Designing a game-based environment for enhancing rational number knowledge

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Rational number knowledge is an important factor in students' mathematical development. However, many students face difficulties with rational number concepts. This study describes a new educational game, NanoRoboMath, which has been designed to support students' rational number knowledge. Our goal is to examine the design and the nature of the mathematical practice in the game. Four comprehensive school students aged between 11 and 13 years individually played a prototype version of the game. Video recordings and log data of these game sessions revealed that the game was able to elicit mathematical activities related to rational numbers. This version of the game seemed to enable a variety of strategies matching different skill levels and supported arithmetic activities related to different aspects of rational number conceptual knowledge.

Learning to understand rational numbers is a crucial part of mathematical development (Booth & Newton, 2012). Fraction knowledge has proved to be an important predictor of later success with school mathematics (Booth et al., 2014; Siegler et al., 2012). However, many students struggle with rational number content in the classroom, especially that content which is in conflict with features of natural numbers (McMullen et al., 2015; Ni & Zhou, 2005; Van Hoof et al., 2018). Yet, a small number of students are also highly capable of reasoning about rational numbers beyond traditional classroom activities, such as having exceptional adaptive rational number knowledge (McMullen et al., 2020). Thus, there is the potential for interventions to both (a) improve the basic skills of students who have the most challenges and simultaneously (b) improve more-skilled students' adaptive expertise with rational numbers (Hatano & Inagaki, 1986; Moss & Case, 1999).

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Previously, game-based learning environments have proven effective for supporting students' mathematical development at a wide range of initial skill levels (Brezovszky et al., 2019, 2015). The present study aims to examine the design of one such game, NanoRoboMath, and investigate how the game elicits mathematical activities with rational numbers.

Supporting rational number knowledge

The design of the NanoRoboMath game was guided by two aims in supporting students' rational number knowledge: (a) supporting the transition from reasoning about natural numbers to also having a correct conceptual understanding of rational numbers and (b) supporting adaptive rational number knowledge. These aims are embedded in practice of mental arithmetic with rational numbers.

The transition from reasoning about natural numbers to rational numbers is partially influenced by a natural number bias – the tendency to inappropriately interpret features of rational numbers using natural number reasoning (Ni & Zhou, 2005). It has been extensively described in both the mathematics education and educational psychology literature as a potential cause of some of the difficulties in rational number learning (see a recent review by Vamvakoussi et al., 2018). While some features of rational number knowledge may be considered a continuation of natural numbers, those features that are inconsistent across the two number types may require substantial conceptual change in order to be fully understood (McMullen et al., 2018; Stafylidou & Vosniadou, 2004; Vamvakoussi & Vosniadou, 2004; Van Hoof et al., 2018). The design of NanoRoboMath aims to take into account the following features of rational numbers that have inconsistencies in comparison with natural numbers: representations of rational numbers, the magnitude of a rational number, the effects of arithmetic operations on rational numbers, and the density of the set of rational numbers.

Unlike natural numbers, rational numbers can be represented in multiple ways. This is true not only within representations (e.g. 1/2 = 2/4; .5 = .50), but also across representations (e.g. .5 = 1/2). The magnitude of a rational number cannot be interpreted from the symbolic representation the same way as natural numbers. Longer decimal representations or bigger numerators and denominators do not directly imply larger magnitudes. For instance, the magnitude of a fraction must be determined based on the relation between its numerator and denominator. With natural numbers, multiplication always leads to a greater magnitude in the outcome and division leads to a smaller magnitude. With rational numbers this is not automatically the case, as numbers less than one lead to the opposite effect on numerical magnitude. The set of natural numbers is discrete on the real line, whereas the set of rational numbers is dense. The next larger natural number can always be determined and there is a finite number of natural numbers between any two natural numbers. In contrast, the next larger rational number with respect to the natural ordering does not exist and there are infinitely many rational numbers between any two rational numbers.

Explicit reference to the aspects of rational numbers that are incongruent with natural numbers is rare in mathematics instructional material (Van Dooren et al., 2019) and teachers themselves can struggle with the concepts (Depaepe et al., 2015). Without explicit material to support instruction on these topics, they may be lightly covered throughout the main period of instruction on rational numbers in primary school. Even instructional interventions aimed at eliciting conceptual change with density knowledge appear to have limited effects (Vamvakoussi & Vosniadou, 2012). One approach that has appeared successful in encouraging conceptual change has been to force students to explicitly confront their misconceptions (Mikkilä-Erdmann, 2001). Encouragingly, some studies indicate that a game-based learning environment may be a useful avenue for supporting conceptual change with rational number knowledge, though this has mostly been found with concepts of the size of rational numbers (Kiili et al., 2017).

