Swedish primary teacher education students' perspectives on linear equations

PAUL ANDREWS

Linear equations, connecting arithmetic to the symbolism of formal mathematics, represent a key topic of mathematics. However, the understanding primary teacher education students bring to their studies has been rarely examined. In this study, students were invited to explain in writing how an unannotated solution to $x + 5 = 4x - 1$ had been conceptualised by the hidden solver. Data, coded against an iteratively derived framework, showed that most students were familiar with linear equations, able to articulate an objective for equation solving and offer solution strategies, typically based on either doing the same to both sides, swapping the side swapping the sign or both.

The literature on equation solving typically distinguishes between two forms of equations. Firstly, there are equations like $13 = 3x + 1$, which have the unknown on one side of the equation only. Such equations can always be solved by means of a series of operation reversals (Herscovics & Linchevski, 1994). Secondly, there are equations with unknowns on both sides that cannot be solved by such approaches (Filloy & Rojano, 1989, Kieran, 2004). Interestingly, highlighting a definitional problem not uncommon in mathematics education, there is no accepted vocabulary for this distinction, with the two forms being described as arithmetical and non-arithmetical (Filloy & Rojano, 1989), arithmetical and algebraic (Andrews & Sayers, 2012), procedural and structural (Kieran, 1992), operational and structural (Sfard, 1995) and manipulation and evaluation (Tall et al., 2014) respectively. Of these, some have a more natural resonance with the mathematical knowledge necessary for solving the two forms of equations than others. For example, operational, procedural and manipulation may be problematic because any equation solving involves

---

Paul Andrews
Stockholm University

Andrews, P. (2020). Swedish primary teacher education students' perspectives on linear equations. *Nordic Studies in Mathematics Education*, 25 (2), 29–48.

some form or procedure, operation or manipulation, while evaluation invokes no mental image of either equation. The labels arithmetical and non-arithmetical offer a clearer sense of the distinction, although our view is that arithmetical and algebraic offer clearer indications as to the mathematical understandings necessary for each form of equation.

In broad terms, being able to solve algebraic linear equations draws on a number of related understandings and competences. It requires a relational rather than an operational understanding of the equals sign (Alibali et al., 2007), whereby the equals sign should be seen as representing equality between two expressions and not as a command to operate (Falkner et al., 1999; McNeil et al., 2006). It requires that learners understand and can manipulate the symbols in which equations are represented (Huntley et al., 2007). Thus, solving an equation requires not only that learners ”understand that the expressions on both sides of the equals sign are of the same nature” (Filloy & Rojano, 1989, p. 19) but also that they are able to operate on the unknown as an entity and not a number, insights that acknowledge the structural symmetry of equations (Kieran, 1992).

Didactically, the distinction between arithmetical equations and algebraic equations is of significant interest. On the one hand, students who are first exposed to arithmetical equations, solved by means of operations reversal, may experience difficulties when confronted with algebraic equations that cannot be solved in such ways. On the other hand, students whose first exposure is to algebraic equations, involving a process applicable to both forms of equation, have the advantage of not having learnt a set of procedures that will be superseded. However, algebraic equations have historically been solved by one of two methods, derived from Viete and Euler respectively (Filloy & Rojano, 1989). The former, the conceptual basis of a ”swap the side swap the sign” (SSSS) procedure, is based on the transposition of terms from one side of the equation to the other. The latter, conceptually underpinning a ”do the same to both sides” (DSBS) procedure, relies on operations undertaken on both sides of the equation simultaneously. Interestingly, it is not unreasonable to assume that the former is no more than a procedure inferred from the latter, although, as is highlighted later, SSSS has been subject to more critisism than DSBS, not least because DSBS supports the later learning of algebra, particularly the theory of groups (Wasserman, 2014).

Most intervention studies have exploited the balance scale as a means of facilitating a relational understanding of the equals sign and warranting a DSBS procedure, whether in Australia (Warren & Cooper, 2005), Belgium (Vlassis, 2002), Chile, (Araya et al., 2010) or Turkey (Caglayan & Olive, 2010). In similar vein, studies of didactics have found teachers exploiting the balance scale in Finland, Flanders and Hungary (Andrews

& Sayers, 2012) and Poland (Marschall & Andrews, 2015). Moreover, these same studies have shown that such teachers' didactical sequences usually begin with the general, algebraic equation, rather than the particular, arithmetical equation, because students who can solve an algebraic equation can necessarily solve an arithmetical equation. In short, the balance scale, despite criticisms that it fails to represent negative values in anything but a contrived way (de Lima & Tall, 2008; Pirie & Martin, 1997), seems positively viewed by both teachers and mathematics education researchers. Other intervention studies, undertaken by psychologists working in the traditions of cognitive load theory, have challenged the didactical importance of the balance method. These studies, based on introducing students to both operations reversal and DSBS for the solution of arithmetical equations, find students preferring operation reversal. Their authors (see, for example, Ngu et al., 2015; Pawley et al., 2005) conclude that students' preferences are due to the lower working memory demands of operations reversal. However, in focusing only on the procedurally simple arithmetical equations, fail to consider either the cognitive or the didactical implications for the solution of algebraic equations.

