Understanding the relationship between length and area when changing the size of a two-dimensional geometric figure

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When scaling up or down two-dimensional geometric figures, students tend to believe that if the lengths are doubled, the area is doubled as well. Although a lot of effort has been made to study and overcome this illusion of linearity, previous research reports that the illusion often remains after teaching. We add to this research by studying students' experiences of the relationship between change in length and change in area when enlarging or reducing two-dimensional geometric figures, identified in a learning study aimed at finding powerful ways of teaching scale to 14-year-old students. The aim of this study is to contribute to a deeper understanding of students' experiences of the relationship and how it can be taught. Teaching the change in length and the change in area simultaneously was found to be one key to students' learning.

Studies on students' learning about enlarging and reducing two-dimensional geometric figures have shown that students often overgeneralize linear relationships to nonlinear situations. This improper use of linearity is referred to as "the illusion of linearity" (De Bock, Van Dooren, Janssens & Verschaffel, 2002; De Bock, Verschaffel & Janssens, 1998; De Bock et al., 2003). A series of experimental studies, for instance De Bock, Verschaffel and Janssens (1998), found that there is a widespread and strong tendency among 12- to 16-year-old students, when enlarging or reducing multi-dimensional geometric figures, to intuitively assume that if all sides are twice as long, the area and volume also become twice as large. This means that when working with a scale factor of 2:1, students may take for granted that when a length is doubled, the area and the volume are doubled as well. This phenomenon is found among students of different ages and in different areas of mathematics (Ayan & Bostan,

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2016; De Bock et al., 2002). Fernández et al. (2009) suggest that if students have limited access to strategies needed to enlarge and reduce geometric figures when they interpret scale drawings, these types of understandings can occur. Previous research on this topic has focused primarily on students' difficulties, conceptions, and strategies. However, such research pays limited attention to how to overcome the difficulties (Ayan & Bostan, 2016). We add to this research by studying 14-year-old students' experiences of scale identified in whole-class and small-group discussions during lessons in a learning study (Huang, Gong & Han, 2016). We show how a team of teachers gained insights into students' experiences of scale, and how these insights were later enacted in a lesson. The aim of this paper is to contribute to a deeper understanding of how students experience the relationship between change in length and change in area when enlarging or reducing two-dimensional geometric figures in lessons, and furthermore to illustrate how teachers learn about students' experiences from taking part in a practice-based collaborative and iterative process. Our research questions are:

> How are students' experiences of the relationship between change in length and change in area, when enlarging or reducing two-dimensional geometric figures, shown in lessons?
>
> How is the knowledge that teachers gain about the students' experiences reflected in their teaching?

Students' learning of enlargement and reduction

There is a vast amount of research investigating students' difficulties in understanding proportional and non-proportional relations when enlarging and reducing geometric figures. A major problem identified is the illusion of linearity (e.g., De Bock et al., 2002; De Bock et al., 1998; Gagatsis, Modestou, Elia & Spanoudis, 2009). A majority of students tend to wrongly apply a linear model when solving non-linear problems about relationships between lengths and areas of enlarged and reduced plane geometric figures. Several studies have shown that students often succeed in solving problems that involve figuring out enlargement and reduction where the relationship is linear, but perform poorly on problems where the change is quadratic or cubic. Studies show that the dimension of the figure is crucial and that it is the transition from a one-dimensional figure to a multi-dimensional figure that can cause problems, rather than the shape of the figure (De Bock et al., 2002; De Bock et al., 1998; Modestou, Gagatsis & Pitta-Pantazi, 2004; Van Dooren et al., 2004). However, others argue that the circle may be particularly difficult due to its shape. If the diameter of a circle is doubled, the area is more than doubled, but this can be hard to show, which makes it difficult to discover and highlight the incorrectness of a linear method when working with the circle (De Bock et al., 2002).