The second aim in the design of NanoRoboMath is supporting the development of student's high-level knowledge and skills with rational numbers, in particular adaptive rational number knowledge. Adaptive number knowledge is defined as having a richly connected understanding of numerical characteristics and arithmetic relations (McMullen et al., 2017) and has been found to be distinct from students' routine knowledge of rational numbers (McMullen et al., 2020). Importantly it distinguished those students with an exceptional knowledge of rational numbers, even among those students with high levels of routine procedural and conceptual knowledge. Those students with high adaptive rational number knowledge appeared to be able to more successfully integrate disparate aspects of conceptual knowledge of rational numbers with their procedural skills in order to solve novel tasks. In particular, they appeared able to flexibly switch between fraction and decimal notations in a fluid manner to solve the task at hand.

Previous evidence suggests that the same game-based learning environment may be able to support advanced knowledge, such as adaptive rational number knowledge, while also supporting more basic skills among those who need them (Brezovszky et al., 2019). For instance the *Number navigation game* appeared to be able to meet the player at their initial level of knowledge and support basic procedural fluency among lower prior knowledge students and more advanced knowledge, such as adaptive number knowledge, among students with more prior knowledge.

Design principles of NanoRoboMath

In this section, we describe how the educational content is embedded within the core game features of the NanoRoboMath digital game. This aspect of integrating the learning content into the design of educational digital games instead of just placing it as something extra on top of the game has been seen vital for a successful educational game design (Brezovszky et al., 2019; Devlin, 2011; Habgood, 2007; Young et al., 2012). The player of the NanoRoboMath game acts as a super hero who carries out different challenges by navigating a Nanorobot along the number line using the four basic arithmetic operations.

In the prototype game version piloted in this study, the challenge of the player was to clean polluted water by finding and destroying bacteria. The position of the nanorobot and the position of the target (bacterium) are shown on the number line. The upper screenshot of figure 1 shows a player at the initial position 4.2 and a target at the position 9. In order to move the nanorobot, the player may freely choose one of the four arithmetic operations and enter a rational number as a second operand. The current position of the nanorobot acts as the first operand. In the screenshot, the player chooses to multiply by 2 in order to get closer to the target. By pressing the equals sign, the result of the arithmetic operation is calculated automatically, and the nanorobot moves along the number line to this new value. If the new position of the nanorobot equals the position of the target, the target is destroyed and the game continues with a new target or the level is completed. If there is a difference between the player position and the target position, the player chooses new operations until the nanorobot reaches the target. As the player moves closer or further from the target, the number line rescales (zooming in or out) based on the distance between the location of the nanorobot and the target; see the lower screenshot of figure 1.

In order to provide diverse experiences with rational number arithmetic, there are two playing modes in the game. In the power mode, the player should reach the target by minimizing the magnitudes of the numbers used in the arithmetic operations. This mode is designed to enhance multiplicative reasoning, including the use of multiplicative inverses. For example, moving by addition directly from 4.2 to 9 would consume 4.8 power points (4.2 + **4.8** = 9). In contrast, moving first closer to the target by multiplication (4.2 x **2** = 8.4) and then continuing

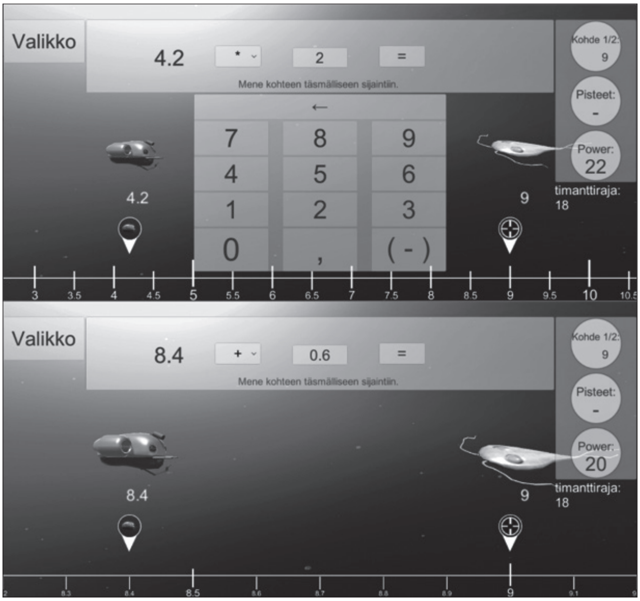

Figure 1. *Two screenshots of NanoRoboMath*

exactly to the target by addition (8.4 + **0.6** = 9) would need altogether only 2 + 0.6 = 2.6 power points. Moreover, multiplication by two could be replaced by division by half (4.2 ÷ **0.5** = 8.4), which reduces the power consumption to 0.5 + 0.6 = 1.1. Hence, there often exists a multiplicative strategy or a combination of both multiplicative and additive operations (i.e. mixed strategy) which consumes less power than the sole use of additive operations (additive strategy). In many cases, it is even more beneficial to use inverse operations and multiplicative inverses of whole numbers (inverse operation strategy).