Despite the balance scale-related consensus, the limited assessments of students' strategies for equation solving have typically found an SSSS procedure, as with high school students in the US (Huntley et al., 2007) and New Zealand (Linsell, 2009). Such an approach, reflecting a rote-learnt and arbitrary transposition whereby the unknown finishes on the left-hand side and a value on the right (Filloy & Rojano, 1989), not only perpetuates an operational conception of the equals sign but fails to support students' understanding that such movement does not change the equation's equality. It masks mathematical understanding (Star & Seifert, 2006), may lead to various later difficulties (Capraro & Joffrion, 2006; Kieran, 1992) and become a "magical" (de Lima & Tall, 2008) procedure that reduces students "to performing meaningless operations on symbols they do not understand" (Herscovics & Linchevski, 1994, p. 60). Finally, teacher education students, who may be expected to have a better developed understanding of equations than school students, typically offer solutions indicative of an incomplete understanding of their conceptual and procedural bases (Andrews & Xenofontos, 2017; Casey et al., 2018; Ellerton & Clements, 2011; Isik & Kar, 2012, Stephens, 2008; Tanisli & Kose, 2013).

Linear equations in the Swedish national curriculum

The Swedish national curriculum for compulsory school is structured by the school years 1–3, 4–6 and 7–9. It asserts that by the end of year 3, students will understand "mathematical similarities and the importance of the equals sign" (Skolverket, 2011, p. 60), which is an understanding necessary for the solving of algebraic equations. It adds, more explicitly in relation to linear equations, that by the end of year 6, students will be familiar with "unknown numbers and their properties and also situations where there is a need to represent an unknown number by a symbol; simple algebraic expressions and equations in situations that are relevant for pupils; methods of solving simple equations" (Skolverket, 2011, p. 61). Finally, by the end of year 9 students will understand the meaning of the concept of variable and its use in algebraic expressions, formulae and equations; algebraic expressions, formulae and equations in situations relevant to pupils; methods for solving equations; functions and linear equations" (Skolverket, 2011, p. 63).

With respect to post compulsory education, or upper secondary school, Swedish students opt for one of a number of vocationally- or academically-focused tracks of three years' duration. Depending on their track choice, students may study up to five mathematics courses, each of one semester's duration and representing an increasing sophistication. That being said, all students, irrespective of track, are obliged to follow at least the first of these courses, which is designed to complement and extend students' earlier mathematical experiences and includes further exposure to linear equations. Thus, all Swedish students, by the time they complete upper secondary school, would have had multiple exposures to linear equations. However, as is the case more broadly, little is known about the knowledge that students take into their adult lives. Moreover, while equations-related competence is addressed in both national and international tests, such studies typically concern themselves only with whether or not an answer is correct. Consequently, little is known about the equation-solving procedures students employ, particularly with respect to any interactions between different approaches and conceptualisations. In part, this may be due to scholars' tendency to report students' use of either a balance scale-induced DSBS or, more generally, a rote learnt SSSS. Typically, they seem not to have considered the possibility that students may combine the two. The aim of this paper, acknowledging the significance of linear equations as a key transitional topic of mathematics, is to explore the equations-related knowledge of Swedish students as they progress beyond the educational system. In addressing this, the study is framed by the following questions:

> How do Swedish primary teacher education students construe linear equations?
>
> How do these different construals interact?

In framing this study, and acknowledging that the mathematical qualifications of Swedish teacher education students has been falling since the 1980s (Björklund et al., 2005), it seems reasonable to assume that the mathematical competence of this group, while clearly not comparable with the elite knowledge necessary for the study of numerate university disciplines, is unlikely to be at the lower end of the competence spectrum. In other words, it is not unreasonable to assume that the equations-related competence of this group of young adults would not be atypical of the cohort in general. Finally, with respect to warranting this study and as we discuss in detail below, this paper represents a first attempt at scale to test a simple to operate, low inference, framework for analysing students' equations-related knowledge.

The current study

This paper draws on data derived from Swedish primary teacher education students' written responses to the solution of the algebraic equation shown below. Presented on paper with no annotations to indicate the hidden solver's thinking processes, additional written instructions asked students to imagine, first, that they had a friend who had been absent when their class had been shown how to solve such an equation and, second, to consider what they would say to help their friend understand the given solution. Further oral instructions confirmed that the task related to when participants had first learnt about algebraic equations at school.

$$
\begin{aligned}
x + 5 &= 4x - 1 \\
5 &= 3x - 1 \\
6 &= 3x \\
2 &= x
\end{aligned}
$$

Providing mathematical explanations is a familiar process for Swedish students for whom an oral component is an integral part of all national assessments. Moreover, explaining to another person, whether real or fictitious, facilitates the development and demonstration of both understanding (Fiorella & Mayer, 2014) and competence (Denancé & Somat, 2015), particularly from the perspective of mathematical content knowledge (O'Neil et al., 2014) and problem solving (Wetzstein & Hacker, 2004). Consequently, it was believed the approach would be particularly applicable to Swedish students.

Shortly after the start of their programmes, all six classes of one cohort of first year teacher education students from a large Swedish university were visited and invited to participate in the study, with unwilling students leaving the room for an early coffee break. At the time of the study, students, all of whom were aiming to become generalist primary teachers, had not yet received any mathematics instruction, although all would have completed at least the first mathematics course of upper secondary education. Students were given a sheet of paper on which was presented the task and its instructions concerning the fictitious student. Participants wrote their responses on the same piece of paper and, on completion of their account, left the room.

The particular equation was chosen for several reasons. First, algebraic equations cannot be solved by operation reversal. Second, it should uncover students' conceptual and procedural knowledge and the relationship between them as, at each step, they would need not only to interpret and explain the solver's hidden thinking but decide what needs to be made explicit to the hypothetical friend. Third, the equation and its solution were unhindered by conceptually unnecessary complications like brackets or fractions. Fourth, although this is not the focus of this paper, the task was designed to elicit students' underlying didactical inclinations, whether conceptually or procedurally focused, an issue of particular relevance to a teacher education cohort, an aim successfully addressed in a study of Greek and Cypriot primary teacher education students (Andrews & Xenofontos, 2017).

