Exploring engineering students' participation in flipped mathematics classroom: a discursive approach

HELGE FREDRIKSEN AND SAID HADJERROUIT

This paper explores first-year engineering students' participation in flipped mathematics classroom. The work uses Sfard's commognitive framework both as a lens for conceptualizing learning as participation in mathematical discourse and as a methodology for analysing the data generated by the activities that build the mathematical discourse. Data was collected mainly by video recording of classroom activities of first-year engineering students enrolled in several mathematics courses at a Norwegian university in 2016/2017. The aim of the study is to add to the lack of research on participation in flipped mathematics classrooms at the university level. The paper argues that engagement in the videos out-of-class enhances students' participation in the mathematical discourse. The commognitive analysis comparing out-of-class videos and in-class activities show that there are indications of student learning through expansion of the discourse in the videos and enhanced participation in mathematical activities.

Students participate in various ways in mathematical activities, for example in the context of the classroom, when they listen to the teacher or take notes, ask questions to clarify the correctness of solutions, actively engage in discussions, reflect on and explain their understanding, and work with peers. Students also participate in mathematical activities when using textbooks individually or in classroom settings. With advances in digital technology and online resources (Juan, Huertas, Trenholm & Steegmann, 2012), new forms of participation in mathematical activities emerge, e.g. students watching videos, doing exercises or quizzes out-of-class without the direct presence of the teacher. Today, flipped classroom (FC) as a technology-supported instructional approach has gained attention in mathematics education. FC is characterized by

Helge Fredriksen, *University of Agder*
Said Hadjerrouit, *University of Agder*

Fredriksen, H. & Hadjerrouit, S. (2020). Exploring engineering students' participation in flipped mathematics classroom: a discursive approach. *Nordic Studies in Mathematics Education*, 25 (1), 45–64.

its course structure, which consists of out-of-class activities where videos take the place of direct instruction and in-class activities where the students focus on key mathematical concepts (Bergmann & Sams, 2012). The most innovative part of a FC, in contrast to video-based learning and similar approaches, is the use of videos for preparatory homework combined with group work in classroom. As students meet in class having prepared for the topic through specially tailored video-homework, there are ample opportunities for student participation in challenging problem-solving tasks during class sessions (Strayer, Hart & Bleiler, 2015).

The research goal of this paper is to explore students' participation in mathematical discourse when FC is employed in university level mathematics courses for first-year engineering students enrolled in several mathematics courses in a Norwegian university in 2016/2017. Guiding research questions are presented in the methodology section.

The work is grounded in a sociocultural view of mathematics which frames learning as participation in mathematical discourse. We rely on Sfard's commognitive framework (Sfard, 2008) that conceptualizes mathematics as a discourse to study students' participation in flipped mathematics classroom. Our argument for using this framework is that it provides a conceptual apparatus for analysing fine-grained aspects of students' participation in the mathematical discourse developed in a FC setting.

Participation in flipped mathematics classroom

A crucial feature of flipped mathematics classroom is its potential to free up time in-class for facilitating participation in mathematical activities focusing on key concepts of the topics that are introduced in the videos out-of-class. The research literature reports on studies that refer to a wide range of learning approaches that are meant to stimulate students' engagement in flipped mathematics classroom, such as active learning (Adams & Dove, 2018; Cilli-Turner, 2015; Kerrigan, 2018), self-paced learning (Weng, 2015), self-directed learning or problem-based learning (Tawfik & Lilly, 2015; Wan, 2015), inquiry-based learning (Capaldi, 2015; Dorier & Maass, 2014; Love, Hodge, Corritore & Ernst, 2015; Love, Hodge, Grandgenett & Swift, 2014), inquiry-based and cooperative learning (Overmyer, 2015), learner-centred pedagogy (Rufatto et al., 2016), student-centred learning (Kuiper, Carver, Posner & Everson, 2015), student thinking (Strayer et al., 2015), or flipped learning (Ouda & Ahmed, 2016). Our understanding of the research literature is that most of these approaches are rather theoretical perspectives on what constitutes learning and are applied to argue why FC is a suitable instructional approach to organizing mathematical learning.

A large number of these studies report on effects of flipped mathematics classrooms in terms of students' perceptions, performance, achievement, attitudes, or satisfaction in comparison to traditional non-flipped courses. For example, Love et al. (2015) argue for turning a traditional classroom into an engaging, inquiry-based learning (IBL) environment by moving the acquisition of basic course concepts outside the classroom and using class time for active problem-based learning. The results describe students' perceptions of the flipped/IBL classroom model rather than participation in mathematical activities. Rufatto et al. (2016) report on increased performance for students participating in a course redesigned as FC, but no clear definition of the notion of participation is given. Weng (2015) describes a developmental mathematics course design that uses flipped instruction and self-paced learning. The author argues that this design suits the students well, and that the learning outcome is better than traditional classes and student satisfaction is high. Capaldi (2015) included inquiry-based learning in a flipped classroom and reported that both styles emphasize active learning and critical thinking through activities such as group work and presentations, while minimizing lectures. However, these studies do not inform research as to how participation is operationalized.

