Algebraic thinking and level of generalisation: students' experiencing of comparisons of quantities

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This article explores grade 1 students' different ways of experiencing quantity comparisons after participating in teaching designed as a learning activity using tasks from the Davydov curriculum. A phenomenographic analysis generated three hierarchical ways of experiencing comparisons: counting numerically, relating quantities, and conserving relationships. The first category comprises arithmetic ways of thinking, whereas the second and third categories comprise algebraic ways of thinking. Algebraic thinking was identified as reflections on relationships between quantities at different levels of generalisation. The implications of these results in relation learning activity theory are discussed.

Previous research has highlighted problems regarding the development of algebraic thinking when teaching initially focuses on specific numerical examples, only later in the students' school career emphasising general structures in the concepts and their relationships (Cai & Knuth, 2011; Nunes, Bryant & Watson, 2009). Although general arithmetic structures and relationships can be regarded as the core of algebra (Bednarz, Kieran & Lee, 1996), and be seen as operations with numerical abstractions (Krutetskii, 1976), there are still difficulties with the transition between arithmetic and algebra (Hitt, Saboya & Zavala, 2016). However, some research treats algebra as a topic applicable to primary students (e.g. Cai & Knuth, 2011; Davydov, 1990, 2008; Kaput, 2008; Kieran, 2004; 2018; Kieran, Pang, Schifter & Ng, 2016; Nunes et al., 2009). If so, algebra can be used from the beginning of the students' mathematics education to enhance their reflection on general arithmetic structures and relationships (e.g. Davydov, 1990; Hitt et al., 2016). This is in line with the national curricula in Sweden, where algebra is to be taught from the

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earliest grades (Skolverket, 2018a). However, recent research shows that very little algebra is used in Swedish primary textbooks (Bråting, Hemmi & Madej, 2019), implying that it is up to the teachers themselves to understand what algebraic thinking can encompass.

When discussing algebra for younger students, Davydov's work has been cited as a central reference (see e.g. Cai & Knuth, 2011; Kaput, 2008; Kieran, 2004; Schmittau, 2004). Davydov and his colleagues developed the so-called Davydov mathematical curriculum for the youngest students, framed by learning activity theory that treats mathematical concepts in a theoretical way rooted in a specific cultural–historical tradition (Davydov, 2008; Roth & Radford, 2011; Schmittau, 2003). In this tradition, the teaching is oriented towards theoretical, abstract, and general mathematics (Davydov, 1990; Schmittau, 2003; Van Oers, 2001; Zuckerman, 2007), including the development of algebraic thinking as foundational knowledge (Davydov, 1990, 2008). Researchers following this tradition conclude that more research is needed into, for example, what task designs facilitate young students' algebraic reasoning (Eriksson & Jansson, 2017), and, following Davydov's terminology, how the "transition" from algebra to arithmetic can be understood (Hitt et al., 2016). Exploring young students' ways of experiencing various theoretical phenomena when involved in teaching intended to develop algebraic thinking may yield new knowledge in the mathematics education research field. With this as a backdrop, the aim of this study is to explore the possible impact of an algebraic learning activity on grade 1 students' different ways of experiencing comparisons of numerical quantities. The following questions are addressed:

1 What qualitatively different ways of experiencing the phenomenon comparisons of quantities can be identified among young students who have taken part in an algebraic learning activity?

2 What indications of emerging algebraic thinking are embedded in the students' different ways of experiencing this phenomenon?

Background

Algebraic thinking and levels of generalisation

Algebra and algebraic symbols can be used to develop a special way of thinking generally, abstractly, and theoretically, namely, algebraic thinking (Davydov, 1990, 2008; Kinard & Kozulin, 2008; Schmittau, 2011). Internationally, algebraic thinking has been approached in various ways intended to make teaching and learning meaningful at different levels of different school systems (e.g. Blanton & Kaput, 2005; Kieran, 2004). As such, algebraic thinking has been proposed to entail the generalisation of geometric patterns and numerical relationships, solving problems and equations using models, introducing functions, and modelling mathematical phenomena (Bednarz et al., 1996). This way of thinking can be summarised as operations with relationships and expressions of generalities to track the structures of arithmetic (Blanton & Kaput, 2005; Hitt et al., 2016; Kaput, 2008; Mason, 2017; Radford, 2010, 2013). It also includes abilities such as analysing relationships between quantities, noticing structures, studying changes, generalising, problem solving, modelling, justifying, proving, and predicting (Cai & Knuth, 2011). When education is intended to concentrate on structures and relationships within arithmetic instead of just on specific numerical examples, actions in education should shift from arithmetic to algebraic thinking (Kieran, 2007). Such a shift can be described as focusing on:

> 1) relations and not merely on the calculation of a numerical answer; 2) operations as well as their inverses, and on the related idea of doing/undoing; 3) both representing and solving a problem rather than on merely solving it; 4) both numbers and letters, rather than on numbers alone; and 5) the meaning of the equal sign from a signifier to calculate to a symbol that denotes an equivalence relationship between quantities. (Cai & Knuth, 2011, p. ix)

The shift to an algebraic way of thinking can be understood as a shift to general, abstract thinking (Davydov, 1990; Krutetskii, 1976). Regarding algebraic thinking, four levels of abilities to generalise can be identified in students' actions: (1) cannot generalise material according to essential attributes even with help; (2) generalise material according to essential attributes but make particular errors; (3) generalise material according to essential attributes on their own after several exercises; and (4) generalise material correctly and immediately without training in solving problems of a single type (Krutetskii, 1976). Following Krutetskii, students' algebraic thinking can be described as generalisation at these different levels.

Learning activity theory: learning tasks and learning models

When teaching is expected to enhance the youngest students' understanding of general structures in arithmetic, research shows that education should shift from addressing just arithmetic aspects to addressing algebraic aspects as well (Cai & Knuth, 2011; Dougherty, 2008; Kaput, 2008; Kieran et al., 2016; Venenciano, 2017). To treat such aspects, teaching can be designed according to an algebraic tradition (Van Oers, 2001). The Davydov curriculum, framed by learning activity theory, can be seen as a precursor of this tradition of developing algebraic learning activity (cf. Cai & Knuth, 2011; Van Oers, 2001). To understand the Davydov curriculum, some key concepts need to be defined. Since learning activity theory and the Davydov curriculum are grounded in a particular cultural–historical tradition, number sense is assumed to be deve-loped through measurements and comparisons, rather than, as is more usual in traditional mathematics education, through counting and operating with numbers (Davydov, 1975, 2008; Dougherty, 2008; Schmittau, 2003; Venenciano & Heck, 2016). According to Davydov (1975, 1982, 1990), the primary goal for young students should be to develop conceptions of real numbers, based on the concept of quantity. In this algebraic measurement tradition, students are invited to work with the concepts equal, greater than, and less than. Measurable attributes such as length, area, volume, mass (i.e. continuous quantities), and the number of physical things (i.e. discrete quantities) serve as quantities to compare when working with these concepts (Davydov, 1975; Dougherty, 2008; Venenciano & Heck, 2016).

In a learning activity, theoretical knowledge can be addressed through object-oriented actions intended to accomplish specific learning tasks that address measurable attributes. Learning tasks require that students, first, analyse factual material to discover general relationships and construct abstractions and generalisations (Davydov, 2008). Second, students should be able to derive particular relationships, constructing a cell (i.e. the theoretical or abstract content – the object of knowledge) and unifying it with the holistic object. Third, in such a task, the students are also challenged to derive a general method for constructing the specific object. Hence, when students develop and complete a learning task, they should be able to discover the origin of the theoretical or abstract content within the object of knowledge in focus. When accomplishing the task, concepts are manifested as object-oriented cognitive actions whereby the students can develop theoretical knowledge. To manifest the concepts, students are invited to model the theoretical knowledge in object-oriented graphic or letter form (Davydov, 1982, 2008; Kozulin & Kinard, 2008). Here, the notion of modelling concerns theoretical modelling, using so-called learning models, which enable students to work with and reflect on theoretical knowledge (cf. Arievitch, 2017; Davydov, 2008; Zuckerman, 2004). Regarding quantity comparisons, suggested learning models include graphic models such as line segments, symbols such as algebraic and numerical symbols, spoken language, and gestures (Davydov, 1975, 1982, 1990, 2008). Line segments can be used as learning models to solve a task (see figures 1 and 2):

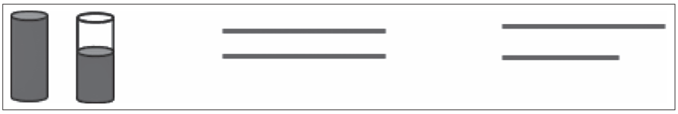

Figure 1. *Compare the above quantities according to the given models (inspired by Davydov, Gorbov, Mikulina & Savaleva, 2012, p. 19)*

In the above example, the students are encouraged to model the values of the two volumes using the learning models. The students are asked to reflect on what is equal, in light of the two equal line segments in one of the presented learning models; then, they are asked to reflect on what is unequal, in light of the other learning model presenting two different lines.