Table 1. *Working definitions, frequencies and percentages of each code*

Code	The student writes something about...	Present	%
Mentions the unknown	the unknown or variable	28	18
Conceptual objective	finding the "value of x"	60	38
Procedural objective	getting x alone or x on one side	88	56
SSSS General	the general SSSS movement of objects	9	6
SSSS particular addition	transposing a particular object additively with a consequent sign change	53	34
SSSS particular division	transposing a particular object by division with a consequent sign change	7	4
DSBS General	doing the same to both sides in general terms	27	17
DSBS General addition	adding to both sides with no reference to the particular objects of the equation	8	5
DSBS Particular addition	adding to both sides with reference to the particular objects of the equation	77	49
DSBS General division	dividing both sides by the number in front of x	9	6
DSBS Particular division	dividing both sides by 3	51	33
Unspecified division	dividing by 3; divide 6 by 3 etc., where it is not clear that both sides are divided	53	34
Equality of both sides	both sides of the equals sign being equal	38	24
Checks solution	checking the solution	20	13
Does not solve equation	the codes above but does not complete a solution	26	17

The framework developed by Andrews and Xenofontos (2017) was taken as a starting point, but found lacking due to none of their Greek-speaking participants making reference to DSBS, which initial readings of the Swedish data had shown to be widespread. Consequently, the framework was developed further. While space prohibits a detailed summary of this process, three refinements of the original framework were undertaken. At each point, the framework was revised, evaluated against responses from 300 Swedish and Norwegian teacher education students, and further refined (Andrews & Larson, 2019). The goal was to develop a set of low inference codes that would be age- and culture-independent, while picking up all perspectives identifiable in the two data sets. Importantly, low inference codes have two major advantages over other forms of coding. First, they should be simple to operationa-lise and, second, while not implying value judgements about mathematical hierarchies, their interactions may yield insights that other analyses cannot (Andrews, 2007). The process outlined above, once applied to the data, yielded the figures shown in table 1, which includes working definitions of the codes.

Operationalising the framework

Although code frequencies have already been shown, it is important to demonstrate how, in practice, the codes were applied. In the following, three students' scripts are presented and discussed in relation to how the final set of codes was applied to them. In practice, almost any script could have been used, but it is hoped that these will give the reader confidence in the framework and its applicability. Isabelle's script, shown in figure 1, is followed, table 2, by the rationale for the codes applied to it. This is followed by scripts and the derived codes for Johanna and Alice, shown in figures 2 and 3, and tables 3 and 4 respectively. All names are pseudonyms.

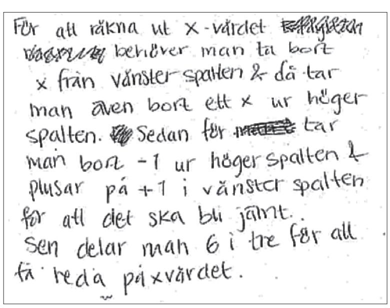
För att räkna ut x-värdet behöver man ta bort
x från vänster-spalten & då tar
man även bort ett x ur höger
spalten. Sedan tar
man bort -1 ur höger spalten &
plusar på +1 i vänster spalten
för att det ska bli jämnt.
Sen delar man 6 i tre för att
få reda på x-värdet.

Figure 1. *Isabelle's script*

Table 2. *The codes inferred from Isabelle's script*

Swedish	Translation	Assigned code
För att räkna ut *x*-vardet	In order to calculate the *x*-value	*Conceptual objective*: Shown in the reference to calculating the value of *x*.
behöver man ta bort *x* från vänster spalten och då tar man även bort ett *x* ur höger spalten	one needs to take away *x* from the left-hand side and then also take away *x* from the right-hand side	*DSBS Particular additive*: Shown in the statement concerning subtracting *x* from both sides.
Sedan tar man bort -1 ur höger spalten och plusar på +1 i vänster spalten	Then one takes away -1 from the right-hand side and adds +1 to the left-hand side	*SSSS particular addition*: Shown in the statement concerning the change of sign when an object is transposed[1].
för att det ska bli jämt	in order that it (the equation) should be equal	*Equality of both sides*: No comment necessary
Sen delar man 6 i tre för att få reda på *x* värdet	Then one divides the six by three in order to get the *x* value	*Unspecified division*: The statement could imply either DSBS or SSSS.
		In an earlier version of the framework, due to the statement following a DSBS particular addition, this would have been coded DSBS induced SSSS. But this was ultimately rejected as requiring too high an inference.