The research literature on FC motivates the present study in two ways. Firstly, although there are large number of studies that report on students' learning of mathematics in terms of engagement, perceptions, performance, or satisfaction of FC in contrast to non-flipped courses, none provides insight into forms of students' participation in flipped mathematics classroom. Secondly, the theoretical approaches employed in flipped mathematics classroom research do not offer a sufficient conceptual apparatus for analysing "fine-grained aspects of student participation in the mathematical discourse" (Nardi, Ryve, Stadler & Viirman, 2014, p. 185). As a result, these approaches are not adequate to properly explore students' participation characterizing flipped mathematics classroom. Since a key feature of flipped mathematics classroom is to facilitate student participation in the mathematical discourse, we argue that using the commognition framework, its discursive notions and view of learning as participation in collective activities will advance knowledge about teaching and learning in a FC context.

The commognitive framework and mathematical discourse

Sfard (2008) proposed "to combine the terms *communicational* and *cognition* into the new adjective *commognitive*" (p. 83), in order to stress the idea that cognitive processes and interpersonal communication are but different manifestations of basically the same phenomena. Furthermore, Sfard develops her ideas by clarifying that mathematical learning emerges in specific forms of discourse, and ”participation in communicational activities of any collective that practices this discourse” (p. 91). Mathematics as a discourse is thus considered as a specific type of communication, and it is not the same as mathematics as language, because this type of communication is much wider than simply language (Sfard, 2008; Wing, 2011). When learning mathematics is conceptualized as developing a discourse, the unit of analysis is to be found in the mathematical discourse itself. Given this background, Sfard (2008, pp. 133–135) defines four characteristics of mathematical discourse:

Word use
Special keywords used in the mathematical discourse, such as ”triangle”, ”function”, ”vertical asymptote”. The uses of these words are well defined in the mathematical discourse, even though they can appear in everyday discourse.

Visual mediators
Visual objects operated upon as part of the discursive process. They include symbolic objects, such as a mathematical formulae or formal notation systems, iconic mediators, such as graphs and pictures, and concrete mediators, such as beads of an abacus. In the context of the flipped mathematics classroom in this study, videos are used as a medium for presenting the discourse by means of graphs, functions, mathematical formulae, etc.

Endorsed narratives
Spoken or written utterances concerning the ”description of objects, of relations between such, or processes with or by objects” (Sfard, 2008, p. 300). Narratives are subject to endorsement or rejection within the discourse. In the case of academic mathematical discourse, the narratives that are approved consensually are called mathematical theories, which include definitions, axioms, and theorems.

Routines
When observed over time, mathematical discourse is repetitive and patterned. Routines are the ”set of meta-rules that describe a repetitive discursive action”. The term routine is broad, and it can refer to many rules, those regulating the properties of mathematics objects (object-level rules), and those less explicit regulating how the participants think about the mathematical objects (meta-level rules). More specifically, routines can be divided into *deeds* (aimed at change in objects), *rituals* (aimed at social acceptance or approval, and alignment with others' routines) and *explorations* (aimed at the production of an endorsed narrative). Explorations themselves can be divided into construction, substantiation and recall.

A particular property of routines in a mathematical discourse involves the realization of discursive mathematical objects. Sfard (2008, p. 170) and later Nachlieli and Tabach (2012, pp. 12–13) illustrate this by an example on how the symbol x^2, a table of values, and a parabolic graph are considered to be realizations of the same discursive mathematical object, the "basic quadratic function". When the discourse contains narratives where such a realization of mathematical objects occurs, the discourse can be said to be *objectified*. Objectification comprises two sub-processes: *reification*, "a replacement of talk about processes with talk about objects" (Sfard, 2008, p. 301), and *alienation*, a "discursive form that presents phenomena in an impersonal way" (p. 295).

Summarizing, the commognitive framework provides operational definitions of discursive notions that are central to mathematics, and a conceptual apparatus for analysing mathematical discourse. In addition, the framework comes equipped with a set of methodological tools suitable for analysing student learning as participation in mathematical discourse when employing FC at the university level.

Commognition and the notion of learning as participation

Sfard (1998) suggested two metaphors of learning: Learning-as-acquisition, and learning-as-participation. The latter views learning as participation in collective activity, while the former regards learning as an individual endeavour. Commognition theory aligns itself with learning-as-participation.

Within this framework, learning is considered as a change in one's discourse. Sfard (2007, pp. 575–576) distinguishes two types of learning. Firstly, object-level learning, which expands the existing discourse of the participants through extending their word use, constructing new routines, visual mediators, and producing new endorsed narratives. Secondly, meta-level learning, which involves the meta-rules of the discourse, for example, defining a word will now be done in a different way, and this "originates in the learners direct encounter with the new discourse", for example, change from arithmetic to algebra (Caspi & Sfard, 2012). Learning as change of one's discourse happens through the "process of scaffolded individualization" (Sfard, 2008, p. 282), which presupposes interaction with the teacher, or people who have already mastered the discourse, for example, a competent student. In the commognition framework, teaching is not defined separately from what Sfard calls learning-teaching agreement (p. 299). Accordingly, a teacher is a person "who assumes the role of the leading discourse while the student is the person who assumes the position of the follower of the discourse" (Tabach & Nachlieli, 2016, p. 301).