Later in the Davydov curriculum, the students are presented with the task shown in figure 2.

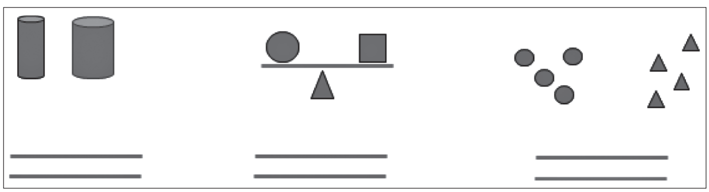

Figure 2. *Compare the quantities according to the given models (inspired by Davydov et al., 2012, p. 19)*

In this task, the students are asked to discuss what is equal, which has been shown with the help of the learning models. Such learning actions are intended to change and develop thinking in relation to specific concepts (Davydov, 1975; 1982; 2008).

Focusing on general relations, the children have opportunities to compare the different quantities. Before assimilating the concept of number, the children are encouraged to record the results of comparisons using letter formulas (e.g. a and b) in comparisons such as $a > b$ and $b < a$. In the following figure, A is compared with D. These quantities are then discussed with help of the line segments, which are used as a learning model.

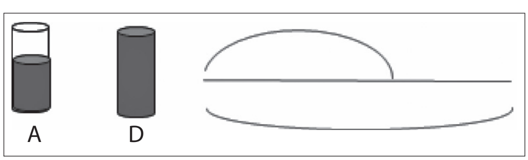

Figure 3. *How can the volumes be compared? (inspired by Davydov et al., 2012, p. 25)*

Here, the students are challenged to suggest various hypotheses and, with the teacher's help, to complete the comparison using the learning model. In this learning activity, the volumes, letter symbols, and learning model are used to conduct the comparison and to reflect on the quantitative relationships.

Methodology

Phenomenography

To find qualitatively different ways of experiencing comparisons, phenomenography was chosen as the analytical tool. The analyses focused on what can be interpreted as forming the students' expressions related to the phenomenon of comparing quantities. This is what is called a second-order perspective in phenomenography (cf. Marton, 1981, 2015). Phenomenography distinguishes between first- and second-order perspectives. A first-order perspective can be an opinion expressed in relation to a phenomenon, while a second-order perspective concerns how a person experiences a phenomenon. What a person sees or discerns in a phenomenon depends on previous experiences. Ways of experiencing thus vary among people; these different ways are described in phenomenographical analysis using qualitatively different categories. Several empirical studies have shown that there are always a limited number of ways, or categories, in which a phenomenon can be experienced (Marton, 1981, 2015; Marton & Booth, 1997). The identified categories form an outcome space in which the categories are usually related to one another in a hierarchical structure (Marton & Booth, 1997). Typical data sources are open-ended interviews and analyses of verbal and written conversations. In the interviews, the informants are encouraged to speak or speculate freely about the given phenomenon, presenting concrete examples to avoid superficial descriptions of how things should be (Marton, 1981; Sin, 2010). To validate the categorisation, it is recommended that the outcome space be reorganised by more than one person familiar with phenomenography (Larsson, 1986; Sin, 2010).

Data

The data for this study comprise transcripts of interviews with grade 1 students (six to eight years old) who were asked to compare different numerical quantities. The interviewed students had all participated in a longitudinal interventional study[1] exploring the Davydov mathematics curriculum in a Swedish intercultural primary school. In total, 146 grade 1 students participated in the interventions.

The interviews were designed to evaluate how the students used the tools suggested by the Davydov curriculum when comparing numerical quantities, and to explore the various ways these students experienced comparisons of numerical quantities. The students were interviewed individually or in pairs in sessions about five to ten minutes long. Each session was video-recorded to document the students' hands, gestures, and verbal reasoning. In total, 42 interviews were captured in recordings. After analysing half of the interviews, no more categories emerged. In relation to the ways of experiencing comparisons in these interviews a theoretical saturation was reached. Thus, the conclusion was drawn that the number of interviews was sufficient to answer the research questions. Some interviews were held in the classroom, and some in a small room next to the classroom.