As a matter of fact, there is a strategy which enables moving from a rational number a/b to a different rational number c/d in one move with power consumption less than one. Namely,

$$\frac{a}{b} \div \frac{a \times d}{b \times c} = \frac{a}{b} \times \frac{b \times c}{a \times d} = \frac{c}{d},$$

where a, b, c and d are whole numbers and the magnitude of either $\frac{a \times d}{b \times c}$ or $\frac{b \times c}{a \times d}$ must be less than one. In the above case, this would mean moving from 4.2 = 42/10 to 9 = 9/1 using division by

$$\frac{42 \times 1}{10 \times 9} = 0.4666\ldots$$

However, we note that in the piloted version of the game, it was not possible to enter ultimately periodic decimal number representations, which restricts the use of this strategy.

In the time mode, the player should choose arithmetic operations allowing the nanorobot to reach a given interval surrounding the target as quickly and effectively as possible. The player gains more points the closer the exact target the nanorobot gets and the faster the given target interval is reached. For example, it suffices to add 10 in order to quickly move from 4.95 close to the number 15. Of course, adding 10.05 would give more points if the player can calculate and enter this operation as quickly as adding ten. Hence, players must make strategic choices between approximate and precise calculations. It can be beneficial to roughly estimate the magnitudes of the rational numbers involved in the time mode and choose operations that arrive close to the target instead of spending time with precise calculations.

Within the game, the player has an opportunity to explore and experiment with many kinds of rational number combinations. The aim of this kind of exploration is to develop well-connected knowledge about the relations between numbers and operations (Lehtinen et al., 2015). There are no categorically right or wrong moves although some moves get the player closer to the target and others do not. In this way, the game differs from the traditional drill-and-practise exercises and may therefore support adaptive rational number knowledge (Brezovszky et al., 2019; McMullen et al., 2020).

In both modes, the player can explore the effects of different operation-number combinations and may therefore be confronted with their misconceptions about multiplication (or division) always resulting in a larger (or smaller) magnitude outcome (Christou & Vosniadou, 2012). Understanding of the magnitudes of rational numbers is supported by indicating the positions on the number line and estimations of magnitudes are especially useful in the approximate calculations of the time mode tasks. Moreover, the dynamic, scalable number line representation used in the game might help students better understand that between any two rational numbers one can always find (infinitely) more and more rational numbers (Vamvakoussi & Vosniadou, 2012). This density concept is manifested in the game by zooming the number line when the nanorobot gets closer to the target. Hence, instead of just whole numbers there are tenths, hundredths, and thousandths shown in the number line depending on the distance between the nanorobot and the target. The multiple ways of representing rational numbers have been seen important in promoting deeper conceptual understanding (Deliyianni & Gagatsis, 2013). Representational flexibility is here supported by the number line acting as an iconic representation, which enables recognizing the same mathematical object in both symbolic and iconic modes of representation. The piloted version contains only levels with decimal notation. Fraction notation is included in the later versions of the game, where mixing decimals and fractions in one calculation will also be required.

In this exploratory case study, we consider some features of a prototype of the NanoRoboMath game in order to provide an understanding of the gameplay processes for further development of the game design. In particular, we aim to examine if students' mathematical activities while playing NanoRoboMath suggest the game may support the desired learning gains. The research questions are the following:

1 To what extent does NanoRoboMath elicit rational number arithmetic practice?

2 What kind of playing strategies do players of NanoRoboMath use in their game play?

Methodology

The testing was carried out in October 2018. Four participants played the game individually using a PC in the presence of the first author. Description of the participants is given in table 1.

In this purposive convenience sampling, the first author asked four children in his circle of acquaintances to participate in the study. All players volunteered to play the game and they told that they used digital games also in their leisure-time. Only player D reported having difficulties in school mathematics. This enabled us to better test the game with students in different mathematical levels. The participants played the game in a location that was convenient for them, which was the reasons for differing playing sites. The cases represented the planned target groups of the game. We assumed that 5th graders already have basic knowledge needed to play the game for strengthening understanding of the operations with decimal numbers. Moreover, 7th grade was assumed

Table 1. *Participants of the study*

Player	Age (years)	School level	Grade	Session setting	Difficulties in school mathematics
A	13	secondary	7th	home	not reported
B	11	primary	5th	home	not reported
C	11	primary	5th	home	not reported
D	11	primary	5th	school	reported

to be favorable as in the beginning of lower secondary school students are expected to comprehend rational number arithmetic and the teaching of rational numbers in 7th grade is focused on increasing conceptual understanding and procedural fluency.