From this perspective, mathematical proficiency is a matter of participating in a discourse characterized by its own words or vocabulary, visual mediators, set of routines, and narratives. The students become familiar with the ways of doing and thinking which are specific to mathematics. In a flipped mathematics classroom, the leading discourse will first of all be found in the out-of-class videos that students use in preparing for in-class activities. In-class, students participate in different ways in the mathematical discourse, while the teacher takes a more orchestrating and guiding role.

Commognition in mathematics education at the university level

Several studies have used the commognition framework to investigate mathematical discourse at the university level, for example, regarding textbook discourse (Park, 2016), in-service teachers' mathematical discourse (Berger, 2013), the discourse of limit (Güçler, 2013, 2016), discursive shifts in calculus (Nardi et al., 2014), the discourse of functions (Viirman, 2014), undergraduate mathematics students' first encounter with subgroup test (Ioannou, 2018), and comparison of English and Korean speaking university students' discourses on infinity (Kim, Ferrini-Mundy & Sfard, 2012). Looking at five empirical papers, Presmeg (2016, p. 423) suggests that commognition is broad enough to be a useful theoretical lens for research in diverse settings. However, some important issues are not yet investigated thoroughly, indicating that there is unrealized potential for use of the theory, for example, communication in the form of gestures and body language (Ng, 2016), the affordances and constraints of digital tools (Berger, 2013), and the compatibility issue with other theories (Nardi et al., 2014). Further to this, and despite growing interest in the theory, there is a lack of studies using commognition to explore students' participation in flipped mathematics classroom. Hence, the contribution of this study is in the use of the commognitive framework in the investigation of an alternative undergraduate mathematics teaching setting, that is flipped mathematics classroom.

Methodology

This article is based on data collected as part of a study conducted at a university campus in Norway, where we follow several classes of first-year computer engineering students. These students participate in two or three mathematics courses during their study and the results presented here are based on data from the 2016/2017 cohort of 25 students, while they follow a course in calculus and linear algebra called Mathematics-1.

The course was conducted utilizing a flipped classroom (FC) design with two sessions per week, each consisting of out-of-class and in-class components, where students were asked to prepare for the in-class session by watching 3–5 videos, each 8–15 minutes long. The in-class sessions were spent on activities related to the material presented in the corresponding out-of-class videos. These could either be exam-related text-book tasks meant for rehearsing the procedures learnt in the videos, or more open-ended tasks with the purpose of modelling or investigating mathematical phenomena. The students were arranged in groups, and the in-class sessions were 90-minutes long.

The first two weeks of teaching of the study year were part of a joint research project between University of Agder and San Diego State University (SDSU). Due to this, teaching and video lecturing were conducted in English, where both the first author and the SDSU graduate student in mathematics education appeared as teachers in the sessions. Our data collection from the classroom sessions consisted of one camera focusing on a single group, in addition to a camera at the back of the classroom, filming most of the class activity in addition to the whole-class discussions taking place. During this period, we focused on filming the same group of students all the time, in case we later would look for longitudinal features of the group's activities that could be traced throughout this group's activity. The students in this group were picked due to their fluency in spoken English language to allow the SDSU researcher to be involved in the analysis. In all five session were filmed during these two weeks. The video recordings were first analysed utilizing descriptive accounts (Miles & Huberman, 1994). In these accounts, each session was broken into separate episodes of activity, where we highlighted the characteristics of the episode, in addition to noticing special features about the episode that might shed light on the research goal, which was to explore students' participation in the mathematical discourse.

The results presented in this article stem from the first session held for the class. We chose to focus our analysis on this particular session for two reasons. Firstly, the task for this session was similar to the task utilized in a design-based research presented in Wawro et al. (2012). This was appealing, since our study rests on previously published task design which reportedly spurred a collaborative atmosphere in the group work. Secondly, the earlier mentioned pre-analysis via descriptive accounts indicated a rich variety of student participation in the mathematical discourse, probably due to the open-ended problem formulation in the task. This was advantageous for the purpose of shedding light on the research goal.

One of the out-of-class preparatory videos related to this in-class session was made subject for analysis utilizing Sfard's commognitive framework. This video was chosen since it introduced all the important concepts of vector representations in $\boldsymbol{R}^2$ that the in-class task was based upon, including scaling, addition, and subtraction. The other videos in this out-of-class session showed various examples. One was dedicated to a real-life geographical situation, others demonstrated how to calculate the length, the direction, determine unit vectors, and scaling of vectors, in addition to showing examples like how to determine parallelism and calculating midpoints. Some examples also extended the concepts into $\boldsymbol{R}^3$.

According to Morgan and Sfard (2016), building an *analytical scheme* is a natural extension of a general discursive theory, such as commognition. The purpose of this scheme is to provide an in-depth analytical tool attuned towards a particular area of research. The scheme for this study is built on the notion of how various aspects of the discourse can guide researchers in asking in-depth research questions and how we might operationalize these questions by providing textual indicators. This is not the same as coding in the sense of grounded theory but is rather a tool to enrich our research goal on exploring student participation in more detail. The following scheme (table 1) was developed for, and utilized in, our analysis with the specific aim of finding indicators of students' participation in the mathematical discourse within the context of flipped mathematics classroom. Some of the questions and textual indicators were inspired by similar analyses found in Morgan and Sfard (2016) and Viirman and Nardi (2018).