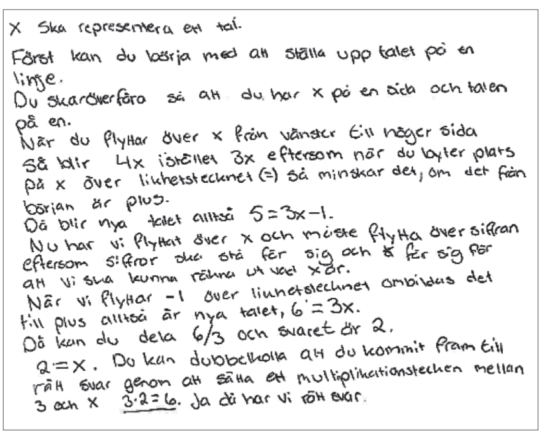
X Ska representera ett tal.
Först kan du börja med att ställa upp talet på en linje.
Du ska överföra så att du har x på en sida och talen på en.
När du flyttar över x från vänster till höger sida så blir 4x istället 3x eftersom när du byter plats på x över likhetstecknet (=) så minskar det, om det från början är plus.
Då blir nya talet alltså 5 = 3x − 1.
Nu har vi flyttat över x och måste flytta över siffran eftersom siffror ska stå för sig och x för sig för att vi ska kunna räkna ut vad x är.
När vi flyttar −1 över likhetstecknet ombildas det till plus alltså är nya talet, 6 = 3x.
Då kan du dela 6/3 och svaret är 2.
2 = x. Du kan dubbelkolla att du kommit fram till rätt svar genom att sätta ett multiplikationstecken mellan 3 och x 3·2 = 6. Ja då har vi rätt svar.

Figure 2. *Johanna's script*

Table 3. *The codes inferred from Johanna's script*

Swedish	Translation	Assigned code
x ska representera en tal	x should represent a number	*Mentions the unknown*: No comment necessary
Först kan du börja med att ställa upp talet på en linje	First, you begin by setting up the equation on a line	No code applied as this is not directly related to the solution
Du ska överföra så att du har x på en sida och talen på en	You should transfer so that you have x on one side and the number on one	*Procedural objective*: Shown in the statement concerning the separation of unknowns from knowns
När du flyttar över x från vänster till höger sida så blir $4x$ istället $3x$ eftersom när du byter plats på x över likhetstecknet (=) så minskar det	When you move the x from the left to the right side, the $4x$ becomes $3x$ because when you move the x over the equals sign it is subtracted (literally diminished or reduced)	*SSSS particular additive*: Shown in the reference to moving over the equals sign and the sign of the particular object, x, changing
Då blir nya talet alltså $5 = 3x - 1$	The task is now $5 = 3x - 1$	No code applied as this is just a statement of derived fact
Nu har vi flyttat över x och måste vi flytta över siffran eftersom siffror ska stå för sig och x för sig för att vi ska kunna räkna ut vad x är	Now we have moved over the x, we must move over the figure, because the figures should stand alone and the x should stand alone so that we can calculate what x is	*Conceptual objective*: Shown in the statement concerning the calculation of the value of x. *Procedural objective*: Shown in the reference to the figures and the x being able to stand alone. *SSSS particular additive*: Here there is reference to moving over the particular, albeit unspecified, figure.
När vi flyttar -1 över likhetstechnet ombildas det till plus alltså är nya talet, $6 = 3x$	When we move -1 over the equals sign it transformed to a plus, and the new task is $6 = 3x$	*SSSS particular additive*: Shown in the movement of the -1 over the equals sign leading to a change of sign
Då kan du dela 6/3 och svaret är 2. $2 = x$	Next you can divide 6 by 3 to get 2, $x = 2$	*Unspecified division*: It is not clear if the specified operation is based in DSBS or SSSS
Du kan dubbelkolla att du kommit fram till rätt svar genom att sätta ett multiplikationstecken mellan 3 och x. $3 \cdot 2 = 6$. Ja, då har vi rätt svar	You can double check that you have arrived at the correct answer by inserting a multiplication sign between 3 and x. $3 \times 2 = 6$. Yes, we have the right answer	*Checks solution*: Although the solution is complete, the intention is clear

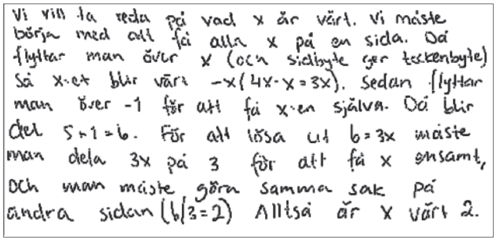
Vi vill ta reda på vad x är värt. Vi måste
börja med att få alla x på en sida. Då
flyttar man över x (och sidbyte ger teckenbyte)
Så x-et blir värt −x (4x-x=3x). Sedan flyttar
man över -1 för att få x-en själva. Då blir
det 5+1=6. För att lösa ut 6=3x måste
man dela 3x på 3 för att få x ensamt,
och man måste göra samma sak på
andra sidan (6/3=2) Alltså är x värt 2.

Figure 3. *Alice's script*

Table 4. *The codes inferred from Alice's script*

Swedish	Translation	Assigned code
Vi vill ta reda på vad x är värt	We want to find out what x is worth	*Conceptual objective*: Finding the value of x
Vi måste börja med att få alla x på en sida	We must begin by getting all the x:s to one side	*Procedural objective*: A statement relating to getting the x alone
Då flytta man över x (och sidbyte ger tecken-byte) så x:et blir värt -x ($4x - x = 3x$)	Then one moves x over (and changing the sides means changing the sign) so that the x becomes worth -x	*Particular additive SSSS*: The statement in brackets is trans-parent
Sedan flyttar man över -1 för att få x:en själva. Då blir det 5 + 1 = 6	Then one moves the -1 over in order to get the x alone, which gives 5 + 1 = 6	*Procedural objective*: A further statement relating to getting the x alone *Particular additive SSSS*: the combination of the moving over of -1 and the result being +1 is construed as a SSSS action
För att lösa ut $6 = 3x$ måste man dela $3x$ på 3 för att få x ensamt, och man måste göra samma sak på andra sidan (6 / 3 = 2)	In order to solve $6=3x$ one must divide $3x$ by 3 in order to get x alone and one must do the same thing to the other side (6 / 3 = 2)	*DSBS Particular division*: Shown in the desire to perform the same particular division (by 3) to both sides *Procedural objective:* A further statement relating to getting the x alone
Alltså är x värt 2	And so x is worth 2	No code applied as this is just a statement of derived fact

Results

Of the students involved in the study, six either left their sheets blank or apologised for their lack of equations-related knowledge. This latter group was well represented by Kerstin's rather poetic view that it was "altogether too long since I wrestled with equations" and Aida's somewhat blunt, "No idea, I barely understand them myself". Otherwise, 156 students, 128 females and 28 males, offered mathematically interpretable responses, the initial analyses of which can be seen in table 1.