The players were asked to play the game around 30–45 minutes. The basic mathematical task in NanoRoboMath game was to choose a suitable arithmetic operation and an operand in such a way that the nanorobot moves along the number line from the starting position closer to the target position. In order to reach the target, the player had to use non-integer rational numbers at some point. Arithmetic fluency (calculation speed) and ability to do approximate calculations were involved in time mode levels. The power mode tasks challenged the players to adapt their moves between additive and multiplicatively strategies in order to minimize the magnitudes of the operands. The players had to deal with the increasing complexity of the target numbers from whole numbers to tenths and hundredths.

In order to ensure that the player understood the game mechanisms, the researcher first introduced the game by collaboratively playing with the participant (one introductory level each for power and time). After two introductory levels, the participant started playing the game independently. During the game play, the researcher would remind the players about such game features as power consumption, assistance of the number line, and possibility of moving gradually towards the target. The researcher also gave suggestions about which level and mode the player could choose next, although the final decision was given to the player. In addition to the introductory levels, the tested game contained 6 power mode levels with a total of 28 targets and 6 time mode levels with 29 targets. The levels were designed to be gradually more difficult. The power mode levels started with two levels involving halves, then two levels including quarters, one level dealing with tenths, and the final level involving hundredths. Time mode levels were designed similarly having two levels with targets close to whole numbers, two levels with targets near halves, one level where targets involved also quarters and the final level with tenths.

Data was collected using video-recorded observation, gameplay log data, and post-play interviews. Sessions were recorded using a digital video camera and Wondershare Filmora Scrn 2.0.1 screen recorder. The mathematical activities of the players were tracked using the gameplay log data and the video recordings. The analysis of the quality and quantity of the rational number arithmetic practice in the game is based on the gathered gameplay data including playing time, completed targets and levels, and the different number–operation combinations used by the players.

The game strategies of the players in power mode tasks were analyzed by comparing their power consumption to three different reference values: 1) the additive limit L_a, 2) the multiplicative limit L_m and 3) the inverse operation limit L_i. These reference values for a power mode task are defined as follows. Let P be the starting position of the player and let T be the position of the target. Then

$$
\begin{aligned}
L_a &= |T - P|, \\
L_m &= \min\left(L_a, \max\left(\left|\frac{T}{P}\right|, \left|\frac{P}{T}\right|\right)\right), \\
L_i &= \min\left(L_a, \left|\frac{T}{P}\right|, \left|\frac{P}{T}\right|\right).
\end{aligned}
$$

If the player moves in a direct path from the starting position to the target position using additive operation(s), the nanorobot will consume power exactly the amount L_a. Note that this moving towards the target can also be done in several steps. The limit L_m corresponds to a playing strategy where the target is reached using one multiplicative operation instead of the additive ones (L_a), when beneficial (min-function). However, the operand chosen by the player for a multiplicative operation in this strategy is always equal or greater than one (max-function). For obtaining the inverse operation limit, the player is capable of using multiplicative operands of magnitude less than one. This strategy corresponds to the playing performance described in the previous section where a task is completed in one move with power consumption less than one.

The game strategies in the time mode tasks were analysed by calculating the accuracy of the player when reaching the target interval $[T-1, T+1]$, where T is the target position. Denote by x the difference of the target T and the final position of the player. For the analysis, this difference was divided into four categories of accuracy: $x=0$ (precise), $0<x\leq 0.1$ (high accuracy), $0.1<x\leq 0.5$ (medium accuracy) and $0.5<x\leq 1$ (low accuracy). Here low accuracy indicates that the final position was closer to either $T-1$ or $T+1$ than the actual target T. Since the game uses decimal representations, one decimal difference from the precise target was chosen to indicate high accuracy. The frequencies of these categories in the time mode tasks were calculated for each player.

Results

We answer research question 1 by analyzing the amount and variety of arithmetic practice visible in students' game play. In research question 2, the playing strategies refer to different types of sets of moves the players use in order to complete the game tasks. We analyze the use of additive and multiplicative moves as well as precise and approximate calculations.

NanoRoboMath elicits rational number arithmetic

Time, number of completed levels, targets and moves along with moves and time per target for each playing session and game mode are given in table 2.