The in-class session and the video were transcribed verbatim, and utterances in the transcripts were coded according to Sfard's four characteristics of discourse: endorsed narratives, visual mediators, routines, and vocabulary (word used), in addition to being informed by utilization of the analytical scheme. The out-of-class video and the in-class session were then analysed in connection to each other to look for similar patterns.

The task for the in-class session

The task given to the students in-class was to work with a problem related to movement using two modes of transportation, one with a "magic carpet" travelling along the direction [3,1], and the other one with a "hoverboard" along the direction [1,2] to be able to reach the location of the "old man Gauss' cabin" (Wawro et al., 2012, p. 581). The students were initially asked if it was possible to travel to his location at (107,64), using these means of transportation, a task most groups were able to

Table 1. *Analytical scheme for analysing FC discourse*

Aspects of the discourse	Guiding research questions	Textual indicators
Vocabulary	To what degree are specialized mathematical words in use?	Use of mathematical words like vector, negative, unknowns etc. to encapsulate mathematical meaning in contrast to more everyday words like direction, go and line.
Visual mediators	How does the discourse make use of visual mediators? What kind of mediators?	Presence of tables, diagrams, graphs, algebraic notation etc. Symbolic (mathematical symbols) versus iconic (drawings).
Endorsed narratives	What is the degree of reification and alienation?	Do we observe replacement of talk about processes with talk about objects? Is there talk about phenomena in an impersonal way, referring to abstract mathematical objects?
Routines	What role do explorations have?	Is it expressed routines aiming for solution of the posed mathematical problem, contrary to the "blind" application of a procedure?
	What kind of rituals can be observed?	Do we see examples of students' discussions in terms of acceptance, approval, or alignment with others' routines?
	What deeds can be seen?	Do we see prominence of examples in the out-of-class videos? Do students focus on attaining numerical results? Is the mathematics "tool"-like, that is, used to get concrete answers to tasks?

formulate an equation for, and find the numerical answer to, by solving the vector equation

$$a\cdot[3,1]+b\cdot[1,2]=[107,64].$$

After each group had a presentation of their solution in plenary, the groups were given another task:

> Describe the places that you can reach using a combination of the hover board and the magic carpet and those you cannot. Specify these geometrically and algebraically. Include a symbolic representation using vector notation. Also, include a convincing argument supporting your answer.

The idea behind this task was to help students go beyond the mere processual way of finding correct answers to vector calculation problem and aid them in developing the notion of span in a two-dimensional setting. Focusing on locating specific points in the plane that could be impossible to reach, would force the students to explore all possible locations in the plane. This could hopefully provide the opportunity to aim for a deeper exploration towards the properties of linear independence of vectors. We choose to zoom in on the group's discussion that took place after they were given this last problem, since we found the conversation between the students illustrative for shedding light on the research goal.

Results

Results are presented from analysing both out-of-class and in-class parts of the FC session described previously, based on transcripts. Specifically, the analysis is grounded in textual indicators from the analytical scheme, which informs us on the guiding research questions. Utilizing the same analytical scheme for both in-class and out-of-class transcripts aids us in finding similarities in phrasing of mathematical ideas and concepts, giving evidence on student participation in the leading discourse from the videos.

The video "Introduction to vectors" provided the students with a formal tutorial on vectors in $\boldsymbol{R}^2$. It showed how to add/subtract them in a geometrical sense and how to represent them with unit vectors and in a polar form. As such, the video was dominated by narratives about vectors, focusing on properties of vectors, and procedures on how to relate these through summation and subtraction. We provide some excerpts from this video below, where the first example shows a demonstration of the head-to-tail method of adding vectors. The sequence starts by considering two vectors ***u*** and ***v*** as shown in figure 1 (V.L. is short for video lecturer).

25 V. L.: Let's say I want to go in the direction ***v*** and then I would like to go in the direction ***u***.

26 V. L.: We have a method for doing this, which is called the head to tail method for adding vectors. And so you'll see that geometrically I

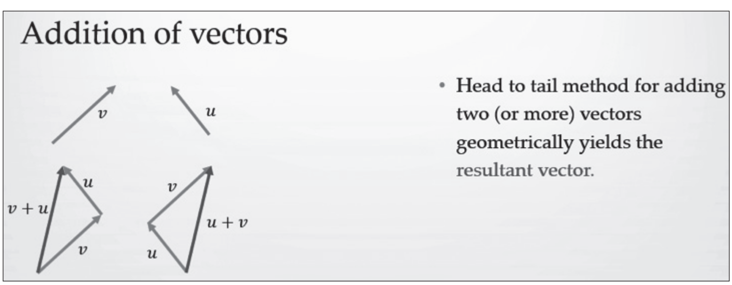

Figure 1. *A typical slide from the preparatory video on the introduction to vectors*

can add two vectors using this method, and I get what is called the resultant vector.