Importantly, the reader is reminded that the results here are based on a third coding of the full data set, following the iterative development of the schedule described above. If, within a student's account, the same code was repeated then only one incidence was recorded. That said, most accounts attracted several codes, with an average of 3.6 being applied to each. Analyses of variance showed no influence of either gender or age on students' responses, with the latter being particularly interesting; 99 of the 156 students (63 %) were 22 years or older and, of those, 43 (28 %) were 27 years or older, indicating that whatever students had learnt at school had been retained for several years. Also, four of the codes shown in table 1 were observed in fewer than ten cases each. Of these, three represent high levels of generality with the consequence, particularly when students might be expected to solve an equation by means of a series of particular operations, that their rarity is probably unsurprising. Therefore, these are not discussed further as the intention is to focus on the commonly occurring codes. That said, it is interesting to note that of these four, the emergence of SSSS particular division, which involved students describing a form of transposition whereby multiplication on one side of the equation became division on the other, was surprising, not least because it seemed obscure in relation to the natural and rather obvious emergence of the equivalent additive strategy.

Of course, frequencies alone offer only a partial picture of students' construal of the equation solving process. In the following, drawing on the ease by which the interactions of low codes can be determined, are presented various cross-tabulations intended to highlight the relationship between the commonly occurring codes. However, as the reader will be aware, space limitations prevent a complete presentation, as 15 codes would yield $15 \cdot 14 / 2 = 105$ interactions. Two commonly occurring codes concerned statements of objectives, where 60 students (38 %) wrote something coded for conceptual objective, focused on identifying the value of x, while 88 (56 %) indicated a procedural objective, typically about getting unknowns on one side or alone. When the two codes were compared, as in table 5, the scripts of 40 students (26 %) yielded both conceptual and procedural objectives, indicating, overall, that $20 + 40 + 48 = 108$ individual students (69 %) wrote something coded as an objective for the equation solving process.

With respect to students' solution strategies, the most frequently identified were DSBS particular additive and SSSS particular additive. The cross-tabulations of table 6 show that the scripts of only 12 students were coded for both strategies, with 65 (42 % of all students) writing uniquely of a DSBS approach and 41 (26 % of all students) writing uniquely of a SSSS strategy. Thus, 118 individual students (76 %) wrote something

Table 5. *Cross-tabulation of conceptual objectives against procedural objectives*

		Procedural objective		
		Absent	Present	Totals
Conceptual objective	Absent	48	48	96
	Present	20	40	60
	Totals	68	88	156

Table 6. *Cross-tabulation of DSBS against SSSS additive strategies*

		DSBS particular addition		
		Absent	Present	Totals
SSSS particular addition	Absent	38	65	103
	Present	41	12	53
	Totals	79	77	154

Table 7. *Cross-tabulation of SSSS particular division against unspecified division*

		Unspecified division		
		Absent	Present	Totals
DSBS particular division	Absent	52	53	105
	Present	51	0	51
	Totals	103	53	156

recognisable as a *conventional* additive strategy. In similar vein, table 7, shows a complete lack of any interaction of the two commonly occurring division-related strategies. Taken with the 7 scripts coded for SSSS particular division, these figures show that 111 students (71 %) wrote something intelligible as an awareness of the division process, albeit articulated in different ways.

Other questions of interest concern the relationship between the forms of objective and dominant solution strategies. It was shown earlier that 88 scripts were coded for a procedural objective, while 60 yielded a conceptual objective, a ratio of 88 : 60 or 1.47. If students' objectives were independent of their preferred additive approach, then it would be reasonable to expect this ratio to persist across these approaches. In this respect, table 8 shows the cross-tabulations of each objective with each of the dominant additive strategies. With respect to DSBS particular addition, the ratio is 40 : 29 or 1.38, while for SSSS particular division it is 34 : 25 or 1.36. In both cases, if one fewer script had been coded for conceptual objective, then the ratios would both have been 1.46. In other words, students' objectives were distributed proportionally across the two additive strategies, indicating that however students construe the objective of equation-solving, no inference can be made with respect to the additive strategy they invoke.

Table 8. *Cross-tabulations of the two particular addition strategies against the two objectives*

		Conceptual objective			Procedural objective		
		Absent	Present	Totals	Absent	Present	Totals
DSBS particular addition	Absent	48	31	79	31	48	79
	Present	48	29	77	37	40	77
	Totals	96	60	156	68	88	156
SSSS particular addition	Absent	68	35	103	49	54	103
	Present	28	25	53	19	34	53
	Totals	96	60	156	68	88	156

In similar vein, table 9 shows similar results for the comparisons between the two objectives and the dominant division strategies. Here, the respective ratios were 32 : 26 (1.23) and 27 : 20 (1.35), neither of which is significantly different from the 1.47 shown above. In other words, knowing students' objectives for equation-solving will yield no insights with respect to their division strategies.

