Developing a frame for analysing different meanings of the concept of variable mediated by tasks in elementary-school mathematics textbooks

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Pupils' studies in arithmetic can support the development of their algebraic thinking if arithmetic is taken as a starting point for generalising in sense-making discussions. One of the most prominent concepts in algebra is that of the variable, which can have many different meanings, depending on its context. In this paper, we develop a frame for content analysis of tasks in elementary-school mathematics textbooks. New categories for the meaning of variable are added to previous summaries, based on the literature review and the analysis. The developed frame can be used for analysing curricular materials, especially at the elementary-school level.

Algebra can be perceived as a gatekeeper to more advanced mathematics and full participation in many fields in society (Capraro & Joffrion, 2006; Stephens, 2005). Research in mathematics education has shown that the transition from arithmetic to algebra causes problems for a large number of pupils (Kieran, 1992; Linchevski, 1995; Rojano & Sutherland, 2001). On the other hand, numerous studies involving elementary-school children and even pre-school children show that from an early age, children are capable of algebraic thinking and using algebraic concepts. It is a matter of providing them with appropriate instruction (Blanton et al., 2017; Carraher & Schliemann, 2007).

Research on early algebra is a developing area of mathematics education studies. Early-algebra educators aim to give their arithmetic students a

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good foundation for understanding algebraic concepts and principles by guiding them to generalize, symbolize and reason in sense making discussions. According to Kieran, Pang, Schifter and Fong Ng (2016), the proper use of traditional symbolism is not necessary for early algebra. Rather, the focus is on developing mental habits, where generalization and mathematics as a science of structures are essential and mathematical communication is utilized in sense making. They describe how interest in algebraic thinking with younger pupils has also influenced algebra research involving older students.

We are developing early-algebra teaching materials for grades 1–6, in line with the new Finnish National Core Curriculum for Basic Education 2014 (Finnish National Agency of Education, 2014), as part of the national LUMA Finland development project funded by the Finnish Ministry of Education and Culture. The project offers in-service training courses for teachers in different parts of Finland and online. To be efficient in this work, we have to know what kinds of entry points for early algebra are offered by the current mathematics textbooks in Finland.

One of the most important concepts in algebra is that of the variable (Carraher & Schlieman, 2007), which has many different meanings, depending on its context (Usiskin, 1988). Do Finnish elementary-school mathematics textbooks include tasks that have the potential for paving the way for the different meanings of the concept when brought into the early algebra-type of discussions? We are planning to perform a content analysis of the tasks in the main textbook series in Finland. In this article, our aim is to develop a suitable frame for analysing the potentials of the tasks in elementary-level mathematics textbooks to mediate the meanings connected to the concept of variable.

Earlier summaries on the meaning of the variable in school algebra have been done for secondary-level mathematics (Philipp, 1992; Usiskin, 1988) or have not been broad and sophisticated enough for our purposes (Blanton et al., 2011). The frame developed in this research responds to the cited need and can be used to analyse curricular materials, especially at the elementary level.

Theory

In this section, we present a literature review on early-algebra research on variables and on the different meanings of the concept of variable in school algebra. Based on the review, we present the original frame for content analysis (table 1). We also shortly conceptualize textbook in mathematics instruction and content analysis of tasks in textbooks, and formulate the research questions.

Early algebra and variables

The concept of variable plays a seminal role in algebra (Blanton et al., 2015; Carraher & Schliemann, 2007; Ely & Adams, 2012). For example, Blanton and colleagues have identified it as one of the five big ideas in organising algebra content for elementary teachers (Blanton et al., 2011; Blanton et al., 2015). The other four big ideas (Blanton et al., 2015) in early algebra are *generalised arithmetic; equivalence, expressions, equations and inequalities; functional thinking;* and *proportional reasoning.* These ideas should not be considered mutually exclusive. For instance, variables perform an important role in all the other four contexts of algebraic thinking.

The concept of variable can pose a major obstacle for students in transition from arithmetic to algebra (Ely & Adams, 2012; Philipp, 1992). This is due to several new uses of algebraic letters, which require students to expand their understanding beyond thinking of variable as only representing an unknown but fixed quantity (Ely & Adams, 2012). Nonetheless, young kindergarten and elementary-school children have shown their capability for algebraic thinking with varying variables (Blanton & Kaput, 2004; Blanton et al., 2017; Brizuela et al., 2015). Davydov (1975) and Davydov, Gorbov, Mikulina and Saveleva (1999) have developed a curriculum where young children first learn general quantitative relationships and express them with letter symbols, and only later do they apply these relationships to numbers of different types. Dougherty (2008) applied Davydov's approach, and her third-grade pupils learned to use algebraic symbols and diagrams evolving from measuring situations.