27 V. L.: So first I laid down the vector *v* and I aligned the vector *u* so that the head of the vector *v*, so that the arrow, the tail of the previous vector lines up with it [while this is stated, the adding of the two vectors is animated in the video].

The statement in line 25 could be considered an explorative routine, stating the problem in a manner that prompts the listener for what's to come, it aims for a solution of the problem, according to the analytical scheme (table 1). The statement in 26 is a narrative about the geometrical routine of adding vectors, which is later substantiated by another narrative in 27 on how the procedure should be acted out. Although this is a devised procedure for how to add vectors, it can also be considered to be highly alienated. There is no real-life attachment to the procedure, and as such it is a discourse on abstract mathematical objects.

Later on, the video lecturer goes on to demonstrate subtraction of vectors in 31 and 32.

31 V. L.: Subtraction of vectors work very similar except for subtraction is not communitive.

32 V. L.: So what we have to look at is the same process, let's say I have my vector *v*, and I want to look at subtracting *u*, well, I look at negative *u*, *v* minus *u*, so that direction is reversed, we notice here that we let *u* be negative, in this direction [pointing with cursor].

Another mentioning of adding vectors can be seen from the introduction of unit vectors in the videos later on in the video.

42 V. L.: And what these are is the unit vectors along both the *x*- and the *y*-axis so this *i* ["*i*" being a vector in this setting] down here is a unit vector, so it's of length one, and this *i* hat [ˆ placed on top of the vector *i*], hat is what we refer to the denotation of it, is the unit vector along the *x*-axis.

Dominant words used in the video were *vector* and *direction*. Iconic visual mediation through animated drawings of vectors was used constantly in the video to provide a graphical impression on mathematical vector operations.

During the in-class session, we filmed a group consisting of four persons: Einar, Moses, Ian and Pepin (names are pseudonyms). Before coming to class, all these students had watched the corresponding out-of-class videos. Before starting their work, they were given a sheet of A3 paper and pens with different colours for collaborating purposes.

In the excerpt below, the students are in the beginning of the discussion just after the task was given, trying to figure out which parts of the plane the two modes of transportation would reach.

3 Ian: [...] if we only go the magic carpet vector the line would go like this, and the line would go like this [drawing two lines on the sheet of paper for each mode of transportation]. So no but the thing is ...

4 Einar: So that would also be the opposite direction.

5 Moses: But we can't reverse, with the other.

6 Ian: If we go here, then we can reverse with this one to get here, which gets us here, and we can do the same thing here, and then go here [statements illustrated by iconic mediation at the group's sheet of paper].

In this excerpt, we can notice word uses like *direction* and *reverse*, which also were used in the first video the group had watched beforehand (turn 32 from the video mentioned above). In turn 3 and 6, Ian is working on describing a process for adding the two vectors (the two modes of transportation) to point to various concrete positions. During this initial phase, there were little endorsed narratives, rather, students were expressing their views through explorative routines. Ian is seen to construct the explorative routine statements in 3 and 6. This is in accordance to our analytical scheme (see table 1), since Ian is using his own reasoning in progressing towards a solution of the posed problem. Einar and Moses pose critical questions in 4 and 5. Again, utilizing the analytical scheme, these would be labelled ritual routines, meant to support and deepen the arguments of Ian's exploratory routines.

Visual mediation was performed through utilizing the A3 paper, representing the two modes of transportation with scaled versions of the "modes of transportation" vectors in the previous task. The arguing of Ian in 6 refers to this geometrical representation, where he is substantiating his claim by zigzagging towards an imagined placement of Gauss (see figure 2).

The next step in the group's discussion is a push towards more abstraction.

10 Einar: We need a general formula for it.

11 Ian: What I am proposing is that *X* times *h* plus *Y* times *m* can equal any position, given that *X* and *Y* can be negative numbers and decimals [Moses is talking along with Ian when uttering the formula].

12 Moses: We have too many unknowns, we can't solve that.

In turn 10, Einar prompts the group to make a mathematical formulation that would address the problem given in the task. In turn 11, it seems clear that Ian reifies the mathematical process that the group had been

Figure 2. *Students working on visualizing a process on how to reach an arbitrary position in the plane using the two vectors*

working on. He was able to generate a narrative combining the vectors ***m*** and ***h*** in arbitrary fashion using *X* and *Y* factors. The vectors ***m*** and ***h*** are invented to describe the two modes of transportation. From working process-wise, highlighting incremental steps to visualize how the transportation process towards Gauss cabin could be performed, Ian answers to Einar's generalization challenge in turn 10, uttering a mathematical statement about the linear combination of vectors that is completely alienated from the case in front of him. His narrative was also symbolically mediated on the group's sheet of paper. Another important property of the statement is how he is able to utilize mathematical terms like *negative numbers*, *decimals* and *position* in his expression, which is another alienated, impersonal utterance.

Moses, who was still thinking process-wise, was looking for a way to "solve" this equation in turn 12 and found it hard to deal with the general expression for the generalized linear combination that Ian depicted. Later on, the students continued their line of reasoning.

39 Ian: [...] if we are just multiplying them by something, and we also know that *X* times *Y* can equal any number. So if we do *X* plus *Y* we can get any number, and if we do *X* times *Y* we can get any number. So then, should not *X* times ***d*** plus *Y* times ***e*** also equal any number? I am pretty sure you can get any number.