Table 9. *Cross-tabulations of two division strategies against the two objectives*

		Conceptual objective			Procedural objective		
		Absent	Present	Totals	Absent	Present	Totals
DSBS particular division	Absent	71	34	105	49	56	105
	Present	25	26	51	19	32	51
	Totals	96	60	156	68	88	156
Unspcified division	Absent	63	40	103	42	61	103
	Present	33	20	53	26	27	53
	Totals	96	60	156	68	88	156

Finally, the figures of table 10 show the distribution of the two dominant division strategies against the two dominant additive strategies. It is here, for the first time, that variation in students' perspectives begins to emerge. For example, it seems clear that scripts coded for DSBS particular addition were more likely to invoke a DSBS particular division than scripts coded for SSSS particular addition. It is also clear that these same scripts were more likely to be coded for DSBS particular division than unspecified division, indicating a relatively strong overall DSBS conception. On the other hand, students coded for SSSS particular addition were more likely to be coded for unspecified division than DSBS particular division.

Table 10. *Cross-tabulations of two additive strategies against two division strategies*

		DSBS particular division			Unspecified division		
		Absent	Present	Totals	Absent	Present	Totals
DSBS particular addition	Absent	66	13	79	53	26	79
	Present	39	38	77	50	27	77
	Totals	105	51	156	103	53	156
SSSS particular addition	Absent	67	36	103	73	30	103
	Present	38	15	53	30	23	53
	Totals	105	51	156	103	53	156

Discussion

In this paper, by drawing on the application and analysis of low-inference codes applied to beginning primary teacher education students' written interpretations of a solution to $x + 5 = 4x - 1$, I have examined the understanding of linear equations that Swedish primary teacher education students bring to their undergraduate studies. The analytical framework, developed from one used in an earlier study of Cypriot and Greek teacher education students (Andrews & Xenofontos, 2017), had been subjected to four iterations of development, each yielding codes of lower levels of inference than the previous (Andrews & Larson, 2019). This framework has made it possible to address the two research questions, concerning students' construals of linear equations and, importantly, the interactions of those construals.

The analyses show that linear equations were familiar to the great majority of students, with only six of the 162 participants effectively denying any knowledge. The remaining students, 126 females and 26 males, offered mathematically interpretable responses indicative, for the majority, of clearly-remembered equation solving principles. Indeed, acknowledging that almost two-thirds of respondents were 22 years or older and recalling material learnt several years earlier, it is encouraging to see the high proportions of students offering equation-solving objectives (69 %), alongside clearly articulated equations-related addition (78 %) and division strategies (71 %). In comparison with their international peers, the results of this study present a relatively positive picture of Swedish students' equation-related competence. For example, of the Greek and Cypriot primary teacher education students who had completed the same task as that reported here (Andrews & Xenofontos, 2017), only 6 % (compared with 38 %) offered a conceptual objective, while not one wrote anything related to DSBS, referring only to SSSS. By way of contrast, the scripts of 102 Swedish students (65 %) yielded statements in receipt of at least one DSBS code, while not one student wrote anything interpretable as an arbitrary transposition whereby the unknown finishes on the left-hand side and a value on the right (Filloy & Rojano, 1989). Such results, even though this was made explicit in only a quarter of the submitted scripts, suggest that the students of this study, even those whose solutions were dominated by SSSS, were largely aware of the equality of the two sides of the equation (Capraro & Joffrion, 2006; Star & Seifert, 2006), another characteristic missing in the Greek-speaking students' reasoning.

More generally, the above indicates a conception of equations and equation solving different from that found in other studies of primary teacher education students undertaken in ways unrelated to that reported here. For example, Stephens (2008), investigating American primary teacher education students at the mid-point of their programme, found "a collective conception of algebra as a school subject matter dominated by symbols and symbol manipulation" (Stephens, 2008, p. 44), results reflected by Tanisli and Kose (2013) with respect to Turkish preservice primary teachers. In similar vein, Ellerton and Clements' (2011) study of more than 300 middle school teacher education students, also mid-way through their programmes, found that "most of the students seemed to be unwilling, or unable, to go beyond mere symbol manipulation" (p. 400). Even when compared with studies of secondary teacher education students – students who could reasonably be expected to have a secure understanding of linear equations – the evidence reported here compares very favourably. For example, while Alvey et al. (2016) found US secondary teacher education students reporting similar procedures to those reported here, other studies have found poor procedural fluency and a lack of awareness of the objectives of equation solving in both the United States (Casey et al., 2018) and Turkey (Isik & Kar, 2012). In sum, the Swedish primary teacher education students of this study seemed to have a more sophisticated perspective on linear equations at the start of their programme than many of their international peers at points much later. More importantly, perhaps, it seems that Swedish teacher education students' knowledge of linear equations confounds the received perception, a perception founded on international tests of mathematics achievement, that, in relation to its economic peers, Sweden performs poorly educationally.