Although efforts aimed at eliminating these tendencies have been made, research shows that the phenomenon survives and returns, regardless of the age of the students and experimental settings. Even with significant support, such as images, meta-reasoning and authentic problems, only a few students showed that they mastered both relationships (linear and non-linear) (Van Dooren et al., 2004; Modestou et al., 2007). Hilton, Hilton, Dole and Goos (2013) showed that 60–80 % of the students (age 10–16) in their study answered that the area of a butterfly was doubled when the length and width of the butterfly were doubled. Less than 10 % of the students, regardless of age, could answer the question correctly. Furthermore, it was found that when students work with tasks involving non-linear relationships, they subsequently have greater difficulties when given a task where there is a linear relationship (De Bock et al., 1998; Paic-Antunovic & Vlahovic-Stetic, 2011). The conclusion drawn by Paic-Antunovic and Vlahovic-Stetic (2011) is that the feedback to students resulted in "the illusion of linearity" decreasing, but it did not make it possible for the students to see the difference between these two relationships or kinds of problem, nor did it increase students' ability to understand the mathematics that underlies the problems. Ayan and Bostan (2016) found in a study in Grades 6, 7, and 8 that the

> obstacle of linearity was revealed in all of the problems related to the area and volume regardless of the variation of didactical variables such as including a figure or not, including a square/cubic number or not, including a direct or indirect measure, and using real-life connections. (p. 516)

Theoretical framework

The variation theory of learning (Marton, 2015) is the theoretical framework used in this study. The theory stems from more than thirty years of phenomenographic research investigating students' conceptions of different phenomena (Marton, 1981; Marton & Booth, 1997). How learners *experience* an "object of learning" is central in the theory. What students say or do reflects how they experience the phenomenon at hand (Marton, 2015). The object of learning refers to what students should learn, and "critical aspects" relate to what the learner needs to discern in order to learn. In the learning study that is reported on in this paper, the teachers identified critical aspects for their students' learning (see Methods section below). In order to discern the critical aspects, students need an opportunity to experience variation in relation to the aspects (Huang & Li, 2017; Runesson, 2005). Consequently, for an aspect to be discerned, the students must experience variation, through contrast (by seeing difference) or generalization (by seeing sameness) (Marton & Pang, 2013). For example, when the teacher directs students' attention towards the difference between linear and non-linear relationships, a contrast between them is made (Pillay, 2013). When the teacher on the other hand compares two linear relationships, it may be possible to discern the similarity between the relationships. Both teachers and students can create these patterns of variation and invariance in lessons and tasks (Kullberg, Runesson Kempe & Marton, 2017; Watson & Mason, 2006). The critical aspects can be different for different learners, and the object of learning is therefore dynamic and can change in the process of the learning study (Mårtensson, 2015; Pang & Ki, 2016), since the meaning of identified critical aspects changes for the teachers and new critical aspects for students' learning emerge from the learning study process (Runesson, 2007).

Method

The data analyzed in this paper was generated from a learning study (Cheng & Lo, 2013; Huang et al., 2016; Marton & Tsui, 2004), a type of lesson study (Lewis, Perry & Murata, 2006; Yoshida, 1999) in which a team of teachers plan a lesson collaboratively and refine it several times in an iterative process. In a learning study, the teachers use a learning theory (e.g., variation theory) as a tool for designing lessons and enacting them. The learning study (LS) in this study included three cycles (lessons) with planning and revision (see figure 1). The lessons were planned collaboratively by a team of three experienced teachers and a facilitator (Svanteson Wester, 2014). During meetings, the teacher team reflected upon the nature of the object of learning, for instance: What

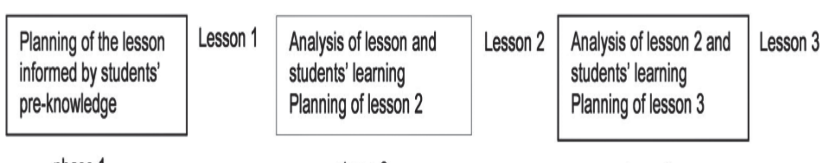

Figure 1. *The iterative process of planning, evaluating and revising teaching and learning in a learning study (Kullberg, Mårtensson & Runesson, 2016, p. 313)*

does it imply to have the ability to enlarge and reduce two-dimensional geometric figures? What must be learned in order to be able to handle (length) scale correctly? How do students experience the object of learning? Before the initial planning of the lesson, a written pre-test was used to gain knowledge about students' understanding of the topic. The students' learning from the lesson was tested by a post-test some days after the enacted lesson.