40 Einar: You can, like unit vectors [...], you can get anywhere with them. So it makes sense that you can also do exactly the same if they are not parallel. So, I think that's true.

Ian was still doubtful if his mathematical object really expressed all possible points in the plane and tried to verify mathematically that all places were reachable upon multiplication and addition of arbitrary numbers and vectors (figure 3). Einar endorsed Ian's narratives and did so by

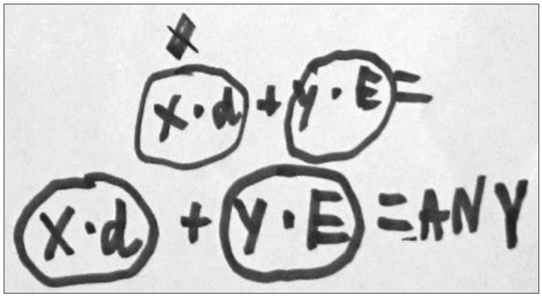

Figure 3. *Ian's symbolic mediation for reasoning on how all point in $\boldsymbol{R}^2$ should be reachable based on decomposition of any number in turn 39*

referring to unit vector additions described in the video mentioned above (see line 42). This indicates that the students had become familiar with the leading discourse in the videos and were able to expand and develop it through explorative routines.

Discussion

The main principle of FC is to enhance student participation through carefully designed sessions that synthesize out-of-class and in-class components into a consistent whole. The research presented in this article sought to explore how students participate in mathematical discourse in such flipped classroom (FC) sessions from a commognitive perspective. Specifically, we sought to address the research goal and the guiding questions raised in the analytical scheme presented in the methodology section.

Firstly, there is evidence that students utilize mathematical terms found in the videos to discuss mathematical ideas in the tasks. Terms like *unit vectors* and *reverse/opposite direction* can be related to the video demonstrating how sums of scaled unit vectors can form a resultant vector. We cannot prove that the students extended their vocabulary from watching these videos, however, there is reason to believe that the videos had an impact. We conjecture that since English is a foreign language for these students, it provides an even stronger evidence that the similarity in words being used support this claim.

Secondly, the videos seemed to contribute to students' formulations of endorsed narratives. The group was mathematizing actively, stating narratives that reified the process of how to reach a certain point in the first part of the task towards properties of vectors in general in the second part. Similar narratives were found in the video we analysed.

Thirdly, the task design seemed to trigger many explorative routines among the students, although rituals were observed to have a dominant role through students' supportive collaboration. The routine of adding several increments of vectors to obtain a resultant vector is relatable to similar discourse on summing unit vectors in the videos. Ian was leading in the process of reifying the group's initial attempts at iconic mediation of vector drawings into an objectified discourse about linear combinations. This can be seen as evidence that students indeed were able to extend the discourse in the videos towards a formulation of their own narratives about the linear independence of vectors.

These results show that the commognition theory allows a fine-grained analysis and a rich description of the mathematical discourse, which cannot be studied through approaches that consider learning as individual acquisition of knowledge.

We now consider some factors that could have influenced students' participation in the mathematical discourse. Firstly, videos in a mathematics FC take the role of direct instruction and aim at introducing the key mathematical words of the discourse, visual mediators to complement word use, and other discursive elements (routines and narratives). We could see clear signs that the ideas from the videos came through in the students' mathematical discourse in-class. We may highlight how the students used the idea of reversing a vector, the idea of adding scaled unit vectors to form a resultant vector, and the concept of using vectors to reach a point in the plane. This aligns with Sfard's (2008) notion of leading discourse (p. 282), where the teacher usually provides this in-class. However, in a FC setting, the students engage with the leading discourse through preparatory videos out-of-class, a discourse they have the opportunity to extend through the enactment of specially tailored in-class tasks.

This leads to the second factor that contributed to students' participation in the mathematical discourse. The tasks in the FC in-class session aimed at exploring and extending the mathematical concepts introduced in the videos, which were the basics of vectors. Furthermore, an additional purpose of the tasks was to bridge the out-of-class mathematical activities of the videos with those in-class in a meaningful way. The results indicate that the tasks in the session enabled the students to expand the discourse of the videos.

Group work in-class was, we posit, a third factor that enabled students to participate in the mathematical discourse. Even though Sfard's metaphor of learning-as-participation should not be equated with advocating a lot of discussion or collaborative work, there are indications that the students working together in the session provided opportunities for participation in the mathematical discourse in terms of approval and alignment with each other's routines.

In summary, these three factors together may have contributed to students' participation in the mathematical discourse. The results do not suggest that students changed their discourse, or that the findings can be generalized to other flipped mathematics classrooms. Nevertheless, this study provides ideas on central issues of the commognitive framework that can be applied to other FC contexts to explore student participation in mathematical discourse.