Table 2. *The basic parameters of the playing events for each player*

	Intro	Power mode						Time mode						Total
Player	Time	Time	Levels	Target	Moves	Moves/ Target	Time/ Target	Time	Levels	Target	Moves	Moves/ Target	Time/ Target	Time
A	0.07.40	0.23.20	5	24	38	1.6	0.00.58	0.14.28	5	21	31	1.5	0.00.41	0.45.28
B	0.07.20	0.45.27	6	24	50	2.1	0.01.54	0.11.23	3	10	14	1.4	0.01.08	1.04.10
C	0.08.29	0.09.26	1	3	7	2.3	0.03.09	0.15.10	4	17	29	1.7	0.00.54	0.33.05
D	0.14.12	0.27.50	1.5	5	25	5.0	0.05.34	-	-	-	-	-	-	0.42.02
Total	0.37.41	1.46.03	13.5	56	120	2.1	0.01.54	0.41.01	12	48	74	1.5	0.00.51	3.04.45

The variation of the playing time reflects to some extent the player's motivation and commitment to play the game. In particular, player B engaged quickly and eagerly in the game, especially aiming to improve his scores in the power mode. He even replayed some of the previous levels in order to make new records and wanted to play the most difficult power mode level after the requested 45 minutes. By contrast, player C reported being tired and stopped playing already after 33 minutes. He was more inspired by the time mode levels and less interested or able to improve his scores on the power mode. In any case, based on the video analysis it is apparent that all the players were concentrating on the game and actively trying to complete the levels they played.

According to table 2, players A and B were highly involved in doing mathematics, since they were able to complete most of the targets and levels of the game. Also player C was able to complete almost half of the levels. The fact that the number of moves for player C is more than 40 % less than that for players A and B, is partly due to his shorter playing time. In order to compare the performance of the players, the moves per target as well as the time per target were calculated. We notice that the seventh grader A was able to play the game more efficiently than the fifth graders, needing fewer moves and less time per target. In the total sample, the power mode targets required more than twice as much time as the time mode targets and almost one and half times more moves than the time mode targets. The differences between the playing modes in moves and time per target varied across the players, being smallest for player A.

Player D, although positive in her assessment of the game, had difficulties with understanding the rational number arithmetic in the game, rather relying on whole numbers to navigate on the number line. Based on the parameters of table 2, the performance of player D was clearly lower than the performance of the other players. The first two power mode levels took a long time for her. She was not able to complete the second level and, in addition to one introductory time mode level, she did not have enough time to play time mode tasks later on. Nevertheless, the video recordings show that this is not due to lack of interest in the game but rather describes her ability to manage the rational number arithmetic in the game. Although player D was able to navigate towards a favorable direction and gradually approach the target with some scaffolding given by the researcher, moving along the number line with steps of less than one seemed to be a novel challenge to her.

The number of moves and different number–operation combinations were used to examine player's mathematical activity during game play. Table 3 shows that around three quarters of the total amount of 194 moves were additive. There was a difference in the use of additive and multiplicative operations between the players ($df = 3$, $\chi^2 = 14.77$, $p < .002$, $\Phi_C = .28$). Player C used less multiplicative operations than would be expected (standardized residual = -2.5). Also player D did not use many multiplicative operations but her total number of moves was low as well. The use of multiplicative operations was highest for player A. One third of her moves were multiplicative.

Table 3. *Number of different types of moves and number–operation combinations in the sample*

	Moves	Additive		Multiplicative		Different combinations		
Player	Total	+	–	×	÷	Total	unique in session	unique in sample
A	69	25	21	14	9	51	42	22
B	64	24	23	14	3	48	41	24
C	36	23	12	1	0	28	24	17
D	25	9	13	2	1	16	8	2
Everybody	194	81	69	31	13	104	N/A	65

The players used altogether 104 different number–operation combinations, among which about 60 % were used only once in the whole sample. Each player contributed to this set of unique combinations ($n = 65$). However, there was a statistically significant difference between the players in the proportion of combinations unique in the whole sample to player's total number of combinations ($df = 3$, $\chi^2 = 10.15$, $p = .017$, $\Phi_C = .27$). Whereas the number of combinations unique in sample used by the players A, B, and C was over 40 % of their total number of combinations, this percentage for player D was only 12,5 %, which is lower than expected (standardized residual = -2.0).

Similar difference was observed when number–operation combinations occurring only once in the player's game session were examined ($df = 3$, $\chi^2 = 8.64$, $p = .035$, $\Phi_C = .25$). In the individual sessions of players A, B and C less than 20 % of the combination were used more than once, whereas half of the combinations of player D were used at least twice (standardized residual = 2.4). The above mentioned differences between player D and the other players might be due to the small total amount of moves and combinations of player D. Nevertheless, we conclude that the game elicits a good variety of arithmetical calculations, where the number–operation combinations are not extensively repeated in individual games or between different players.

Those combinations which were overall used more than once ($n = 39$) are represented in the four-set Venn diagram of figure 2 in greater detail. There were no such combinations that were used by all four players. The largest overlap (17 joint combinations) was between players A and B who both played most of the targets of the game. This suggests that the game does not induce the players to find particular solutions for the tasks. Instead, it appears that the game design enables a lot of flexibility in choosing different possibilities to reach the target.