Early algebra aims to develop algebraic ways of thinking among young pupils who have not yet started their proper studies in the subject. Whether symbolic forms of communicating variables should be involved in early algebra was an intensively debated topic during the decades before the year 2000 (Kieran et al., 2016). Regarding the emphasis on *algebraic thinking* in early algebra, Kaput, Blanton and Moreno (2008) argue that a critical aspect, which makes an activity algebraic, is deliberate generalisation. According to them, even the use of numbers can be considered algebraic if special cases are used as examples of general principles.

Fujii and Stephens (2001) have introduced the concept of *quasi-variables*. They present a numerical example of a true equation ($78 - 49 + 49 = 78$), arguing that it can be used as an example of a group of equations, which is true whatever the first number or the subtracted and the added numbers are. Such type of equations can be symbolised as $a - b + b = a$. However, even without symbolising, the principle can be generalised in words and thoughts, starting from the numerical example. This kind of quasi-variables can build a bridge between arithmetic and algebraic thinking.

Elementary-school mathematics textbooks include different types of numerical tasks or groups of tasks, which can be interpreted as examples of general underlying mathematical relationships or principles. These tasks invite pupils and teachers to discuss general principles, or teachers can deliberately use the tasks to point out generalisations. When practicing with these tasks, and in discussions about them, pupils can gain more or less explicit experiences of the different meanings of the concept of variable.

Different meanings of the concept of variable

The mathematical community lacks an agreement about the terms for different usages of letters in mathematical expressions and equations. In line with its historical origins, Ely and Adams (2012) reserve the term *variable* for situations where it can assume any value from a large set of values and co-varies together with another quantity. According to Kieran (1989), the mathematics reform movement in the late 1950s and the early 1960s influenced the search for unifying concepts in the US mathematics curriculum. The variable was taught in a very general form, and almost all literal symbols in mathematics became referred to as variables. The alternatives for the term *variable* in this broad sense would be *letter symbol* or *literal symbol*. Because these terms point more to the symbol itself instead of the concept, we follow Philipp (1992), Usiskin (1988) and Blanton and colleagues (2011) in using the term *variable* in the broad sense and *varying variable* or *varying quantity* in the context of functions.

In the following paragraphs, we discuss the different meanings of the concept of variable based on the literature and our own experience as mathematics teachers. Our discussion is summarised in table 1, which constitutes our original frame for the content analysis of tasks in mathematics textbooks.

Usiskin (1988) has analysed different conceptions of school algebra and has recognised different roles of the concept of variable, depending on the context. First, if algebra is perceived as generalised arithmetic, the variable performs the role of *pattern generaliser* or *generalised number*. For example, the equation $a \times b = b \times a$ generalises and symbolises the commutativity law of multiplication of real numbers. In the equation, the letters a and b represent any real number; their role is that of a generalised number. Other researchers also recognise this meaning of the variable (Blanton et al., 2011; Ely & Adams, 2012; Philipp, 1992).

The second area of school algebra in Usiskin's (1988) analysis involves solving problems by means of equations. In this context, the variable is most often interpreted as an *unknown number*. This is a widely agreed meaning of the letter symbol in equations (Blanton et al., 2011; Ely & Adams, 2012; Philipp, 1992), especially in elementary-school mathematics. To conceptualise equations and equation solving also in the later phases of mathematics education, we agree with Carraher and Schliemann's (2007) view that an alternative way of interpreting the letter symbol in equations as an unknown is to *see it vary*. In this case, imagine that the value of x goes through all the numbers in its domain. The solutions of the equation are those values of the variable that make the equation true, while others make it false. This framing would help pupils learn to know *varying variables* at an early stage and prevent them from constructing too restricted views of symbols, which would be difficult to change later (Carraher & Schliemann, 2007; Ely & Adams, 2012).