Three Grade 8 classes from a school located in a large Swedish city were involved in the study and each class had one double lesson (2 x 40 minutes) in the learning-study cycle. The students had written consent from their parents to participate. The teachers were selected because they were familiar with the variation theory of learning, and two of them had participated in a LS on the same topic before.

The teacher team planned, analyzed and revised the lesson three times in the learning study. Initially, the teacher team had a hypothesis about what students needed to discern in order to master the specific content. Four critical aspects for students' learning had been identified on the basis of teachers' experiences and from research in relation to the object of learning: i) to discern the meaning of uniform image, ii) to discern the lengths in a geometric figure, iii) to discern the change in length when enlarging and reducing two-dimensional geometric figures, and iv) to discern the change in area when enlarging and reducing two-dimensional geometric figures. The critical aspects were enacted in the lesson through tasks that the teacher discussed with the students. During the process of the learning study, the understanding and meaning of the critical aspects deepened for the teachers (Mårtensson, 2015). The knowledge gained about students' experiences of the object of learning, we suggest, made the teachers change the teaching in lesson 3.

The tasks

Several tasks were designed by the teacher team and used in the lessons in order to make the critical aspects noticeable for the students. For example, "The photograph task" (see figure 2) and "The plus sign task" (see figure 3) were used to discuss what a uniform image is, and to make it possible to discern the change in lengths when enlarging by scale factor 2:1. In the photograph task, one photo was compared to three alternative images in which one or two lengths were enlarged.

Other tasks used in the lessons were "The square" (see figure 6), "The paper" (see figure 4, 1:2, 1:4, 1:3), and "The pizza/circle" (1:2, 2:1). For example, the paper task was about reducing (1:2) the size of a sheet of paper by folding it. In the pizza task, the students were asked to compare

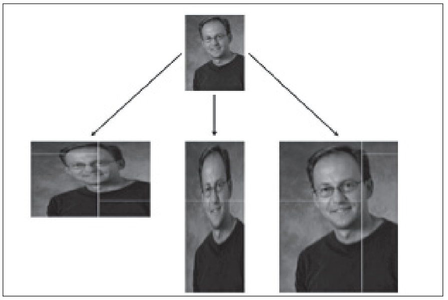

Figure 2. *The photograph task. Enlarge a photo (2:1) as a uniform image*

the amount of pizza in two small pizzas to one large pizza. The small pizza (circle) was the same shape as the large pizza (circle), but reduced by scale factor 1:2. In these tasks, the scale factor (e.g., 1:2, 1:4, 1:3 or 1:2, 2:1), or the shape being enlarged or reduced (square, rectangle, circle) varied between tasks.

Process of analysis

In order to study students' experiences of the relationship between change in length and change in area when enlarging or reducing two-dimensional geometric figures in the learning study, we used video-recordings of both whole-class and group discussions. A one-camera approach was used to capture the whole-class discussion, and one small-group discussion per lesson. The analysis of data proceeded in the following steps: First, we analyzed transcripts from lessons in order to search for sequences where change in length and change in area were present at the same time during teaching in whole-class and group discussions. We identified 24 sequences (in total in the three cycles) in which change in length and in area were present simultaneously. When one student during a whole-class discussion said; "Is the change in area always the double change in length?", this was coded as a sequence in which the change in length and change in area were present at the same time, and the relationship between them is focused upon by the student. Second, we used variation theory to identify the different ways of experiencing the relationship shown by students in the lessons, and how the content was handled by the teacher. The analysis of students' comments (answers to specific tasks and their discussion about the tasks) made it possible to categorize different ways of experiencing the same phenomenon (in this case the relationship). When analyzing the teaching, we analyzed the patterns of variation (e.g., contrast) used by the teacher.

Results

First, we show how students' experiences of the relationship between change in length and area when enlarging or reducing two-dimensional geometric figures were shown in lessons, and second how the teacher team enacted their insights about one critical aspect of the object of learning in lesson 3.

