The quality of conceptual change in mathematics

The case of number concept

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In this article, the main results are presented from a number concept test in which 538 students from 24 randomly selected Finnish upper secondary schools took part. The test included identification, classification and construction problems in the domain of rational and real numbers. In addition, the students were asked to explain their answers and estimate their certainty about them. The theories of conceptual change and of mathematics concept formation were used to categorize students' explanations into different levels. The results indicate the clearly restricted nature of students' prior thinking of whole numbers and of their everyday experiences of counting and continuity. On the basis of the results, we claim that the problems that students have with these difficult concepts are not only due to the complexity or abstract nature of the concepts to be learned, but also to the quality of their prior knowledge, which is not sufficiently taken into account in traditional teaching.

The notion of real numbers is one of the most complex and profound concepts in mathematics. Mathematicians have various rigorous constructs for these numbers; the most familiar ones are those where the real numbers are presented as limits of a sequence or a cut on the number line. In this article, we deal with the problems upper secondary students have when they struggle to understand the basic components of real numbers: the concept of limit in the context of the number line and of the function. These concepts have a long and troubled history and they are difficult for students today (e.g. Boyer, 1949; Cornu, 1991; Fischbein, Jehiam & Cohen, 1995; Merenluoto & Lehtinen, 2002; Sierpinska, 1987; Szydlik, 2000; Williams, 1991).

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The constructs of real numbers are, however, relatively new in the history of mathematics: the rigorous definition for real numbers was developed at the end of the 19th century. The core problem in this slow development of this concept was the dichotomy of discrete and continuous quantities; in the integration of two fundamentally different knowledge domains, the advanced thinking of numbers along with the domain in which the characteristic feature is continuity (Boyer, 1949). In upper secondary mathematics, the concept of continuity[1] is based on the limit of a function, and the limit in turn is explained as a function f having a limit y when x tends towards the value x_0, that is to say that the values taken by the expression $f(x)$ get close to y when x gets close to x_0. In this definition the presumption of numbers as an everywhere dense set is embedded, and thus represents to the learner a fundamental cognitive conflict that is briefly explained in the following section.

Initial thinking of numbers based on discrete quantities

Many researchers (e.g. Carey & Spelke, 1994; Gallistel & Gelman, 1992; Spelke 1991; Starkey, Spelke & Gelman, 1990) argue that human reasoning is guided by a collection of innate domain-specific systems of knowledge. What is important in this article is the fact that, concerning the development of numbers, this ”mechanism” is based on the idea that quantities are separate, discrete. This separate nature of small quantities seems to be one of the basic ontological presumptions of the naive framework theory of numbers. By the terms discrete and separate we mean the instinctive feeling connected with numbers and quantities that there is always the ”next number”, ”next quantity” and that there is some kind of space between them. This quality of numbers is also found in the writings of Aristotle (see Boyer, 1949). The notion of the next number or a successor is a basic feature in the set of natural numbers (Landau, 1951/1960), and together with the principle of one to one correspondence with objects, they ”are woven into the very fabric of our number system” (Dantzig, 1954). In the domain of rational and real numbers the successor is not defined but infinite successive division is possible. At a secondary level, the rational numbers are defined as a relation of two integers, where the denominator is not zero, and the real numbers are defined as points of the number line where there are no gaps. Thus, the underlying but fundamental difference between the number domains is the discrete and dense nature of numbers resulting in different rules of order and operations performed with these numbers. For mathematicians, the hierarchical construction of numbers is logical and coherent because they are already familiar with the structure. For students, however, it looks fragmented and inconsistent because, at the stage of their learning, when they are dealing with continuity and limit, they do not have enough structural knowledge to recognize the logic of the hierarchical structure.

Moreover, small natural numbers and the concept of a successor are among those special concepts which have a high unconditional certainty attached to them. This kind of certainty seems to be derived from at least three different sources. Firstly, in several empirical studies, it has been found that even infants have an intuitive conception of small cardinalities as discrete objects (e.g. Starkey, Spelke & Gelman, 1990). The second source of the certainty comes from everyday experiences and linguistic operations (e.g. Wittgenstein, 1969) in counting objects. Later, in formal mathematical instruction, these prior conceptions are strengthened, and this is important in order to teach and learn the notion of natural numbers. Hence, the conception of numbers as discrete objects is based on innate cognitive mechanisms, powerful experiences of everyday counting, and on formal mathematical instruction. Thus, we claim that the change from using the discrete numbers to the use of rational numbers, where the next number is not defined, means a radical conceptual change for the learner and requires metacognitive knowledge about strategies to be able to choose the rules of operation depending on the task at hand. Thus, every extension of the number concept demands new rules to be learned for operations and the use of a new kind of logic often leading to many different, but systematic problems and misconceptions in mathematics learning (c.f. The multiplier effect, see Verschaffel, DeCorte & Van Coillie, 1988).