Solving equations also requires the ability to simplify expressions (e.g. add $2x$ and $3x$); in this activity, the literal symbol again functions as a *generalised number* (Philipp, 1992). In addition, equations may include letters representing constants, such as the letter b in the equation $bx + 5 = x - 3$. If x is the variable to be solved, then b represents any possible *constant* (Usiskin, 1988) or *coefficient* (Ely & Adams, 2012).

The third conception of algebra (Blanton et al., 2015; Carraher & Schliemann, 2007; Usiskin, 1988) includes the study of the relationships among quantities, such as in functions and formulae. In this context, variable means *varying variable* or *varying quantity*. Küchemann (1981) states that in this case, "the letter is seen as representing a range of unspecified values and a systematic relationship is seen to exist between two such sets of values" (p. 104). For example, we can ask what happens to the value of function $f(x) = 1/x$ when x approaches zero. In the context of functions, we talk about the *independent variable* or the *argument* and the *dependent variable* or the *function value* (Usiskin, 1988). Also in the formula $A = b \times h$ for calculating the area of a rectangle, b and h are interpreted as *varying*. The formula applies to all the possible values of base and height. The value of the area depends on both of them. There is another role performed by the variable in this conception of algebra. The category and the type of the functions $y = ax^2 + bx + c$ depend on the values of the constants a, b and c, called *parameters* (Usiskin, 1988).

The fourth conception of algebra (Usiskin, 1988) pertains to the study of structures at the university level. Because of the abstract nature of this context, we do not include it in our frame.

In his discussion on the different roles of variables in school algebra, Usiskin (1988) also predicts that computer science will become a vehicle through which many pupils will learn about variables. According to him, in programming, a pupil learns to know a variable as an argument or a varying variable far sooner than is customary in algebra. In computer science, the uses of variables also cover all the other uses of the variables that he mentions in his discussion. The new *Finnish national core curriculum for basic education 2014* (Finnish National Agency of Education, 2014), applied for the first time in the autumn of 2016, requires that pupils should study programming from grade 1 to grade 9. Many other countries are starting to teach programming to their pupils, even the young ones (Benton, Hoyles, Kalas & Noss, 2017; Mannila et. al., 2014). An important question concerning algebra and early algebra is how pupils' experiences about variables in programming interfere with their conceptions about variables in algebra (Kilhamn & Bråting, 2019). An additional role of a variable in programming is its use for *storing* a piece of information, whether numerical or a string of characters. This concrete content of the storage can change during the execution of the program, according to the instructions given to the computer.

Table 1. *The original frame*

Conception of algebra	Role of variable
A. Generalised arithmetic	1. Generalised number
B. Study of procedures of solving certain kinds of problems by means of equations	1. Unknown
	2. Varying variable
	3. Generalised number
	4. Coefficient
C. Study of relationships among quantities, formulae and functions	1. Varying variable
	2. Parameter
D. Programming	1. Generalised number
	2. Unknown
	3. Varying variable
	4. Coefficient or parameter
	5. Storage

Mathematics textbooks and content analysis

The textbook's role in mathematics instruction can be conceptualised by Rezat and Strässer's (2012) socio-didactical tetrahedron. For our purposes, it is sufficient to consider only the top of the model, which is also a tetrahedron. The bottom triangle of this top consists of three line segments joining three vertices, which represent the *student*, the *teacher* and *mathematics*. All those vertices are connected to the top vertex *artifact*.

The authors draw from the Vygotskian perspective (Vygotsky, 1997) and point out that any encounter of students and teachers with mathematics is mediated by artefacts, which include both psychological and technical tools. Artefacts (e.g. mathematics textbooks) have certain affordances and impose specific constraints on the user. They also restructure the didactical situation as a whole and must be considered its fourth constituent, thus comprising the fourth vertex in a model describing it (Rezat & Strässer, 2012).

Especially in the Nordic countries, the research on mathematics textbooks has most often focused on analysing their contents. In this case, the main research method has been content analysis (Rezat & Strässer, 2015). According to Krippendorff (2013), content analysis is "a research technique for making replicable and valid inferences from texts (or other meaningful matter) to the contexts of their use" (p. 19). Mathematical tasks in textbooks can be perceived as "meaningful matter" for content analysis, whose predominant aim is to draw inferences from the tasks about the latter's impact on students and teachers. However, content analysis of tasks can only reveal opportunities to learn and raise discussions in the lessons. The utilisation of those potentials depends on the actions of teachers and students.