Finally, reflecting on the methods of this study, two key issues stand out. First, the task proved effective in eliciting students' understandings of the equation solving process. In this respect, the instruction to explain the solution to a friend was effective in encouraging students to justify their interpretations of the actions of the hidden equation-solver. Indeed, this study has confirmed the potential of explanation to uncover both understanding (Fiorella & Mayer 2014) and competence (Denancé & Somat, 2015). Second, the analytical framework, with its low inference codes and no hierarchy, went deeper into solvers' conceptualisations than other studies, whose approaches would not have distinguished between approaches based on DSBS and SSSS (see, for example, Foster, 2018; Star & Seifert, 2006; Vaiyavutjamai & Clements, 2006). Indeed, in many such studies, little attention has been paid to the underlining conceptualisation of solvers' thinking. Indeed, those studies with such a focus have typically drawn on a *standard algorithm* (Hästö et al., 2019; Star & Siefert, 2006; Xu et al., 2017) or a *canonical approach* (Buchbinder et al., 2015; Buchbinder et al., 2019), each of which draws effectively on DSBS and a well-defined sequence of steps reflecting the rules of arithmetic found in typical textbooks (Alvey et al., 2016). Moreover, large-scale international tests of achievement, even when they expect students to show their solutions, pay no attention to the particular approaches adopted by students (See, for example, the released items from TIMSS 2011 in Foy et al., 2013). Thus, due to its flexibility and ease of use, the analytical framework used in this study has the potential to transform the study of students' equation solving competence, not least because the interactions of the codes, as indicated above, offer insights typically hidden from other approaches.

References

Alibali, M., Knuth, E., Hattikudur, S., McNeil, N. & Stephens, A. (2007). A longitudinal examination of middle school students' understanding of the equal sign and equivalent equations. *Mathematical Thinking and Learning*, 9 (3), 221–247.

Alvey, C., Hudson, R., Newton, J. & Males, L. M. (2016). Secondary pre-service teachers' algebraic reasoning about linear equation solving. *Issues in the Undergraduate Mathematics Preparation of Teachers*, 1 (1), 1–16.

Andrews, P. (2007). Negotiating meaning in cross-national studies of mathematics teaching: kissing frogs to find princes. *Comparative Education*, 43 (4), 489–509.

Andrews, P. & Sayers, J. (2012). Teaching linear equations: case studies from Finland, Flanders and Hungary. *The Journal of Mathematical Behavior*, 31 (4), 476–488.

Andrews, P. & Larson, N. (2019). *The development of a set of low-inference codes for uncovering students' understanding of linear equations: facilitating comparative analysis*. Paper presented to the Seventh Conference on Research in Mathematics Education in Ireland, Dublin.

Andrews, P. & Xenofontos, C. (2017). Beginning teachers' perspectives on linear equations: a pilot quantitative comparison of Greek and Cypriot students. In T. Dooley & G. Gueudet (Eds.), *Proceedings of CERME 10* (pp. 1594–1601). Institute of Education, Dublin City University.

Araya, R., Calfucura, P., Jiménez, A., Aguirre, C., Palavicino, A. M. et al. (2010). The effect of analogies on learning to solve algebraic equations. *Pedagogies: An International Journal*, 5 (3), 216–232.

Björklund, A., Clark, M., Edin, P.-A., Fredriksson, P. & Krueger, A. (2005). *The Market comes to education in Sweden: an evaluation of Sweden's surprising school reforms*. Russell Sage Foundation.

Buchbinder, O., Chazan, D. & Fleming, E. (2015). Insights into the school mathematics tradition from solving linear equations. *For the Learning of Mathematics*, 35 (2), 2–8.

Buchbinder, O., Chazan, D. & Capozzoli., M. (2019). Solving equations: exploring instructional exchanges as lenses to understand teaching and its resistance to reform. *Journal for Research in Mathematics Education*, 50 (1), 51–83.

Caglayan, G. & Olive, J. (2010). Eighth grade students' representations of linear equations based on a cups and tiles model. *Educational Studies in Mathematics*, 74 (2), 143–162.

Capraro, M. & Joffrion, H. (2006). Algebraic equations: Can middle-school students meaningfully translate from words to mathematical symbols? *Reading Psychology*, 27 (2-3), 147–164.

Casey, S., Lesseig, K., Monson, D. & Krupa, E. (2018). Examining preservice secondary mathematics teachers' responses to student work to solve linear equations. *Mathematics Teacher Education and Development*, 20 (1), 132–153.

De Lima, R. & Tall, D. (2008). Procedural embodiment and magic in linear equations. *Educational Studies in Mathematics*, 67 (3), 3–18.

Denancé, V. & Somat, A. (2015). Learning by explaining: impacts of explanations on the development of a competence. *European Review of Applied Psychology*, 65 (6), 307–315.

Ellerton, N. & Clements, M. (2011). Prospective middle-school mathematics teachers' knowledge of equations and inequalities. In J. Cai & E. Knuth (Eds.), *Early algebraization: a global dialogue from multiple perspectives* (pp. 379–408). Springer.

Falkner, K., Levi, L. & Carpenter, T. (1999). Children's understanding of equality: a foundation for algebra. *Mathematics Teaching in the Middle School*, 6 (4), 232–236.

Filloy, E. & Rojano, T. (1989). Solving equations: the transition from arithmetic to algebra. *For the Learning of Mathematics*, 9 (2), 19–25.

Fiorella, L. & Mayer, R. (2014). Role of expectations and explanations in learning by teaching. *Contemporary Educational Psychology*, 39 (2), 75–85.

Foster, C. (2018). Developing mathematical fluency: comparing exercises and rich tasks. *Educational Studies in Mathematics*, 97 (2), 121–141.

Foy, P., Arora, A. & Stanco, G. (2013). *TIMSS 2011 user guide for the international database. Released items. Mathematics eighth grade*. International Association for the Evaluation of Educational Achievement.

Herscovics, N. & Linchevski, L. (1994). A cognitive gap between arithmetic and algebra. *Educational Studies in Mathematics*, 27 (1), 59–78.