The interviews were conducted by the author. The students were offered small cubes, a piece of paper, and a pencil. To begin the interview and to construct an inequality to compare, the students were asked to take an odd number of cubes and place them in two piles in front of them. As an introductory question, all the students were asked "How can we compare the numbers of cubes in the two piles?" which was complemented by several follow-up questions. This task was inspired by the Davydov curriculum (Davydov et al., 2012) and a similar task can be found in the national assessment tests for grade 1 in Sweden (Skolverket, 2018b). Some of the follow-up questions were "How can we represent the quantities in the different groups?" and "What more can we do with the groups, in relation to mathematics?" Gestures, such as pointing at the different piles of cubes and different models, were used by the interviewer to explain the questions.

Data analysis

The analysis was conducted in several steps. A first step was to identify and transcribe the parts of the interview related to the students' quantity comparisons. The interviews were transcribed verbatim, including gestures, what the children wrote, and which material they used. As the sentence structure and sometimes the words used could be grammatically incorrect in the students' verbal language, the transcripts were edited to be readable, and the repetitions and short breaks common in oral language were excluded. In the transcripts, the students were anonymised using numbers instead of names.

In the next step of the analysis, the transcripts were divided into passages that were grouped and regrouped into tentative phenomenographic categories. These categories were developed, as suggested by Marton (2015), based on the most distinctive characteristics of the comparisons, and by identifying the qualitative differences that clarified the students' experiencing of comparisons. One example of a reflection on a comparison that went beyond just giving one sign as a result is illustrated by the following quotation from Dominique: "If it is equal, it is like this, '=' [Dominique removed one cube from the left pile]. Now, these are more [He drew <, 'less than']. If you move this, the other pile is more. Then, it is like this [He drew >, 'greater than' instead]". Here, the reflection was identified in relation to the relationships between the quantities. As a next step in the analysis, such reflections were further analysed in terms of levels of generalisation and evidence of the students' algebraic thinking. Levels of generalisation were identified according to Krutetskii's (1976): *level* 1 – if the students linked the symbols only to a specific number (did not generalise); *level* 2 – if the students reflected on the quantities using a tool in only one specific way (generalised under specific circumstances); *level* 3 – if the students argued about the comparison using symbols connected to some kind of clue, for example, a semantic clue (generalised with insignificant hints); and *level* 4 – if the students combined the tools and found new ways to discuss the relationships between the values. Furthermore, the displayed algebraic thinking was described by the focus of the students' actions (cf. Cai & Knuth, 2011). The actions were interpreted according to: whether they focused on relationships between the quantities instead of just numerically counting the specific things; whether they focused on both the results and how to represent the comparisons; whether they related to other tools than just numerical symbols; and whether they focused on the meaning of inequality and equality. The results were discussed by fellow researchers, especially regarding aspects of algebraic thinking and the level of generalisation in relation to the research questions (cf. Sin, 2010). The outcome space and its categories were also validated by two other researchers familiar with phenomenography. These two researchers reorganised the outcome space according to the category descriptions (cf. Larsson, 1986).

Ethical considerations

All the parents of the students signed a letter consenting to their children's participation in the project. This letter clarified that video- and audio-recordings of teaching situations and interviews as well as the students' worksheets could be used as data in further research (cf.

Vetenskapsrådet, 2017). Since most of the students in the project were newly arrived in Sweden (25 mother tongues were represented in the school), this letter was verbally translated for parents in the school by mother-tongue teachers and interpreters.

Results

The phenomenographic analysis resulted in three qualitatively different categories describing the students' ways of experiencing comparisons of quantities. Briefly, these ways of experiencing comparisons can be described as: 1) counting numerically, i.e. counting things; 2) relating quantities, i.e. establishing relationships between quantities; and 3) conserving relationships, i.e. conserving the existing relationships between quantities regardless of models or symbols.