Research questions

In this study, we ask the following research question:

> What kind of frame is suitable for analysing different meanings of the concept of variable that tasks in elementary-school mathematics textbooks have the potential to mediate?

We divide the research question into two sub questions.

1. How well does the original frame fit the purpose of analysing the different meanings of the concept of variable?
    a. To what degree do the tasks with the possibilities for mediating the meanings of the variable fit the categories in the frame?
    b. How exclusive are the categories?

2 What kinds of elaborations are needed for the original frame?

Methodology

In this research, we constructed, tested and elaborated a frame for content analysis on tasks in elementary-school mathematics textbooks. In the work, we applied parts of the deductive type of qualitative content analysis (Rezat & Strässer, 2015).

Both of us are mathematics teachers and thus familiar with the uses of variables in school algebra in lower and upper secondary schools. We used our knowledge of theory and our professional experience in making decisions concerning the analysis of tasks and development of the frame. It must be acknowledged that we most probably discovered more possibilities for connecting the tasks in the elementary-school textbooks to the concept of variable than an ordinary class teacher would. Our analysis considered the textbook as an artefact, and we tried to reveal the possibilities offered by the tasks for mediation of mathematical meanings. But in the end, the teachers and the students are the ones who realise the mathematics lessons. The first author was responsible for the analysis, and the second author helped in discussing the interpretations.

Method

The original frame was tested by performing a content analysis using a sample of six elementary-school mathematics textbooks (table 2). The books were chosen from three textbook series, with two of them as the main series of the two largest publishers, and the third as the less used series. One book from each grade was randomly selected, starting with grade 1 and continuing through grade 6 so that two books from each series were included in the sample. The unit of analysis was a task as the textbooks presented them with numbers. In some books, the tasks were more like collections of similar tasks; in other books, the single tasks were often identified with separate task numbers. Our data included all the tasks that we found in the books. There were altogether 3311 tasks in the six mathematics books.

Our analysis involved searching for the different roles of the variables among the numerical tasks. Single arithmetic tasks can be used to discuss general principles (Fujii & Stephens, 2001), but we think that more similar tasks than just one are needed to encourage pupils and teachers to reflect and generalise. We considered a numerical task in a textbook to offer possibilities for generalising and thus noticing the variable's role if the larger task had at least three successive and similar single tasks. The line had to be drawn somewhere. For example, task 1 in *Kymppi 3, syksy* (Rinne et al., 2017a, p. 72) presented a five-by-three grid with small squares, and below it was written two products: $5 \cdot 3 = \_\_$ and $3 \cdot 5 = \_\_$

Table 2. *The textbooks used in testing the frame*

Name of the book	Grade	Number of tasks
YyKaaKoo 1A (Hartikainen, Hurmerinta, Häggblom, Sipilä & Väistö, 2017)	1	342
Tuhattaituri 2b (Asikainen et al., 2016)	2	497
Kymppi 3, syksy (Rinne, Sintonen, Uus-Leponiemi & Uus-Leponiemi, 2017a)	3	525
NeeViiKuu 4B (Hartikainen et al., 2017)	4	565
Tuhattaituri 5b (Asikainen et al., 2017)	5	593
Kymppi 6, kevät (Rinne, Sintonen, Uus-Leponiemi & Uus-Leponiemi, 2017b)	6	789

(in Finland, dot between the numbers is used for multiplication). The pupil's task was to find the values of the products, which were both 15. This part of the task was followed by five grids of different widths. However, all of them had the height three because the chapter's topic was the multiplication table of number three and at the same time, the commutativity law of multiplication. In this task, the pupil was expected to write both products and their values under the rest of the grids. We interpreted the task as having the possibility for the pupils and the teacher to generalise that in a product, they could change the roles of the multiplier and the multiplicand, and the value of the product would remain the same. The multiplier and the multiplicand in the generalisation performed the role of a generalised number. If the textbook marked the single numerical tasks with separate task numbers, we looked at their surrounding tasks and followed the three-task principle.

For the roles of variable B4 coefficient in equations and C2 parameter in the context of functions (table 1), we thought that the existence of another but similar equation or function would be enough to invite the teacher and the pupils to reflect on the similarities and the differences between the cases. The role of the coefficient or the parameter would then be in focus. Missing-number equations, where the unknown B1 played a prominent role, were so common among the tasks (382 in total) that we included all of them, either in groups or isolated. New single cases would just strengthen the pupils' earlier experiences. Our data also included tasks that were not numerical in nature. For example, functions were studied based on their graphs. We did not apply the three-task principle to those tasks.