Conclusion

As stated in the literature reviewed, our study aimed to provide a deeper insight on student participation in mathematical discourse during an in-class session, based on the preparatory out-of-class videos. As our evaluation of the current literature showed, other studies have not addressed these aspects of mathematical learning in a FC environment. Simply reporting on individual students' perceptions of learning, attitudes, or performances does not provide insight on participation in mathematical activities. Although this study can be said to form a micro-analysis of a certain episode throughout the whole course, it nevertheless serves to characterize how learning is likely to have occurred through participation in a mathematical discourse supported by the FC pedagogical setting. Furthermore, we cannot claim that these results are unique for a FC setting, since the combination of videos, tasks, and group-work can be utilized in other pedagogical settings as well. However, these findings are significant for FC as it relies substantially on the use of out-of-class resources to prepare for in-class activities. Most students seem to respond positively to this systematic way of addressing students' participation in the mathematical discourse.

The commognitive approach directs attention towards a view of learning as participating in a certain mathematical discourse that is not bound to a specific conversation between discussants but evolves historically and culturally as a unity. As such, the analysis in this article is relevant beyond the situational aspect of the specific FC session considered. Although our data is collected from a class of engineering students, the task provided for the students and the mathematical activities in-class should apply to any discipline in undergraduate mathematics education.

From the discussion above, we see how important it is that students' engagement with the out-of-class leading discourse can develop through in-class participation with the very same discourse. It is of crucial importance for FC designs to take the discursive approach to mathematics seriously so that students do not experience disconnections between out-of-class and in-class activities which can result of course materials that are not discursively coherent.

Acknowledgements

We want to thank Matt Voigt for his valuable contributions during the initial phase of this study. Also, thanks to Olov Viirman for his kind support and advices.

References

Adams, C. & Dove, A. (2018). Calculus students flipped out: the impact of flipped learning on calculus students' achievement and perceptions of learning. *PRIMUS*, 28 (6), 600–615. doi: 10.1080/10511970.2017.1332701

Berger, M. (2013). Examining mathematical discourse to understand in-service teachers' mathematical activities. *Pythagoras*, 34 (1), 1–10.

Bergmann, J. & Sams, A. (2012). *Flip your classroom, reach every student in every class every day*. Washington: International Society for Technology in Education.

Capaldi, M. (2015). Including inquiry-based learning in a flipped class. *PRIMUS*, 25 (8), 736–744. doi: 10.1080/10511970.2015.1031303

Caspi, S. & Sfard, A. (2012). Spontaneous meta-arithmetic as a first step toward school algebra. *International Journal of Educational Research*, 51-52, 45–65. doi: https://doi.org/10.1016/j.ijer.2011.12.006

Cilli-Turner, E. (2015). Measuring learning outcomes and attitudes in a flipped introductory statistics course. *PRIMUS*, 25 (9-10), 833–846. doi: 10.1080/10511970.2015.1046004

Dorier, J.-L. & Maass, K. (2014). Inquiry-based mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 300–304). Dordrecht: Springer Netherlands.

Güçler, B. (2013). Examining the discourse on the limit concept in a beginning-level calculus classroom. *Educational Studies in Mathematics*, 82 (3), 439–453. doi: 10.1007/s10649-012-9438-2

Güçler, B. (2016). Making implicit metalevel rules of the discourse on function explicit topics of reflection in the classroom to foster student learning. *Educational Studies in Mathematics*, 91 (3), 375–393. doi: 10.1007/s10649-015-9636-9

Ioannou, M. (2018). Commognitive analysis of undergraduate mathematics students' first encounter with the subgroup test. *Mathematics Education Research Journal*, 30 (2), 117–142. doi: 10.1007/s13394-017-0222-6

Juan, A. A., Huertas, M. A., Trenholm, S. & Steegmann, C. (2012). *Teaching mathematics online: emergent technologies and methodologies*. Hersey: Information Science reference.

Kerrigan, J. (2018). Active learning strategies for the mathematics classroom. *College Teaching*, 66 (1), 35–36. doi: 10.1080/87567555.2017.1399335

Kim, D.-J., Ferrini-Mundy, J. & Sfard, A. (2012). How does language impact the learning of mathematics? Comparison of English and Korean speaking university students' discourses on infinity. *International Journal of Educational Research*, 51, 86–108.

Kuiper, S. R., Carver, R. H., Posner, M. A. & Everson, M. G. (2015). Four perspectives on flipping the statistics classroom: changing pedagogy to enhance student-centered learning. *PRIMUS*, 25 (8), 655–682. doi: 10.1080/10511970.2015.1045573

Love, B., Hodge, A., Corritore, C. & Ernst, D. C. (2015). Inquiry-based learning and the flipped classroom model. *PRIMUS*, 25 (8), 745–762. doi: 10.1080/10511970.2015.1046005

Love, B., Hodge, A., Grandgenett, N. & Swift, A. W. (2014). Student learning and perceptions in a flipped linear algebra course. *International Journal of Mathematical Education in Science and Technology*, 45 (3), 317–324. doi: 10.1080/0020739X.2013.822582

Miles, M. B. & Huberman, A. M. (1994). *Qualitative data analysis: an expanded sourcebook* (2 ed.). Thousand Oaks: Sage Publications, Inc.

Morgan, C. & Sfard, A. (2016). Investigating changes in high-stakes mathematics examinations: a discursive approach. *Research in Mathematics Education*, 18 (2), 92–119. doi: 10.1080/14794802.2016.1176596

Nachlieli, T. & Tabach, M. (2012). Growing mathematical objects in the classroom – the case of function. *International Journal of Educational Research*, 51, 10–27.