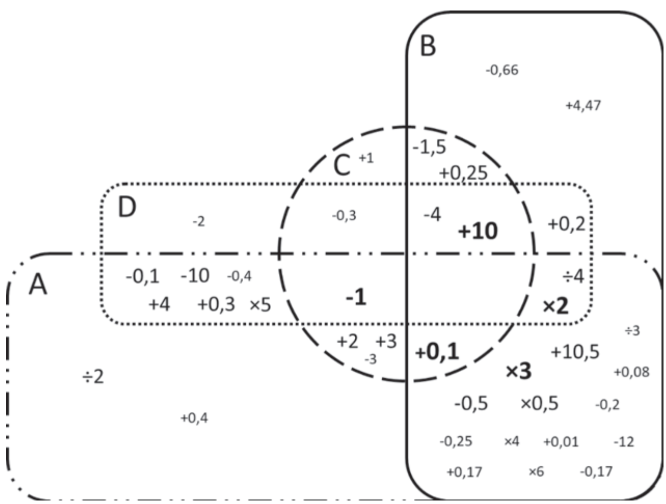

Figure 2. *The distribution of the number–operation combinations*

Note. The combinations used more than once in the whole sample are depicted in a Venn diagram where the sets A, B, C and D contain the combinations played by the corresponding players. The font size of the element describes its frequency f. Small font indicates $f = 2$, medium font $f = 3$, 4, or 5 and large font $f = 6$, 7 or 8.

Among those combinations which were used at least twice, there were 15 different number–operation combinations each for addition and subtraction. Division occurred only in three combinations and multiplication in six combinations. Combinations involving numbers related to the ten-base system (0.1, 1, 10) were used quite often. Also multiplication and division by small whole numbers (2, 3, 4) was frequent. On the other hand, there were some moves using multiplication by non-whole numbers and even by numbers with magnitudes less than one. No statistically significant differences were found in the use of the four arithmetic operations between the two playing modes ($df = 3$, $\chi^2 = 1.252$, $p = .741$). Both modes were able to induce the players to use a multitude of number–operation combinations.

Playing strategies in power and time mode tasks

In order to analyze the different playing strategies we studied the player's power use in power mode levels and accuracy in time mode levels. Figure 3 describes each player's power consumption in power mode tasks. There were altogether 25 out of 57 completed tasks where the power consumption of the player equals the additive limit L_a. In those cases the player moved in a direct path from the starting position to the target position using one or more additive operations. As a matter of fact, only in 4 of these 25 cases did the player use more than one step to reach the target. One of these rare cases was made by player A when she used a strategy where the correct number of tenths and hundredths were reached by the

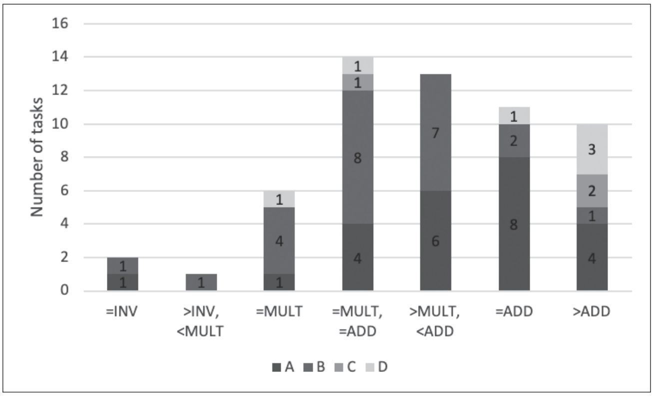

Figure 3. *The power consumption of the players in power mode tasks*

Note. The categories indicate whether the power consumption of the task was less than (<), equal (=) or greater than (>) the inverse operation limit INV, the multiplicative limit MULT or the additive limit ADD. Since INV < MULT ≤ ADD, the categories are in ascending order with respect to power consumption.

first move and the correct number of units in the second move. On the other hand, a frequent strategy of player D was to gradually approach the target. This was also the case in both of her completed tasks with power consumption L_a. Moving from 15 to 7.5 she used three steps to get to the closest whole number (8) and three steps to get to the correct number of tenths.

Tasks where the power consumption is less than the additive limit correspond to game play where multiplicative operations were used. Only players A and B appear to extensively use this strategy. Power consumption greater than the additive limit corresponds to at least one move either going beyond the target or ending up further away from the target; typically due to a miscalculation. For example, player A was once confused by the decimal places and moved with tenths instead of hundredths. Player C used once multiplication in a situation where addition would have required less energy. Player D multiplied by 5 instead of 0.5. In two tasks of player A and in one task of player B and C, the power consumption was greater than the additive limit, because the player mixed up the position of the nanorobot with the position of the bacterium and moved to the wrong direction.