There were 241 tasks that could not be categorised with the original frame, but in which we found hints of the categories of algebra or variables. For example, those tasks included relationships of unknown quantities, letter symbols, the function type of the relationships, inequalities and missing numbers or digits, and we thought that they might have something to do with variables. Those tasks required additional research. We carefully examined and classified them based on the mathematical topic or the type of task. The results of analysing the uncategorisable cases were finally used to elaborate on the frame.

Results

In this section, we answer the items under the first and second research questions and present the elaborated frame.

To what degree do the tasks with the possibilities for mediating the meanings of the variable fit the categories in the frame?

Of the total of 3311 tasks, 928 (28 %) were classified into the categories (conception of algebra) and subcategories (role of variable) of the original frame. We were unable to decide about 241 tasks (7.3 %). Table 3 presents

Table 3. *The number of tasks classified in each subcategory and uncategorisable tasks*

Conception of algebra	Role of variable	Number of tasks
A. Generalised arithmetic	1. Generalised number	284
B. Study of procedures of solving certain kinds of problems by means of equations	1. Unknown	382
	2. Varying variable	81
	3. Generalised number	1
	4. Coefficient	13
C. Study of relationships among quantities, formulae and functions	1. Varying variable	315
	2. Parameter	52
D. Programming	1. Generalised number	0
	2. Unknown	3
	3. Varying variable	11
	4. Coefficient or parameter	0
	5. Storage	0
Uncategorisable tasks		241

the frequencies of each subcategory. The sum of the frequencies is not 928 because some tasks are included in more than one subcategory.

Subcategories D1 generalised number, D4 coefficient or parameter and D5 storage in the context of programming (table 1) were empty subcategories. Subcategory B3 generalised number in equation solving was practically an empty subcategory.

After a careful study of the 241 uncategorisable cases, 155 tasks were used to elaborate on the frame, and 86 tasks were determined as not offering possibilities for mediating the meanings of the concept of the variable.

How exclusive are the categories?

Altogether, 179 tasks were grouped into more than one subcategory, comprising 5.4 % of all the tasks and 19 % of the categorised tasks. Of the total, 146 tasks were coded jointly under 2 subcategories, 32 tasks under 3 subcategories and 1 task under 4 subcategories.

In 90 of the 179 tasks, subcategory B1 (table 1) was included in the two to four subcategories involved. B1 was a subcategory of the unknown in the category of equations. We coded all missing-value equations (e.g. $45 + \_\_ = 76$) and tasks whose logic was similar to that in this subcategory. This is a commonly used type of task in Finnish textbooks. In addition, 34 tasks were coded into the joint subcategories C1 and C2. Multiplication tables, as well as tasks including several compounds of a number (e.g. 1 and 3, 2 and 2, and 3 and 1 for number 4), where one of the parts is known and varies, can be interpreted as functions. If in those tasks, there were several multiplicands or different numbers represented the whole, then the tasks were coded under the common subcategories of C1 and C2.

If we would neglect the above-mentioned situations, we would be left with 55 tasks, accounting for 1.7 % of all the tasks and 5.9 % of the categorised tasks. Among these, 13 tasks were in the joint categories of A1 and C1 and 18 tasks in B2 and C1.

There was considerable overlapping in the subcategories. However, analysing the reasons clarified the overlapping to a great extent. Because of the versatile nature of the tasks in elementary school mathematics textbooks, the subcategories can not be made totally exclusive.

What kinds of elaborations are needed for the original frame?

All the empty subcategories are found in category D Programming. Programming as part of the mathematics curriculum was introduced in Finland in such a hurry that the textbook authors lacked enough time to plan appropriate tasks to support the development of the pupils' computational thinking, concept introduced by Wing (2006). We should not be too eager to leave out subcategories in this context because in the future, there will most probably be more versatile programming tasks in mathematics textbooks. We can easily imagine tasks that employ the meanings of the storage and the coefficient or parameter categories in visual programming environments, the tool for programming in elementary school. It is more challenging in the subcategory D1 generalised number.