Table 1. *Summary of the categories*

Experiencing comparisons as a matter of	The object-oriented actions	Level of generalisation	Indications of algebraic thinking
Counting numerically	Counting the numerical examples of cubes; drawing the specific numbers of cubes	1 – Students do not generalise	
Relating quantities	Relating different quantities to one and the same symbol Relating quantities to letter symbols according to the structure of the alphabet	1 – Students do not generalise; they use symbols other than numbers but not according to essential attributes	
	Relating quantities using different symbols Relating quantities using line segments Relating quantities using line segments and gestures	2 – Students generalise material according to essential attributes under specific conditions	Focus on relationships and not merely on numerical answers Focus on both representing and solving problems
	Representing a quantity with a letter symbol as an empirical or semantic clue	3 – Students generalise according to essential attributes when there are insignificant hints	Focus on letter symbols rather than on numbers Focus on relationships and not merely on numerical answers Focus on both representing and solving problems
Conserving relationships	Conserving the relationships regardless of the tools used; different tools are used to argue about the same relationships	4 – Students generalise independently and correctly	Focus on letters not on numbers or the things Focus on both representing and solving problems Focus on relationships and not merely on numerical answers Focus on the relationship between quantities and their values

The three categories are hierarchically related, with the first category representing the least complex and the third category the most complex experiencing of quantity comparisons. This means that the ways of seeing a comparison in the third category encompass several aspects of the phenomenon. The categories are summarised in table 1, in which the first column presents the categories, the second column describes the students' object-oriented actions related to the comparisons, the third presents the levels of generalisation, and the fourth presents indications of algebraic thinking. In the following, the different categories are described. After a heading naming each category, the category is explained and illustrated by citing examples of the students' actions (i.e. verbal speech, writing, and gestures related to the comparisons).

Counting numerically

The category counting numerically is based on the students' use of numerical symbols, drawings, and gestures to compare specific numerical quantities. In this category, students expressed the comparison mainly in arithmetic terms, and concluded the comparisons by choosing the correct mathematical sign for greater than or less than. An example of comparison in terms of counting is that of Student 1 (S 1).

S 1: [She puts the cubes in two piles. Then she draws the correct number of cubes in each group, like squares] I was drawing just like we did before. We can draw to show how many there are. I can draw squares instead of lengths, as we did before [After that, the student counts the number of squares to compare the two piles].

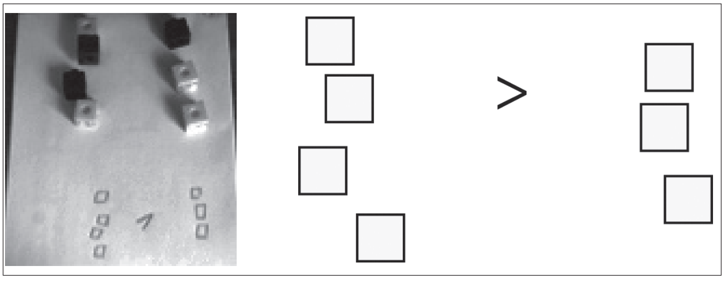

Figure 4. *Left: The worksheet; Right: Reconstruction of the squares she drew*

Here the cubes were drawn as piles and then were counted. The greater than symbol was used to indicate which pile had more cubes. The reflection on the different quantities was related to these specific numbers in this specific situation. The category also includes an example in which students used numerical symbols that did not correlate to the quantities of the piles of cubes. In this case, the student appeared to find it more important to name the pile than to describe it using an accurate numerical symbol. One student said, "We can call this group two", at the same time as she was pointing at the pile of three cubes.

To sum up, in this category the way of experiencing the phenomenon was manifested through students' utterances that contained only numerical examples. The category includes level 1 generalisation and no indications of algebraic thinking.

Relating quantities

The category relating quantities describes student reflections on quantities going beyond merely discussing the numerical examples of specific quantities. Instead, different symbols were used to represent the quantities, and models were constructed by the students and used as means to discuss the differences between the quantities. The symbols and the constructed learning models were used separately. This category includes reflections at different levels of generalisation, as well as different possible foundations of algebraic thinking. In the following, the first examples can be compared to the lowest level of generalisation, level 1, the next examples to level 2, and the final ones to level 3.

At the first level of generalisation, the students apparently knew about letter symbols, but used these symbols without any attempt at generalising. First, this is illustrated by an example in which the student suggested that all quantities could be named using the same letter symbol.

S 2: It is easier if all of them are Y.

If just one symbol is used for all quantities, the symbol cannot be used to discuss the differences between the quantities. The argument that all cubes or all line segments can have the same letter symbol does not indicate any ability to generalise or any foundation of algebraic thinking. Another argument that indicated neither algebraic thinking nor generalisation was uttered by Student 3.

S 3: Å is the biggest one, because it is the last one. J is in the middle, 3 is also in the middle [Å is a Swedish letter near the end of the alphabet, and J is in the middle].