Students' experiences

The point of departure for the teaching in lessons 1 and 2 was first to make it possible for the students to discern the change in length compared to the scale factor and then to discern that the area will not change in the same way as the lengths. The students on the other hand indicated by their comments in the lessons that they were unsure about *how* length and area are connected, not only that they change differently. We demonstrate this with two examples. The first sequence (lesson 1) shows that the teacher focuses on the change in length when scaling and that change in length *is not* the same as the change in area. The teacher started to discuss what was twice as big in the figure when scaling with the scale factor 2:1.

Excerpt 1

1 Teacher:	The question is just what we want to be twice as big? Because there are a lot of things, at least three different things, that could be twice as big
2 Saga:	I thought, wouldn't the area be twice as big as well? If the length is going to be twice as big and the width twice as big

The teacher did not take Saga's comment into account and the change in area was not discussed further in this sequence. Later, during the whole-class discussion, the question about the change in area was brought up again when the teacher asked another student (Nils) what he thought about the idea that the area of a particular picture (no. 5 in figure 3) was twice as big as the area in the small original picture (see The plus sign task, figure 3). Nils answered that the area was four times as big. The teacher asked if this could be the case and asked the students to check if they could put four small figures in the big one and then the teacher said; "Do you see something weird here? That even if we make the lengths double, the area will not be twice as big. The area will be four times bigger".

In lesson 1, the teacher handled the two critical aspects *one at a time*, i.e. first the change in length compared to the scale factor, and then the change in area, with the purpose of letting the students discern that the change in area was not the same as the change in length. The students on

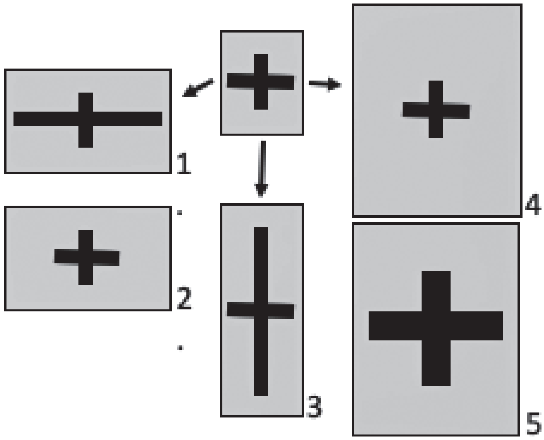

Figure 3. *The plus sign task:* Enlarge the picture by a scale factor of 2:1. *The original picture is in the middle and alternative enlargements of the picture, nos. 1–5, are around it*

the other hand seemed to focus on how change in lengths and area were connected. The teacher made a contrast between the change in length and change in area, however, only by telling what *it was* and what *it was not*, i.e. the change in length is double, when scaling 2:1, but the area is not double, the area is four times as big. However, the students did not get the opportunity to see how the two aspects were connected.

The second sequence (lesson 2) also shows students' ways of experiencing the relationship between change in length and area when enlarging or reducing two-dimensional geometric figures. The task worked on was to let the students reduce a sheet of paper by scale factor 1:4 and thereby make it possible for students to discern that when the lengths were four times longer or four times shorter, the area was not four times as big or four times as small. Two different solutions were brought up by the students from a task worked on during group work and discussed in whole class.

Figure 4. *Two solutions to the paper task about reducing a sheet of paper by scale factor 1:4 in lesson 2*

At the end of the whole-class discussion, one student (Erik) pointed out two different patterns between change in length and change in area, when he explained,

> Since you think two to the power of two, then it will be four [...] on the scale of 1:2. But two, no but four to the power of two is not eight, it is 16. The one with 16 pieces [...] has the right proportions.

The teacher did not discuss it further, but instead continued to focus on change in length and said that in the solution on the left-hand side, the students had only divided the base by four, not the height, which was just split in two, and in the correct solution, on the right-hand side (figure 4), all lengths were divided by four. The teacher was not aware at this point that the solution on the left hand-side (see figure 4) could be explained by the students having over-generalized the result from an earlier task when they reduced a rectangle by the scale factor 1:2 and the change in area was half the change in length i.e. when the scale factor was 1:2, the lengths were half as long and the area four times as small. The teacher directed the students' attention towards the lengths only when she said "it is the lengths we should constantly be thinking about when scaling." The teacher focused on the fact that the change in length and change in area are not the same, but she did not explore further in what way they are different. Hence, some students may still have believed that the connection between the change in length and change in area is linear, or those who were able to see that length and area change differently may not have discerned the relationship.