Subcategory B3 generalised number included only one task. This subcategory is used for transforming expressions in the context of equation solving. However, in many countries, such as Sweden (Kilhamn, 2014), the work with expressions starts in earlier grades than in Finland. In the context of balance-scale tasks, which can be used to pave the way for algebraic equation solving, it is easy to create tasks, which raise the need to combine groups of the same weight. For example, the total weight of two boxes of x grams and three boxes of the same weight is $5 \times x$ grams. We keep this subcategory in the frame.

We classified the 241 uncategorizable tasks based on the data and obtained nine classes (presented in the following subsections): quantitative reasoning, different codes, statistical variables, random variables, simple inequalities, introducing the system of xy-coordinates, recursive discussion of number sequences, tasks emphasising only the values of the multiplication tables, and tasks with missing digits, missing denominators and so on. The first five of these classes helped us elaborate on the frame. In the following discussion, we pay special attention to those classes.

Quantitative reasoning

In both items of the task shown in figure 1, the weights of the barrels with three different colours are unknown but do not vary. The relationships among the barrels' weights are important here. Reasoning with this information is needed to balance the scales at the bottom. In the literature, the kind of activity applied to this task is called quantitative reasoning, which was one of the original five big ideas of Blanton and colleagues (2011, p. 13) that characterised early algebra. It was also one of the entry points to early algebra distinguished by Carraher and Schlieman (2007). Quantitative reasoning was not included in Usiskin's (1988) analysis. The reason may be that at the secondary level of education, quantities and magnitudes are topics in science rather than in school algebra.

The role of the barrels' weights in figure 1 is close to that of a generalised number in generalised arithmetic. The weights of the barrels of different colours represent general weights or general magnitudes of the

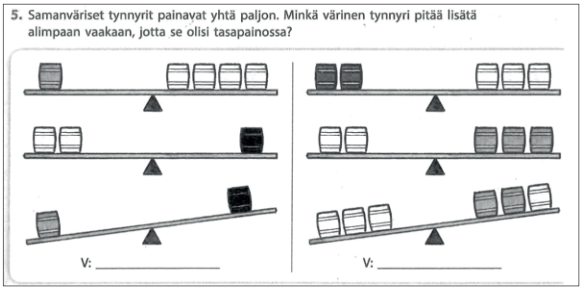


Figure 1. *Task 5 in Kymppi 6, kevät (Rinne et al., 2017b, p. 115)*
Note. Transl. Barrels of the same colour are equal in weight. What is the colour of the barrel that should be added to the scales at the bottom to balance it?

quantity weight. Kaput and colleagues (2008) and Kieran and colleagues (2016) also parallelise quantitative reasoning with generalised arithmetic in describing different content strands of algebra. However, because numbers of different type are such an important topic in elementary-school mathematics, we prefer not to broaden the category of generalised arithmetic in our frame. Instead, we have created a new category, *quantitative reasoning*, where the variable performs the role of a *generalised magnitude*. Moreover, quantitative reasoning usually supposes that quantities have positive magnitudes only, yet the number system is already expanded to negative integers and rational numbers in Finnish elementary schools. None of the categories can be seen as subsuming the other.

Different codes, statistical variables and random variables

Among the uncategorisable tasks, three classes involve a function type of correspondence between the elements in two sets. The elements in those sets are not necessarily numbers. The classes are different codes,

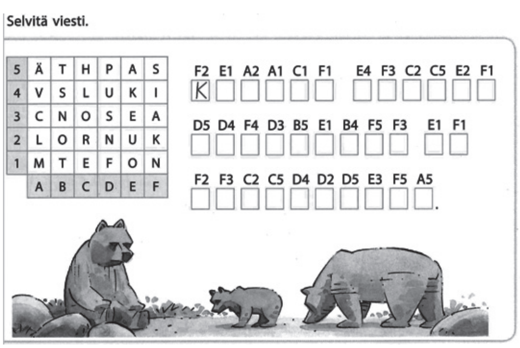


Figure 2. *Task 6 in Kymppi 3, syksy (Rinne et al., 2017a, p. 6)*
Note. Transl. Work out the message.

statistical variables and random variables. In the task shown in figure 2, a message is written in some kind of cryptography. The pupil's task is to substitute the letter-number symbols with the letters found in the table and work out the message. For each letter-number symbol, there is exactly one letter, which is the requirement given for the elements in two sets in the definition of a function.