The structure of the alphabet was highlighted as a means to assign value and the letters were not connected to any attempt at algebraic thinking, so the generalisation can be said to be at level 1.

Related to level 2 generalisation, the students displayed abilities to generalise related to specific conditions. In the following example, the students used different geometric symbols.

S 4a: This can be a circle … [Points at one of the groups of cubes].

S 4b: … and this can be a triangle [Points at the other group]. These [The students are pointing at the symbols for the different quantities] cannot be the same, because [The student is pointing at the groups of cubes] these are not the same. A circle and a triangle are not the same.

Also, the following example in which Student 5 used different letter symbols for each quantity can be understood as illustrating level 2 generalisation (see figure 5). In this example, the student did not connect the letter symbols to the line segments; rather, the letter symbols were just used together with the piles.

Student 5: We can call the piles A and L.

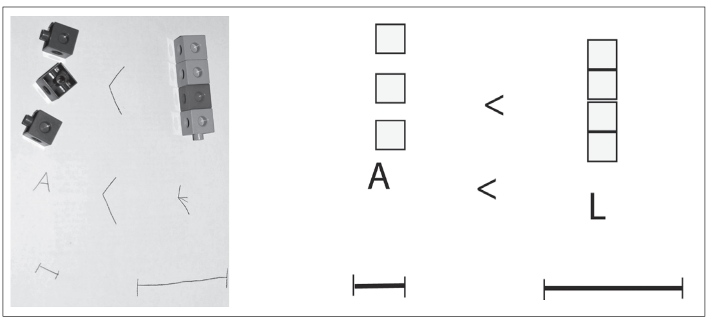

Figure 5. *Left: The worksheet; Right: Reconstruction of the work*

Another example that can be said to represent level 2 generalisation is the following, in which the students constructed and used line segments to analyse and describe the relationship between the quantities.

S 6: [Draws a short line segment beside the group of just a few cubes. The cubes are not in a row or stack but arranged higgledy-piggledy. He draws a longer line beside the bigger group]

S 7: [The cubes are stacked in piles beside each other] A is the biggest one. You can see that by counting, but you can also see that without counting, by comparing the heights of the stacks. We can draw these as lengths – that's easier.

The line segments were constructed in the same way as in figure 5. The students used (i.e. pointed at) these models when discussing the differences between the different quantities. The students used big gestures in relation to the longer line when referring to the larger pile of cubes, and smaller gestures when pointing at the shorter line when discussing the smaller pile. Note Student 8's explanation.

S8: When there are many, I made a long line [He points at the long line, with a big gesture]. When there are just a few cubes, I made a short line [He points with a very small gesture].

In this second level of generalisation, the students analysed the comparisons using symbols other than just numerical ones when they were representing a solution. They used line segments, algebraic symbols, spoken language, and gestures.

In the third level of generalisation, the generalisations are connected to essential attributes, which can also hint at the choice of symbols. The choice of the letter symbols can be connected to semantic aspects as well as to specific contents related to the quantities.

S9: The smallest group and the biggest group, or M and S [Smallest is *minsta* in Swedish and biggest is *största*. The student writes M and S].

The first semantic sounds, or the first letters of the words *minsta* and *största*, were used as symbols.

Another way of choosing letter symbols interpreted as this level of generalisation was when the first letters of names were used as the letter symbols, for example, as Student 10a and Student 10b (two students interviewed jointly) suggested.

S10a: But, they can also be W or F. That's Wera and Fia ... I do have two small sisters. That's me, the oldest sister, so this can be Wera [Wera is pointing at the biggest group and writes "W" by this pile].

S10b: This is F [She writes "F" by the other pile].

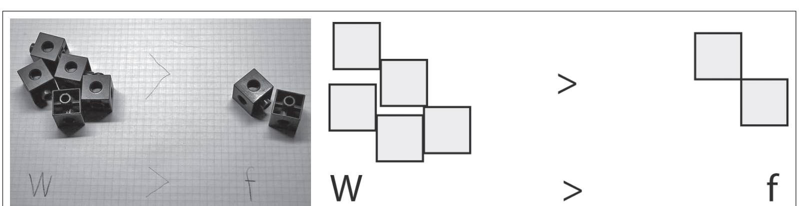

Figure 6. *Left: The worksheet; Right: Reconstruction of the worksheet*

The choice of letter symbol seems to be related to the person's age. As Student 10a was the oldest sister, she argued that since the oldest person had the greatest value of age, the first letter of her name could be used to symbolise the greatest quantity of cubes. The ages had the same relationship with the symbols ”greater than” or ”less than” as did the quantities of cubes.