Student discussion during group work in lesson 2

For the teacher team, the analysis carried out after lesson 2 of the video recordings of students' discussions during group work was a "turning point" that made the teachers change their teaching in regard to how the relationship was taught in lesson 3. Students' comments about the content during group discussions indicated that some students were confused about what the *relationship* between the change in length and change in area was. For example, the teachers identified what was difficult for some students when they had to reduce a sheet of paper by scale factor 1:3. They found that in one student group, two different solutions to the task were discussed (figure 5).

While one student (Helene) argued that the new area should be one-sixth of the original, another student (Anne) instead argued that it was one-ninth. The rationale for Helene's answer, we suggest, may be related to the solution from a previous task, about reducing a sheet of paper by

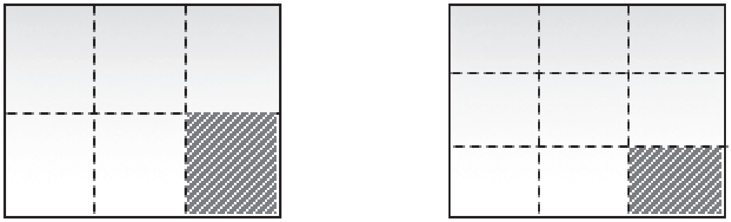

Figure 5. *Two solutions to the task "Reducing a sheet of paper by scale factor 1:3" (lesson 2)*

scale factor 1:2. Helene assumed that when reducing by scale factor 1:2, the change in length was half the change in area, i.e. halving the lengths will generate an area which is one fourth of the original area (excerpt 2, lines 4, 9 and 13). Helene's answer (see figure 5), that the change in area was one-sixth, when reducing by scale factor 1:3, was most likely based on this idea of a pattern of doubling of 2 and 3 in 1:2 and 1:3 respectively. At the beginning of the group discussion, Helene argued: "Listen to me. When we took half of this paper, we got four pieces. Check this out. When we split it in half, we got four" (line 4). At the end of the group discussion, she still insists that there should be six and not nine pieces: "But, really, there should be six pieces" (line 13). Based on the idea of "doubling" the denominator, the change in area should be one-sixth when using scale factor 1:3, since two times three is six. Anne, on the other hand, explained that the solution on the right-hand side seemed to be the correct solution and said that both lengths must be divided by three when the scale factor was 1:3. She most likely discerned that change in length was an important aspect of the object of learning, "and now if we take a third it will be as I said [points to paper with the piece that is one-ninth of the original]" (excerpt 2, line 9). She explained that the change in area in the previous task was one-fourth when reducing by scale factor 1:2. She focused on the change in length when she said; "half of the paper, both the length and width" (line 7) and then she compared it with reducing by scale factor 1:3 (line 9).

Excerpt 2

1 Anne:	I actually think it's this [shows a sheet of paper with the reduced figure corresponding to one-ninth of the original sheet of paper, the solution on the right]
2 Helene:	Yes, but it is not
3 Jacob:	Is it not?
4 Helene:	Listen to me. When we took half of this paper, we got four pieces. Check this out. When we split it in half, we got four

5 Teacher:	Why did you get four pieces?
6 Helene:	Because it is going to be four pieces
7 Anne:	Yes, we did [shows a sheet of paper with the solution to the task from lesson 2] half of the paper, both the length and width
8 Helene:	Yes
9 Anne:	And now if we take a third it will be as I said [points to paper with the piece that is one ninth of the original]
10 Jacob:	What do we do now?
11 Anne:	We take this [holding up the sheet of paper with one-ninth of the original] if we do not find anything else
12 Jacob:	Take that with you
13 Helene:	But, really, there should be six pieces
14 Anne:	Six pieces?

After the group discussion, this particular student group presented two different solutions to the whole class in lesson 2. Helene now argued that the solution on the left-hand side (figure 5) was probably incorrect and said that this was because they had divided only one side by two, and the other side by three (excerpt 3, line 1). At the same time, she pointed out, based on their arguments about the answer to the previous task (reducing a rectangle by the scale factor 1:2), that the change in area should be one-sixth.