In 14 cases the power consumption was equal to the multiplicative limit and the additive limit at the same time. This means that in those cases it was profitable to make a move using additive operations instead of a multiplicative operation with an operand greater than one. For example, moving from 3 to 1.5 using subtraction (– 1.5) requires less energy than division (÷ 2). Hence, multiplicative operations were used to reach the multiplicative limit in only six tasks. Multiplication by a number less than one gave rise to the three cases where the power consumption was less than the multiplicative limit. The inverse operation limit was reached once by player A (3 x 0.5 = 1.5) and once by player B (15 x 0.5 = 7.5). Player B used also a mixed strategy (9 x 0.3 = 2.7, 2.7 + 0.2 = 2.9), which caused the consumption to rise above the inverse operation limit although it still was clearly less than the multiplicative limit.

In the 13 cases where the power consumption was greater than the multiplicative limit but less than the additive limit a combination of additive and multiplicative operations were used in a flexible way. If the multiplication or division by a whole number was chosen wisely in these cases, the consumption was about 7 % – 38 % greater than the multiplicative limit L_m. An example of such mixed strategy used by both players A and B was to move from 5.39 by multiplication by 3 to the position 16.17 and then subtract 0.17 in order to reach the target 16.

Based on these analyses, it is evident that the game play in power mode levels of players C and D clearly differed from the strategies used by players A and B. Players A and B were able to use multiplicative strategies in an effective way. In contrast, player C erroneously thought that splitting additive moves into multiple steps would decrease power consumption. For player D, even the use of additive moves was challenging. If we compare the effectiveness of the playing strategies of players A and B, we notice the following differences. In power mode levels, 17 % of the tasks of player A and only 4 % of the tasks of player B exceeded the additive limit, whereas 33 % of the tasks of player A and even 54 % of the tasks of player B had power consumption less than the additive limit. These differences in the performance between players A and B with respect to the additive limit were not statistically significant (Fisher's exact test = 3.01, $p = .243$). With respect to the multiplicative limit L_m, player B outperformed player A, but this difference was only marginally significant (Fisher's exact test = 5.48, $p = .054$, $\Phi_c = .34$). Whereas player A exceeded the multiplicative limit in 75 % of the power mode tasks, player B exceeded it in only 42 % of the tasks. It seems that player A was at first fluently using additive operations and not paying attention to power consumption. Later on, when reminded of this feature, she also started to use multiplicative strategies successfully. The tasks where the power consumption passed under the additive limit were more frequent towards the end of her session. Player B used multiplicative reasoning throughout his playing session and we could not observe changes in the ways of reasoning for him.

In time model levels, the difference between the players in the accuracy of reaching the target was clear. According to figure 4, player B precisely reached the target in most cases (72 %). Players A and C had a

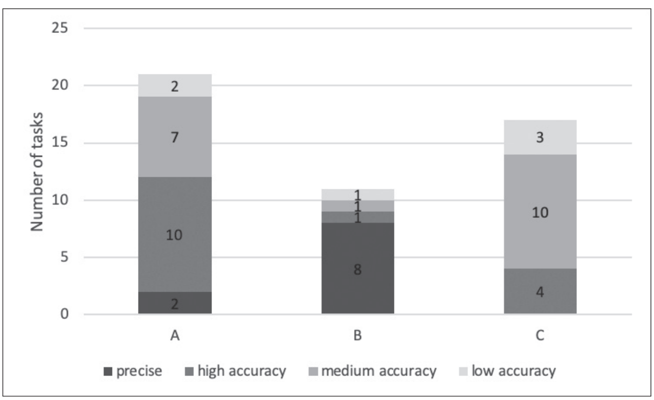

Figure 4. *The accuracy of the players in time mode tasks*

Note. The accuracy is described by the categories $x = 0$ (precise), $0 < x \leq 0.1$ (high accuracy), $0.1 < x \leq 0.5$ (medium accuracy) and $0.5 < x \leq 1$ (low accuracy), where x is the difference between the final position of the player and the target.

different strategy where the difference of the target and the final position of the nanorobot in the target interval varied more evenly between the different categories of figure 4. In 57 % of the time mode tasks of player A, the difference was not more than 0.1. The same accuracy was obtained by player C only in 24 % of the cases, none of them reaching the exact position of the target. The differences between the players A, B, and C in time mode tasks were statistically significant (Fisher's exact test = 23.36, $p < .001$, $\Phi_C = .54$). The amount of tasks where the target was reached precisely was greater for player B than would be expected (standardized residual = 3.8).