Very basic ideas of statistics are included in the core curriculum for Finnish elementary education. In the task shown in figure 3, ten pupils give points to a game they have played. Although the questions focus on the points, the visual presentation of the data emphasises that each person has given a particular number of points. There is a correspondence between the set of persons and the set of possible number of points. Task 14 in *Kymppi 6B* (Rinne et al., 2017b, p. 202) instructs the pupil to throw two dice ten times. For each throw, the sum of the scores should be written on a small line in a sequence. This task includes a correspondence between the throws (or their numbers) and the value of the sum of the scores.

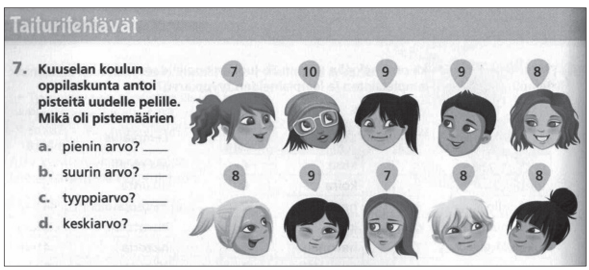


Figure 3. *Task 7 in Tuhattaituri 5 B (Asikainen et al., 2017, p. 144)*

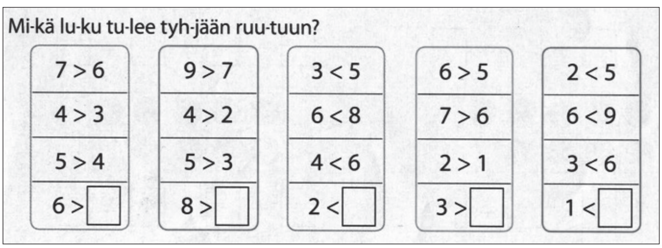
Note. Transl. The pupils' board in Kuusela school shows their points to a new game. Find a) the smallest value, b) the greatest value, c) the mode and d) the mean of the scores

We interpret the tasks in the three classes discussed above as having the potential to mediate a more abstract notion of the concept of the function, that of a *correspondence between two sets*. The role of the variable in those tasks is a *varying variable of elements in a set*. We have added a new category and subcategory to our frame with these labels. Discussing the definitions of a variable in science and in school, Philipp (1992) describes how the mathematics-reform movement in the change of fifties and sixties in the United States changed the definition of a variable. It was no longer associated with a function but with a set. It was defined as a symbol, which could represent any of the members of a particular set.

The definition of a function in Finnish upper-secondary higher-level mathematics books is typically expressed in these terms.

Inequalities

Although inequalities are not mentioned in the core curriculum for the elementary grades of basic education in Finland (Finnish National Agency of Education, 2014), our data includes some tasks with simple inequalities. In those tasks, the variable's role is either the *unknown* or the *varying variable*. Even though the *varying variable* is the appropriate conception when working with inequalities, the books include tasks, such as that shown in figure 4, which clearly states that only one particular number is searched for.


Mi-kä lu-ku tu-lee tyh-jään ruu-tuun?

7 > 6	9 > 7	3 < 5	6 > 5	2 < 5
4 > 3	4 > 2	6 < 8	7 > 6	6 < 9
5 > 4	5 > 3	4 < 6	2 > 1	3 < 6
6 > ☐	8 > ☐	2 < ☐	3 > ☐	1 < ☐

Figure 4. *Homework task + (Plus sign stands for more challenging task) in YyKaaKoo 1A (Hartikainen et al., 2017, p. 119)*
Note. Transl. Which number belongs to the empty box?

The roles of the variables in the simple inequalities found in our data are already included in the category of equations. Furthermore, the mathematical nature of inequalities as open statements is similar to that of equations. We include inequalities and equations in the same category. Blanton and colleagues (2015) also combine equations and inequalities in one of their big ideas: equivalence, expressions, equations and inequalities.

Classes that are not helpful in elaborating on the frame

The tasks in one of the nine classes introduce the system of the *xy*-axis. Since no functional relationships are involved in those tasks, we interpret the roles of the variables in (x, y) as generalised x-coordinate and generalised y-coordinate and regard them as special cases of the generalised number. There is no need to make changes in the original frame.