> Well, I thought it would be like this, dividing in half, then it became four pieces [shows the solution from the task when they reduced a sheet of paper by scale factor 1:2], but when we split it in three then it should be six pieces. (excerpt 3, line 3)

Helene now seems to be arguing both for dividing each of the lengths by the scale factor and also for identifying a pattern in the relationship between change in length and change in area. At the end of the presentation, when the teacher asked Helene if she was satisfied with the group's answer, Helene argued that in the solution on the right-hand side (figure 5), all sides were divided by three (excerpt 3, line 5), and by that she indicated that she had discerned change in lengths. But at the same time Helene argued that the answer should be one-sixth when using scale factor 1:3, because it was four pieces when using scale factor 1:2 (line 3).

Excerpt 3

1 Helene:	Though it will be incorrect because then we have divided this side by three and the other by two
2 Teacher:	Yes

3 Helene:	Well, I thought it would be like this, dividing in half, then it became four pieces, [shows the solution from the task from lesson 2 when they reduced a sheet of paper by scale factor 1:2], but when we split it in three then it should be six pieces
[...]	
4 Teacher:	Are you satisfied with that?
5 Helene:	I don't know. But in the other one we have divided each side by three

Based on these insights about students' experiences of the relationship between change in length and area when enlarging or reducing two-dimensional geometric figures from analysis of the group discussion, the teacher team (see figure 1, phase 3) came to the conclusion that experiencing the relationship between change in length and change in area simultaneously, seemed to be essential in overcoming the illusion of linearity when teaching enlargement and reduction of two-dimensional figures.

Teachers' enactment of knowledge gained about students' experiences

It was not until lesson 3 in the learning study cycle that the relationship between the change in length and area was taught in a way that made it possible for the students (new class) to see how the lengths and area changed in relation to one another. When the teacher team analyzed (see figure 1, phase 3) the group discussions in lesson 2, and listened to and understood how the learners experienced changes in length and area when scaling, they were able to take this into account in the next lesson (with a new group of learners). In lessons 3, the team planned to first handle the change in length regarding the scale factor and thereafter the change in length and change in area simultaneously in order for the learners to experience the difference.

In the third task in lesson 3, about enlarging and reducing a square by scale factor 2:1 and 1:2 (see figure 6) one student (Axel), interrupted the whole-class discussion about the change in lengths and said: "The perimeter is doubled and the area is four times as big" (excerpt 3, line 1).

The comment by the student (Axel) afforded an opportunity to discuss the relationship between length, area and the scale factor. The teacher responded by pointing out that scaling is always about the lengths, but emphasized that the area changes as well, saying "but things happen with the surface as well" (excerpt 4, line 3). The teacher used the knowledge about how the students could experience the object of learning ("if the lengths are doubled, the area is doubled"), and made a contrast between Axel's view ("the area is four times as big") and the other view by saying: "Shouldn't it [the area] be twice as much?" (excerpt 4, line 5). A variation

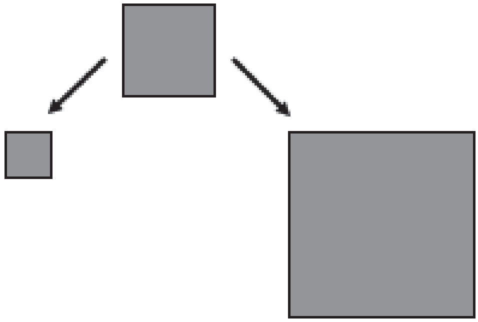

Figure 6. *The square task used in lesson 3*

of two contrasting views of the relationship between side-length and area when scaling up was opened up and brought to the students' attention; however this was most likely insufficient for students to be able to discern the relationship between area and length.

