger in the group of student teachers for grades 5–10. The shift is interesting also considering that Swedish student teachers rather went the other way, from DSBS in the additive steps to other explanations of the final division (Andrews, 2020). However, in Norwegian the notation of "swap signs" naturally refers to changing the sign of the number ("bytte fortegn"), which has a natural connection to additive operations. If the approach of swapping sides and swapping signs is construed rather as "swapping sides yields the inverse operation" (Ngu et al., 2015; Ngu & Phan, 2016b), it is possible the SSSS approach would become more frequent for multiplicative operations too.

Even though the operational steps are crucial in solving equations, the nature of x, the objectives of a solution, and the balance property are essential from a didactical perspective. When teaching equations, it is important to include these parts, such that the explanation consists of more than just rote operations which purpose might not be obvious for students. A vast majority (128 of 146) of the student teachers in this study included a procedural objective which means they explained the procedural goals of the solution, i.e. to step by step separate unknowns from knowns until the equation is solved. Regarding the conceptual objective, less than half of the student teachers (65 of 146) provided that the purpose of the solution is to find the value of x. This can be connected to the large proportion of SSSS as the additive approach. Since SSSS is considered as promoting rote learning (Star & Seifert, 2006), and thus procedural understanding rather than conceptual understanding (Hiebert & Lefevre, 1986), it is not surprising that the procedural objective was more frequent than the conceptual objective. Moreover, although both conceptual and procedural knowledge are important in mathematics, it is common to consider conceptual knowledge as more appropriate and often developed after procedural knowledge (Hurrell, 2021). This study's results support that posture, since almost all student teachers (60 of 65) that included a conceptual objective, included a procedural objective too. Furthermore, student teachers who included a conceptual objective invoked DSBS in their explanations of the operational steps to a greater extent than those who omitted the conceptual objective. Thus, in these data, clarifying that the purpose of solving an equation is to find the value of x was connected to DSBS, i.e. to the operational approach that better promotes learning of algebraic structures (Wasserman, 2014) and conceptual knowledge (Hiebert & Lefevre, 1986).

A comparison of the student teachers for grades 1–7 and the group for 5–10 regarding the procedural and conceptual objectives, shows the student teachers in the latter group included the objectives more frequently. It might not be surprising that this difference was larger for the conceptual objective. The student teachers aiming for grades 5–10 had voluntarily chosen mathematics as one of their subjects, while mathematics was mandatory for the group for 1–7. Thus, it is fair to assume the student teachers in the group for 5–10 had greater interest in mathematics, which, in turn, entails they were more likely to provide explanations beyond a rote learnt procedure.

The comparison of objectives and additive strategies (see table 6) supported the observation that student teachers who included a conceptual objective were more likely to invoke DSBS for the additive steps in the solution, while no connection could be found between the conceptual objective and the SSSS strategy. This result was not found among Swedish student teachers (Andrews, 2020). Conversely, no obvious interaction was found among Norwegian student teachers regarding the presence of objectives and the multiplicative strategy chosen. This was, however, found in the Swedish study, where student teachers presenting a conceptual objective were more likely to invoke DSBS for the multiplicative step and avoid unclear explanations of the same step (Andrews, 2020).

That no interaction between the conceptual objective and DSBS multiplicative was found in the Norwegian scripts, might, however, be explained by that so many student teachers invoked DSBS, and hence interactions to other codes were reduced. An interaction could though be found among the small number of student teachers that gave an unclear explanation of the multiplicative step, where a conceptual objective was provided by only one of these student teachers. Although this is based on a very small number of scripts, it supports the conclusion that student teachers who provide a conceptual objective at least tend to avoid unclear explanations of the multiplicative step.

Implications

Implications about teaching are likely to be dependent on what we expect students to learn. Except for explicit goals about linear equations in the syllabus, there might also be implicit goals that can be achieved by learning equations. If the goal is not only to be proficient in solving equations, but also to understand what you are doing (Andrews & Öhman, 2017), recognise the relational property of the equals sign (Kieran, 1981; Knuth et al., 2006; Stephens et al., 2013), and to facilitate the understanding of algebraic structures (Wasserman, 2014), explanations and solutions by DSBS should be promoted. However, if the prioritised learning outcome is proficiency in solving equations, SSSS might be preferred since it tends to be more successful in giving the right answer (Ngu & Phan, 2016a). To succeed in task solving may also increase the student's interest of mathematics, which is positive for learning. It is also possible that good procedural knowledge might contribute to conceptual knowledge (cf. Hurrell, 2021). That can be arguments for providing SSSS. A further discussion of that is, however, beyond the scope of this paper.

When teaching linear equations, there will be advantages and drawbacks with both SSSS and DSBS (e.g. Andrews & Öhman, 2017; Knuth et al., 2006; Wasserman, 2014). How these aspects are treated must be each teacher's choice. Yet, it is important for teachers to be aware of both methods, since both aspects are relevant for students' learning, although their importance might differ for different students. The results from this study imply that for Norwegian student teachers, the DSBS method needs to be emphasised in the additive steps. Other items to highlight are that student teachers better can stress the role of the unknown, the balance property of an equation, and that the purpose of the solution is to find what value of x that satisfies the equation (the "conceptual objective"). Although some caution should be exercised for a study drawing on data from one university only, these results might be useful for teacher educators as well as for future teachers. Awareness of which aspects student teachers choose and not choose to include in their explanations of the solution of a linear equation, can support teacher educators as well as the student teachers themselves in what the latter need to develop to achieve adequate competency in teaching equations.

In this study, the analytic tool consisted of a set of low-inference codes (Andrews & Larson, 2019). Applying this set of codes was an appropriate way of giving a broad description of how student teachers construe solutions of linear equations, which also consolidates Andrews' (2020) perception of the utility of this tool. The results show what strategies the participants invoked in the additive and the multiplicative steps, but also how different strategies and the presence of the two objectives interacted. Even though the results obtained yielded appropriate information about student teachers' explanations of how to solve linear equations, they still do not tell anything about *the quality* of the explanations provided. The codes reveal the presence of different components of the explanation, but neither how clear the explanation was nor where in the script it appeared. Qualitative analyses could further enhance our knowledge of student teachers' understanding of linear equations. Thus, a qualitative approach to the scripts used in this study is a natural focus for future research.

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