The tasks in the last three classes of uncategorisable tasks hide the variable's role or make it confusing rather than mediate mathematically relevant meanings. Number sequences could be perceived as constituting a function from the set of positive integers to another set, and they would make an ideal context for introducing the concept of varying variable to elementary-school pupils. However, all the number-sequence tasks in our data are discussed in a recursive way. The pupils are just asked to continue the sequence. The last two classes of uncategorizable tasks include tasks emphasising only the values of the multiplication tables (practising to recognise the numbers) and tasks with missing digits or missing denominators instead of missing numbers, for example, 0,_ + _,8 = 3 (in Finland, decimal comma is used instead of decimal point).

Because quantitative reasoning and programming are generally not considered part of algebra, we have to change the title of the first column in the frame to "Context". We unify the labels of the contexts. "Study of the relationships among quantities, *formulae* and *functions*" is transformed to "Function as a relationship among quantities". Table 4 presents the expanded and transformed frame, which is the main result of this study.

Table 4. *The expanded frame after analysing the uncategorisable cases*

Context	Role of variable
A. Generalised arithmetic	1. Generalised number
B. Quantitative reasoning	1. Generalised magnitude
C. Equations and inequalities	1. Unknown
	2. Varying variable
	3. Generalised number
	4. Coefficient
D. Function as a relationship among quantities	1. Varying variable
	2. Parameter
E. Function as a correspondence between two sets	1. Varying variable of elements in a set
D. Programming	1. Generalised number
	2. Unknown
	3. Varying variable
	4. Coefficient or parameter
	5. Storage

Discussion

Based on testing of the original frame, we constructed the expanded frame for performing content analysis on tasks, especially in elementary-school mathematics textbooks. The frame can be used in finding out which meanings of the concept of variable the tasks have the potential to mediate when brought into generalising and sense-making discussions.

Two new subcategories have been added to the frame after the literature review. In equation solving, the variable can also be regarded as a *varying variable* (Carraher & Schliemann, 2007), and when simplifying expressions in equations, the variable is occasionally treated as a *generalised number* (Philipp, 1992). Testing the frame has led to the recognition of two more categories and corresponding subcategories of the frame – first, the context of *quantitative reasoning* and the corresponding role of variable *generalised magnitude* and second, the context of *function as a correspondence between two sets* and the role of the *varying variable of elements in a set*. Inequalities are included in the category of equations. Empty categories are not left out of the frame because the tasks for those categories can easily be constructed.

In line with the idea of early algebra, our expanded frame extends the discussion on the variables' roles to the elementary level. On one hand, earlier summaries have been written to clarify the use of variables in proper school algebra (Ely & Adams, 2012; Philipp, 1992; Usiskin, 1988). On the other hand, the frame developed in this research is more wide and detailed than the summary provided by Blanton and colleagues (2011) in their guidebook for teachers in grades 3–5.

An important aspect of our frame is that the variable's roles are defined according to its context (see also Blanton et al., 2011). Kilhamn (2014) shows how two Swedish elementary-school teachers introduced the topic of variables and expressions to their pupils. The different meanings of the concept of variable – varying variable, unknown and generalised number – were present in the talks of the teachers and the acts of the classes, but they were not developed in a systematic way that would have made it possible for the pupils to internalise the meanings. We recommend that teachers should clarify the different roles of the variable to their students slowly, one by one, in appropriate contexts. Our frame might even help teachers in organising their thinking about variables and deciding which meanings to address and when.

The subcategories in the programming context include very few tasks. However, we anticipate that Usiskin's (1988) predictions will come true. For example, pupils will encounter the meaning of the varying variable in programming earlier than usual in their studies of mathematics. Nonetheless, will mathematics textbooks include tasks that mediate such meanings, or is it the activity of programming itself that offers pupils those experiences? Our opinion as teacher educators is that to promote the development of computational thinking (Wing, 2006), which is emphasised in the new Finnish national core curriculum for basic education 2014 (Finnish National Agency of Education, 2014), some basic structures of programming languages should also be practised through unplugged activities and tasks in textbooks. Future mathematics textbooks may provide tasks that offer the possibilities for mediating the different meanings of the concept of variable in the programming context. They may however, differ from those meanings we are accustomed in algebra. It would be an important topic for future research to focus on analysing the meanings of the variable in the context of programming. Concretising appropriate uses and meanings of the variable in programming is a challenge we want to suggest to task designers.

Our intention is to use the frame for analysing the tasks in the main elementary-school textbook series in Finland. An interesting research topic would also be to compare textbooks in different countries through content analysis of tasks with it. The frame may also be useful for analysing other curricular materials than mathematics textbooks.

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