To sum up this category, the students were reflecting on the quantities, not merely giving answers. The reflections were interpreted as illustrating three levels of generalisation, from level 1 with no generalisation to level 3, at which the choice of symbols was argued for in a sufficiently mathematical way. In levels 2 and 3, there were indications of algebraic thinking through the students' use of symbols, line segments, language, and gestures to analyse structures such as the relationship of the quantities. In this category, letter symbols and line segments were used separately.

Conserving relationships

This category, conserving relationships, is based on the students' reflections on the relationships as they represent the quantities in different ways. Independent of the representation of these quantities, the students used the same non-numerical symbols for the same quantities. If a line segment indicated the same quantity embodied by the pile of cubes, this line model and the pile were represented by the same letter symbol. Thus, when different quantities were involved, the students independently used different symbols and line segments of different lengths. In these reflections, the foundation of algebraic thinking was evident and interpreted as level 4 generalisation illustrated by Student 11.

S 11: The pile of cubes is A [figure 7a]. Then the length can be B. It can be T ... But they are the same [Points at the pile of cubes called A and at the length beside these cubes (see the arrow in figure 7b)], so they must have the same name. I know ... A for both [Now she points at the next group]. This can be B. The model we can call P. But, maybe B [Points at the pile and the B symbol]. Maybe, the model also should be B [figure 7c]? [Then she writes the symbol for greater than (figure 7d)]

Student 11 concluded that the quantity should be represented by the same letter symbol, independent of whether it was represented by the specific things or by the line segments. She argued: ”But, they are the same, so they have to have the same name”. This indicates that the letter symbol is used as a means to represent the quantity, which is presented by the line segments and the numerical things.

Other examples that can be related to this level 4 generalisation include an operation with the two quantities. The letter symbols chosen

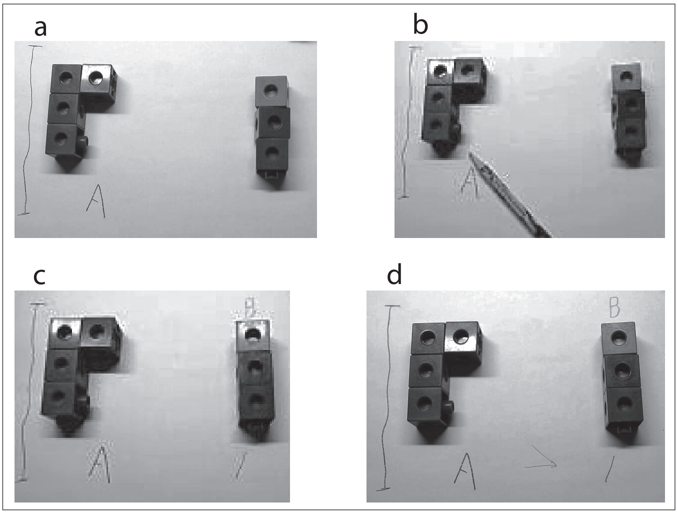

Figure 7. *A sequence of reflections on the quantities*

by the students were used to reflect on the comparisons as well as on how the symbols could be used in an operation.

S 12: This pile can be called B, and the line segment showing B must have the same letter [Points at the small pile and the short line segment]. These are A [Writes A by the big pile and the long line segment. Then he puts groups A and B together]. We cannot call this A or B [Points at the group that is the sum of A and B], because this is A and this is B and neither of them is in that group. A is more than B, and together there are even more. I don't know the new name ... Ö maybe [Ö is the last letter in the Swedish alphabet]? Together they are much more.

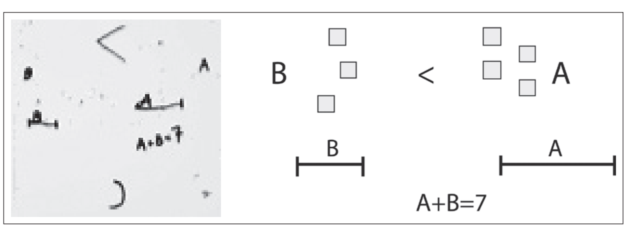

Figure 8. *Left: The worksheet; Right: Reconstruction of the work (with cubes)*

This student's use of the letter symbols clarified which pile of cubes and which line segments were combined. He used the symbols to compare the quantities but also to represent a mathematical operation. When operating with the symbols, he created a new pile and concluded that this new pile could not be represented by the same symbol as the first two piles. The new piles of cubes were independently represented by a third symbol.