Huntley, M., Marcus, R., Kahan, J. & Miller, J. (2007). Investigating high-school students' reasoning strategies when they solve linear equations. *The Journal of Mathematical Behavior*, 26 (2), 115–139.

Hästö, P., Palkki, R., Tuomela, D. & Star, J. (2019). Relationship between mathematical flexibility and success in national examinations. *European Journal of Science and Mathematics Education*, 7 (1), 1–13.

Isik, C. & Kar, T. (2012). The analysis of the problems posed by the pre-service teachers about equations. *Australian Journal of Teacher Education*, 37 (9), 93–113.

Kieran, C. (1992). The learning and teaching of school algebra. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 390–419). Macmillan.

Kieran, C. (2004). Algebraic thinking in the early grades: What is it? *The Mathematics Educator*, 8 (1), 139–151.

Linsell, C. (2009). Students' knowledge and strategies for solving equations. In K. Hannah (Ed.), *Findings from the Secondary Numeracy Project 2008* (pp. 29–43). New Zealand Ministry of Education.

Marschall, G. & Andrews, P. (2015). Polish teachers' conceptions of and approaches to the teaching of linear equations to grade six students: an exploratory case study. *Research in Mathematics Education*, 17 (3), 220–238.

McNeil, N., Grandau, L., Knuth, E., Alibali, M., Stephens, A. et al. (2006). Middle-school students' understanding of the equal sign: the books they read can't help. *Cognition and Instruction*, 24 (3), 367–385.

Ngu, B., Chung, S. & Yeung, A. (2015). Cognitive load in algebra: element interactivity in solving equations. *Educational Psychology*, 35 (3), 271–293.

O'Neil, H., Chung, G., Kerr, D., Vendlinski, T., Buschang, R. & Mayer, R. (2014). Adding self-explanation prompts to an educational computer game. *Computers in Human Behavior*, 30 (1), 23–28.

Pawley, D., Ayres, P., Cooper, M. & Sweller, J. (2005). Translating words into equations: a cognitive load theory approach. *Educational Psychology*, 25 (1), 75–97.

Pirie, S. & Martin, L. (1997). The equation, the whole equation and nothing but the equation! One approach to the teaching of linear equations. *Educational Studies in Mathematics*, 34 (2), 159–181.

Sfard, A. (1995). The development of algebra: confronting historical and psychological perspectives. *The Journal of Mathematical Behavior*, 14 (1), 15–39.
Skolverket. (2011). Curriculum for the compulsory school, preschool class and the recreation centre 2011. Skolverket.
Star, J. & Seifert, C. (2006). The development of flexibility in equation solving. *Contemporary Educational Psychology*, 31 (3), 280–300.
Stephens, A. (2008). What "counts" as algebra in the eyes of preservice elementary teachers? *The Journal of Mathematical Behavior*, 27 (1), 33–47.
Tall, D., Lima, R. de & Healy, L. (2014). Evolving a three-world framework for solving algebraic equations in the light of what a student has met before. *The Journal of Mathematical Behavior*, 34 (1), 1–13.
Tanisli, D. & Kose, N. (2013). Pre-service mathematic teachers' knowledge of students about the algebraic concepts. *Australian Journal of Teacher Education*, 38 (2), 1–18.
Vaiyavutjamai, P. & Clements, M. (2006). Effects of classroom instruction on student performance on, and understanding of, linear equations and linear inequalities. *Mathematical Thinking and Learning*, 8 (2), 113–147.
Vlassis, J. (2002). The balance model: hindrance or support for the solving of linear equations with one unknown. *Educational Studies in Mathematics*, 49 (3), 341–359.
Warren, E. & Cooper, T. (2005). Young children's ability to use the balance strategy to solve for unknowns. *Mathematics Education Research Journal*, 17 (1), 58–72.
Wasserman, N. (2014). Introducing algebraic structures through solving equations: vertical content knowledge for K–12 mathematics teachers. *PRIMUS*, 24 (3), 191–214.
Wetzstein, A. & Hacker, W. (2004). Reflective verbalization improves solutions – the effects of question-based reflection in design problem solving. *Applied Cognitive Psychology*, 18 (2), 145–156.
Xu, L., Liu, R.-D., Star, J., Wang, J., Liu, Y. & Zhen, R. (2017). Measures of potential flexibility and practical flexibility in equation solving. *Frontiers in Psychology*, 8, 1368.

Note

1 This particular statement prompted much debate, as Isabelle's choice of language hindered the development of a set of low-inference codes. Her statement concerning the "taking away of -1 from the left-hand side" and "adding 1 to the right" proved problematic. It could not be coded DSBS, because an explicit reference to the same operation on both sides would be expected. However, while her phrase could be interpreted as a procedure generalised from DSBS, it was interpreted as an SSSS procedure whereby whatever operation was performed on one side of the equation, the opposite was performed on the other. In an earlier version of the framework, a category of DSBS induced SSSS, which would have been applied here, had been considered. However, it was thought to require too high a level of inference and was subsequently rejected.

Paul Andrews

Paul Andrews is a Professor of Mathematics Education at Stockholm University. His research interests are typically focused on comparative analyses of mathematics didactics and participant, both teacher and student, beliefs. Currently he is the director of a Swedish Research Council-funded project on the development of foundational number sense in year one children in England and Sweden. His methodological and theoretical standpoints are varied and determined by the research questions he is pursuing at the time.

paul.andrews@mnd.su.se