Excerpt 4

1 Axel:	The perimeter is double and the area is four times as big
2 Teacher:	We will take a look at this
[...]	
3 Teacher:	It is always about the length [when scaling], but things happen with the surface as well. Axel, will you develop your thought?
4 Axel:	Well
5 Teacher:	Shouldn't it be twice as much [the area]?
6 Axel:	No
7 Teacher:	So, it will be a four times bigger figure?
8 Axel:	Yes
9 Teacher:	In terms of area. The area becomes four times as big, but lengths were twice as large. So, different things happen with lengths and areas. But we want you to focus on the lengths here

Later, when the whole-class discussion was about the perimeter when scaling by the scale factor 4:1, another student (Sofia) also brought up area, asking: "So, the area is always twice as big?". The teacher responded by saying: "the length and perimeter, you notice that the same thing happens. They will be four times as big" and then she said: "But what happens to the area?" (excerpt 5, line 2). Once again, the change in length and change in area were handled simultaneously. Whereas one student (Allan) argued that the area should be "eight times as big" (line 3), another student Heli argued that the area should be sixteen times bigger and explained that it was because ”four times four is sixteen” (line 5). The teacher made a contrast between Allan's and Heli's answers: ”Four times four? You said earlier, Allan. Will it be doubled? Is four plus four eight?” (line 6).

Excerpt 5

1 Sofia: So, the area is always twice as big?

[...]

2 Teacher: But what happens to the area?

3 Allan: Eight times as big

[...]

4 Teacher: Heli, you've brought up this idea

5 Heli: Four times four is sixteen

6 Teacher: Four times four? You said earlier, Allan. Will it be doubled? Is four plus four eight? Yes, it's eight. [Pointing to the scale factor of 4:1] But it's not 16, right? It is four times four

In the last task in lesson 3, the teacher provides an opportunity to simultaneously discern the change in length and change in area and how they affect one another. When the students were working with a task comparing different circles in regard to their different scale, one student (Julia) said: ”The pizza is only half as big around, but it is four times as small in area” (excerpt 6, line 1). The teacher's question: ”How did that four come up?” (line 2) made it possible for the students to explore and articulate how length and area were connected and how they affect one another. Another student (Allan) explained how these two aspects were connected,

> You take the scale factor and multiply by itself. If it is 1:2 you take two multiplied by two, so it will be four. If you take the scale factor 1:3, you take three multiplied by three, so it will be nine. (line 3)

In the sequence, the change in length and change in area were handled simultaneously.

Excerpt 6

1 Julia: The pizza is only half as big around, but it is four times as small in area

2 Teacher: How did that four come up?

[...]

3 Allan: You take the scale factor and multiply by itself. If it is 1:2 you take two multiplied by two, so it will be four. If you take the scale factor 1:3 you take three multiplied by three, so it will be nine

The analysis of the lessons shows that the students experience the change in length and change in area in three different ways. The ways of experiencing are the same for different scale factors used. For example, for scale factor 1:2, when lengths decrease by a factor of 2, the students experience that: i) the area decreases by a factor of 2, or ii) the area decreases by a factor of 4 (due to doubling the decrease in lengths), or iii) the area decreases by a factor of 4 (due to multiplying 2 by itself). For scale factor 1:3, when the lengths decreases by a factor of 3, the students experience that: i) the area decreases by a factor of 3, or ii) the area decreases by a factor of 6 (due to doubling the decrease in length), or iii) the area decreases by a factor of 9 (due to multiplying 3 by itself). The same ways of experiencing were also found for 1:4.

In sum, the analysis shows that the teachers change their teaching in regard to their experience of students' learning. Table 1 shows the interplay between students' difficulties in regard to the object of learning and the teachers' teaching in the lessons. It was not until lesson 3 that the teacher brought up and discussed the relationship between the change in length and change in area.

Table 1. *Interplay between students' difficulties and the teachers' teaching*

Lesson	Students' difficulties in regard to the O.L.	Teachers' teaching in regard to the O.L.
Lesson 1 Class A	Why is the change in area different from the change in length	The change in area is not the same as the change in length
	The students focused on the relationship between the change in length and change in area	*The teacher focused on the change in length in relation to the scale factor*
Lesson 2 Class B	Why is the change in area different from the change of length (The students contrasted two different patterns related to the relationship between change in length and change in area)	The change in area is not the same as the change in length
	The students focused on the relationship between the change in length and change in area	*The teacher focused on the change in length in relation to the scale factor*
Lesson 3 Class C	Why is the change of area different from the change of length	The change in length and the change in area in relation to the scale factor are taught simultaneously The relationship between the change in length and change in area is explored Contrasts between different ways of experiencing the relationship
	The students focused on the relationship between the change in length and change in area	*The teacher focused on the relationship between the change in length and change in area*