Nardi, E., Ryve, A., Stadler, E. & Viirman, O. (2014). Commognitive analyses of the learning and teaching of mathematics at university level: the case of discursive shifts in the study of calculus. *Research in Mathematics Education*, 16 (2), 182–198. doi: 10.1080/14794802.2014.918338

Ng, O.-L. (2016). The interplay between language, gestures, dragging and diagrams in bilingual learners' mathematical communications. *Educational Studies in Mathematics*, 91 (3), 307–326.

Ouda, H. & Ahmed, K. (2016). Flipped learning as a new educational paradigm: an analytical critical study. *European Scientific Journal*, 12 (12), 417–444.

Overmyer, J. (2015). Research on flipping college algebra: lessons learned and practical advice for flipping multiple sections. *PRIMUS*, 25 (9-10), 792–802. doi: 10.1080/10511970.2015.1045572

Park, J. (2016). Communicational approach to study textbook discourse on the derivative. *Educational Studies in Mathematics*, 91 (3), 395–421. doi: 10.1007/s10649-015-9655-6

Presmeg, N. (2016). Commognition as a lens for research. *Educational Studies in Mathematics*, 91 (3), 423–430. doi: 10.1007/s10649-015-9676-1

Rufatto, R., Dickin, H., Florescu, A., Lorch, C., Bremigan, R. & Lorch, J. (2016). Learner-centered pedagogy in a liberal arts mathematics course. *PRIMUS*, 26 (9), 863–877. doi: 10.1080/10511970.2016.1193830

Sfard, A. (1998). On two metaphors for learning and the dangers of choosing just one. *Educational researcher,* 27 (2), 4–13. doi: 10.3102/0013189X027002004

Sfard, A. (2007). When the rules of discourse change, but nobody tells you: making sense of mathematics learning from a commognitive standpoint. *The Journal of the Learning Sciences*, 16 (4), 565–613.

Sfard, A. (2008). *Thinking as communicating: human development, the growth of discourses, and mathematizing*. New York: Cambridge University Press.

Strayer, J. F., Hart, J. B. & Bleiler, S. K. (2015). Fostering instructor knowledge of student thinking using the flipped classroom. *PRIMUS*, 25 (8), 724–735. doi: 10.1080/10511970.2015.1031306

Tabach, M. & Nachlieli, T. (2016). Communicational perspectives on learning and teaching mathematics: prologue. *Educational Studies in Mathematics*, 91 (3), 299–306. doi: 10.1007/s10649-015-9638-7

Tawfik, A. A. & Lilly, C. (2015). Using a flipped classroom approach to support problem-based learning. *Technology, Knowledge and Learning*, 20 (3), 299–315.

Viirman, O. (2014). The functions of function discourse – university mathematics teaching from a commognitive standpoint. *International Journal of Mathematical Education in Science and Technology*, 45 (4), 512–527.

Viirman, O. & Nardi, E. (2018). Negotiating different disciplinary discourses: biology students' ritualized and exploratory participation in mathematical modeling activities. *Educational Studies in Mathematics*, 101 (2), 233–252. doi: 10.1007/s10649-018-9861-0

Wan, N. (2015). Technology integration and the flipped classroom. In N. Wan (Ed.), *New digital technology in education* (pp. 149–170). Cham: Springer International Publishing.

Wawro, M., Rasmussen, C., Zandieh, M., Sweeney, G. F. & Larson, C. (2012). An inquiry-oriented approach to span and linear independence: the case of the magic carpet ride sequence. *PRIMUS*, 22 (8), 577–599. doi: 10.1080/10511970.2012.667516

Weng, P. (2015). Developmental math, flipped and self-paced. *PRIMUS*, 25 (9-10), 768–781. doi: 10.1080/10511970.2015.1031297

Wing, T. (2011). Book review: purifying the dialect of the tribe. Anna Sfard (2008). Thinking as communicating: human development, the growth of discourses, and mathematizing. *Educational Studies in Mathematics*, 76 (3), 363–369. doi: 10.1007/s10649-010-9283-0

Helge Fredriksen

Helge Fredriksen received his master in physics from University of Tromsø in 1994. In addition to being a PhD Research Fellow at Agder University, Kristiansand, he holds a position as Assistant Professor at The Arctic University of Norway, campus Bodø. Fredriksen has research interests in active learning strategies in mathematics such as flipped classrooms. Additionally he participates in various R&D activities in information and communication technologies.

helge.fredriksen@uia.no

Said Hadjerrouit

Said Hadjerrouit received the MS and PhD degrees in informatics from the Technical University of Berlin (Germany), in 1985 and 1992, respectively. He joined Agder University, Kristiansand (Norway) in 1991. He is currently Professor of Mathematics Education at the Department of Mathematical Sciences. He has been in the teaching profession for 30 years. His research interests include Web design, didactics of informatics, ICT and learning, e-Learning, digital tools in mathematics education, Web 2.0 technologies, and teaching and learning theories in mathematics education.

said.hadjerrouit@uia.no