In summary, the category conserving relationships indicates algebraic thinking in which the students were analysing the relationships between the quantities and representing their reflections using more than one tool. The relationship between two quantities was analysed using a combination of the number of physical cubes, the line segments, the algebraic symbols, verbal language, and gestures. Furthermore, the students' use of more than one tool in this algebraic way of analysing the relationships indicates generalisation at a higher level than when the students used the tools separately, or just used numerical symbols.

Discussion

This study focuses on grade 1 students' qualitatively different ways of experiencing comparisons of numerical quantities and on the indications of algebraic thinking embedded in these ways. The analysis generated three categories: counting numerically, relating quantities, and conserving relationships. The first category included no indications of algebraic thinking, here meaning no signs of general reflection beyond the specific situation or use of other symbols than just numerical symbols. This category can be compared to an arithmetic way of thinking (Cai & Knuth, 2011; Davydov, 1990; Mason, 2017; Radford, 2013).

However, in the other two categories, there were several actions that could be interpreted as signs of reflection in ways that can be understood as an algebraic way of thinking (cf. Davydov, 2008; Kozulin & Kinard, 2008; Radford, 2013). Here, the students used mathematical tools such as artefacts (i.e. the different numbers of cubes), symbols (i.e. letters and geometric symbols), models (i.e. line segments), verbal language, and gestures, suggested as learning models according to an algebraic learning activity. This is in line with the arguments that discerning and discussing general and theoretical concepts (i.e. mathematical concepts) requires the development of learning models incorporating tools for object-oriented actions (Arievitch, 2017; Davydov, 1990). Consequently, when the relationships were conserved independent of the tools used, the highest level of generalisation was reached in the students' algebraic thinking. Similar suggestions can be found in Radford (2013) and Mason (2017), both using the standpoint that students need to model mathematical tools in order to develop mathematical concepts, arguing that such modelling can be compared to an algebraic way of thinking.

Given that research on learning activity theory has identified a need to better understand the type of thinking evoked when working on tasks using learning models (Eriksson & Jansson, 2017), the present results concerning the second and third categories can be used to illustrate algebraic thinking. To manage such thinking, Radford (2013) stated that an epistemological distinction between algebra and arithmetic must be implemented. As shown in the second category, the use of letter symbols does not automatically develop into algebraic thinking. Previous studies have concluded that students, for example, need to operate with unknowns without giving the symbols any numerical values to develop algebraic thinking (Radford, 2013; see also Kaput, 2008; Kieran et al., 2016). Furthermore, in the last two categories, the students' actions appear to focus on the results and on justifying, representing, and reflecting on the solutions. Such a focus may enhance algebraic thinking by allowing reflection on relationships and structures subsuming arithmetic numbers and operations (Davydov, 2008; Dougherty, 2008; Mason, 2017; Radford, 2013; Schmittau, 2011). Also, given that there is a need to understand the "transition" between algebra and arithmetic (Hitt et al., 2016), the reflections in the last two categories could be one way to illustrate this transition. In the context of this study, it was therefore important that the relationships were emphasised using the tools when the students jointly reflected on the comparisons, not just the physical things, symbols, or line segments.

As a conclusion of this study, teaching designed as in the interventional study, i.e. developed according to learning activity theory in which tasks and learning models are developed, facilitates students' algebraic thinking. This way of thinking was identified in two of three categories related to the students experiencing of comparisons. However more research is needed to confirm the categories and whether the use of learning models can be generalised to education in other classrooms. Indications from this study also stress that further investigations should explore whether algebraic learning activity can meet the challenges of multicultural classrooms and allow students in a second-language context to reflect on mathematical concepts.

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Notes

1 The longitudinal interventional study was staged as a design-based research project (Van den Akker, Gravenmeijer, McKenney & Nieveen, 2006) epistemologically in line with clinical educational research (Carlgren, 2012). The study was conducted from 2015 to 2018 and involved six teachers, the author of this article as a teacher as well as a researcher, and researchers exploring learning activity theory at Stockholm university. The interviews were conducted when the students had participated in this study for one semester, when they had worked on tasks like those presented in the section on learning activity theory in this article.

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