Discussion and conclusions

Overcoming the illusion of linearity through teaching has been shown in previous studies to be a difficult task (Modestou & Gagatsis, 2013), regardless of the age of the students (De Bock, Verschaffel & Janssens, 1998). The teachers in this learning study initially thought that it was sufficient to teach *that* lengths and area changed differently when enlarging and reducing geometric figures in order for them to learn what was intended. In the learning study process (phase 3), the teacher team discovered that discerning the relationship between how length and area change was troublesome for the students. The relationship (how change in length and area differ) turned out to be a critical aspect for student learning. Although the teachers initially identified the change in length and change in area to be two critical aspects, they were not aware that it was the relationship between them that was problematic for the students, and therefore necessary to address. Our study adds to previous research by showing how 14-year-old students experience the relationship between change in length and change in area when enlarging or reducing two-dimensional geometric figures with different scale factors, in interaction with other students in lessons. The study shows that it was possible for the teachers to identify students' different ways of experiencing when they analyzed video recordings of students' group discussions and whole-class discussions. For example, the teachers found that the students, when reducing a sheet of paper by scale factor 1:2, understood the "four" in "one fourth of the original area" in different ways, as coming in this case from "two plus two", "two multiplied by two", or "two to the power of two", all of which are four. This made some students believe that the area was also reduced by half when encountering other scale factors (e.g., 1:3, 1:4).

The teachers in the learning study came to the conclusion that they needed to handle the change in length and change in area simultaneously in lesson 3, to make the change in length and area, and the relationship between them, come to the fore. The contrast between a set of carefully designed tasks involving first scale factors 1:2 and 2:1, then 1:4 and 4:1, followed by 1:3 and 3:1, proved to be successful, as it allowed the teachers to recognize that students had different ways of experiencing the number two in the scale factor 1:2 (as two multiplied by two or as two multiplied by itself) when finding the change in area. However, it may be a surprise that the teachers did not pay attention to why a scale factor in the length produced an n^2 scale factor in relation to the area from the beginning. This study shows that the teachers had another idea about what was critical for students' learning, but that they changed this view during the process. Furthermore, it is possible that the jointly designed lesson plan may have restricted the teachers' teaching. However, the teachers did not express this during the meetings. The identified critical aspects and the implementations of them have clear implications for teaching practice, as critical aspects can be tried out and revised by other teachers and other students (Kullberg, 2010; Morris & Hiebert, 2011).

The teachers' analysis of the video-recorded lessons, and especially recordings from the group discussion, played a significant role for the teachers in noticing how the students experienced the relationship between change in length and area. The student group discussions, we infer, offered an opportunity for the students to express their own experience of the object of learning, and therefore played an important part in the teachers' opportunities to notice how their students experienced the change in area. Although our study shows that the analysis of the group discussions was of significance, similar findings could very likely be obtained from, for example, whole-group discussion. In our study, the teachers did not notice these different ways of experiencing the object of learning during whole class (see e.g., excerpt 1). There could also be other critical aspects of this object of learning that this study has not identified since this study focused primarily on the relationship between length and area. Our study contributes to research on lesson and learning studies by pointing out the significance of analyzing group discussions when trying to identify critical aspects for students' learning. With the knowledge about students' ways of experiencing identified in group discussions, the teachers were able to address a new critical aspect (the relationship) in lesson 3. The object of learning (what is taught) is in this sense dynamic (Mårtensson, 2015) and can change in the process of a learning study.

Guskey (2002) argues that teachers will not change their practice unless they see how their teaching affects students' learning. During a learning study, teachers have the opportunity to analyze their teaching in relation to their students' learning, and learn from this experience (Cai, Hohensee, Hwang, Robinson & Hiebert, 2017). Learning studies has proven to be a beneficial way of developing both students' and teachers' learning (Cheung & Wong, 2014; Huang & Shimizu, 2016) from lessons. By means of practice-based professional development, such as a learning study, we suggest, teachers learn and become sensitive to their students' learning, which allows them to teach so that the students will have the opportunity to learn; in this sense, the teachers and students learn from one another.

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