Discussion

A goal in the design of game-based learning environments in mathematics is to develop motivating games which offer more than just drill-and-practice with a limited set of skills (Devlin, 2011). In particular, they should aim to fill a gap in existing instruction that may not be easily filled by other methods (Brezovszky et al., 2019). In the present study, we examined the design of a game aimed at promoting rational number knowledge. Our results indicate that the NanoRoboMath game was able to engage the players with rational number arithmetic. All players were actively focused on completing the tasks by moving along the number line with rational number operations. Although the average number of moves and time used for reaching a target varied between the players and game modes, we interpreted that this variation reflected the player's capability rather than motivation to play the game. Some of the participants were able to adapt their strategies with respect to the core game mechanisms, i.e., to take into account the power and time consumption features in their game play. Hence, we found a great deal of variation in playing strategies already across four students, indicating that the game may target a variety of mathematical skills. Moreover, we observed indications that the game may assist the learners in the transition from natural number reasoning to rational number reasoning (Ni & Zhou, 2005) and it may also support their adaptive rational number knowledge (McMullen et al., 2020).

Both playing modes appeared to trigger mathematical activities that are not typical for the mathematics classroom (Lehtinen et al., 2017). On the one hand, time mode tasks elicited quick and approximate thinking, which may indicate the efficient integration of magnitude and arithmetic knowledge. However, while some participants were able to use approximate magnitudes in their time mode estimation strategies, it is not clear how these activities would exactly affect their magnitude representations. On the other hand, good performance in power mode tasks required an advanced understanding of the effects of arithmetic operations. Additive strategies were used by all the players, but multiplicative and mixed strategies were extensively used by two of them. They were even capable of using inverse operation strategies, which should strengthen the correct understanding of the effects of rational number operations on the magnitudes of the results. A common misconception is that multiplication always makes bigger and division smaller (Siegler & Lortie-Forgues, 2015) and the game may be suitable for confronting this misconception and support students' learning about the effect of multiplicative operations with numbers less than one. However, the inverse operation strategy was observed rarely and we note that nobody used division by a number less than one.

It was less clear from these students' behaviors, how the game play would support students' knowledge about representations and density of rational numbers, although these features were central in the game design. The lack of evidence of representational flexibility is due to the limitations of the current game design, and is expected to be better reflected in later versions of the game when both fraction and decimal inputs will be available. Density knowledge is also expected to be more clearly a feature of the gaming context in future versions that will include tasks that require approaching a number without touching. Nonetheless, it is possible that density concepts were tacitly supported during gameplay due to the zooming features on the number line, but this would have not appeared to be central in most gameplay activities investigated in the present study. Future studies that include either more detailed process data such as think-aloud protocols or learning outcome measures will better clarify if these concepts are supported by gameplay.

Both power mode and time mode tasks were able to induce the players to use a large variety of number–operation combinations. The observed combinations varied between the players and in individual games. A major part of the operations were additive, but also multiplicative operations were effectively used by some players. Different players used quite different combinations and the players did not repeat their combinations frequently. Hence, instead of just drill-and-practice, the game seems to enable exploration with rational numbers by giving the player the possibility to discover different ways to reach the target. This kind of exploration aims at developing well-connected knowledge about the relations between numbers and operations. Needing to consider these rich arithmetic relations between rational numbers is expected to support adaptive rational number knowledge, as was found with natural numbers and a previous game-based learning environment (Brezovszky et al., 2019).

It is evident that the small sample size restricts the reliability and generalizability of the conclusions. In the game, knowledge about different aspects of rational numbers was needed for making choices between approximate and precise calculations in time mode tasks as well as between additive and multiplicative strategies in power mode tasks. In future, further studies should be made to confirm that the game elicits such flexibility and adaptivity with rational numbers which also enables students to use rich networks of numerical relations in their arithmetic problem solving outside this game setting (Lehtinen et al., 2015; McMullen et al., 2020). However, this study clearly suggest that the game provides a context for exploring different kinds of relations in a flexible way where it is neither necessary to find out certain pre-determined number–operation combinations nor extensively repeat them. It certainly is out of the reach of this preliminary study to judge upon the learning effects of the game with respect to rational number knowledge. Moreover, the small sample size does not permit us to reliably identify the progress of the players in adapting their strategies with respect to the game modes. Nevertheless, the sample shows promising indications that the core game mechanisms work in a favorable way. Power mode tasks support multiplicative thinking and time tasks approximate thinking concerning the magnitudes of rational numbers. Moreover, the use of inverse operations gives some evidence of triggering changes in the misconception that multiplying always makes bigger. The development of the game continues based on the encouraging results of this preliminary analysis. Above all, explicit and large scale testing of the effectiveness of the game with respect to the learning goals is needed in the future.

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