Initial thinking of continuity

As stated above, in the formal teaching of mathematics, the concept of limit, a central concept of real numbers, is first taught in the context of functions, where the continuity of a function is based on the concept of limit. It is significant to notice that the teaching of mathematical continuity to students occurs on a quite ”untouched” level. Their earlier experiences of continuity come from everyday life, where continuity is related to the continuity of time, motion or direction, not to numbers. The only ”mathematical” continuity so far has been the continuity based on the above described repetition of numbers, which comes first as the linguistic process when children learn to say number words. The fundamental feature of this continuity is its nature as the formation of discrete actions or objects. From a learner’s point of view this kind of continuity is totally different from what that is meant when speaking of everyday continuity, which is attached to the continuity caused by motion. In fact, both the words ”continuity” and ”limit” have significance for the students before any lessons begin (e.g. Schwartzenberger & Tall, 1978) and students continue to rely on those meanings even after the formal definitions have been taught to them (see Table 1). This kind of thinking is further strengthened in the practice where the continuity of function is described as the result of a continuous motion: ”the pencil never leaves the paper”. The general idea of the continuity of motion is one of the basic features of the physical world and even infants grasp it at some intuitive level (Spelke, 1991). This idea is vague, dynamically and instinctively understood in the context of motion or time, but not with numbers (Cornu, 1991).

Table 1. *Different meanings that students attach to the expressions "tend towards", "limit", and "continuity"* [1]

”Tend towards”	”Limit”	”Continuity”
- to approach (eventually staying away from it) - to approach, without reaching it - to approach, just reaching it	- an impassible limit which is impossible to reach, - a point which one approaches, without reaching it, - a point which one approaches and reaches, - a maximum or minimum, - the end, the finish	- ”no breaks or jumps” (cf. ”it rained continuously”) - ”no gaps” (cf. ”the railway is continuously welded”) - ”being in one piece” (confused with connectedness)

Note. [1] Cornu, 1991, 155-158.

Thinking of the ”next” number or next object, continuity caused by motion, and limit as a bound seem to be a form of cognition which presents itself to a person as being self-evident. Self-evidence and credibility seem to be typical of the primitive intuition of ”truths” based on strong everyday experiences. Overconfidence plays an essential role in this intuition. It means that people are inclined to accept with a feeling of absoluteness those statements which are in line with their previous assumptions (Fischbein, 1987).

The crucial role of prior knowledge

In this section we briefly summarize the theoretical framework used to explain the difficulties students have in their struggle to learn mathematics. According to the *theory of reification* (Sfard, 1991; Sfard & Linchevski, 1994), students first learn new mathematical concepts in operations and the difficulties in the learning are then explained as resulting from a cognitive gap in the structural understanding of the concepts. The problematic difference between operational and structural understanding is that the structural is clearly more abstract than the operational stage. At the operational stage, the concept is understood as an operation, where its nature is seen as potential but not actual, whereas to be able to understand something as structural means being able to refer to it as a fully-fledged object where various representations of the concept become semantically unified by this abstract, purely imaginary construct: being both operational and structural. In this theory, however, not very much attention is paid to the role of students' prior knowledge, and thus we combine it with the theories of conceptual change (see Table 2) where the process of learning is examined starting from the learners' perspective and progressing towards the scientifically accepted conceptualizations.

A three-phased theory of reification

The theory of mathematical concept formation, the *theory of reification* (Sfard 1991; Sfard & Linchevski, 1994) is based on three presumptions: first, the mathematical objects are understood as dualities; both operational and structural; and in the learning process the operational phase comes first and the structural later. Several researchers have also explained mathematical concepts either as processes or products, operational or structural, where this two-sided nature is seen as a dichotomy (e.g. Piaget, 1980/1974; Dubinsky, 1991). The theory of reification, in contrast, is based on considering this two-sided nature as a duality, which is also the point of departure from Piagets' three-phased model of learning, where the first two phases had the same name. For example, the number +2 can be understood as an operation 'add two' or as a structural concept where it represents a natural number, an integer and rational number in the hierarchy of real numbers. The second presumption of the theory is that the concepts are first learned in operations (operational understanding), while the structural understanding demands a surpassing of an ontological gap. The third presumption in the theory is that the learning proceeds through a three-phased process of interiorization, condensation and reification, understood as degrees of structuralization. In this process, the first two phases mean operational understanding (Sfard, 1991; Sfard & Linchevski, 1994).

Viewpoint of theories of conceptual change

Theories of conceptual change (e.g. Carey, 1985; Carey & Spelke, 1994; Chi, Slotta, & de Leeuw, 1994; Karmiloff-Smith, 1995; Vosniadou, 1994; 1999) focus on the role of prior knowledge in learning. The term ”conceptual change” in this frame of reference means a radical change or a clear reorganization of prior knowledge. Hence, we speak about the problems of conceptual change, when the learners' prior knowledge is incompatible with the new conceptualization, and where they are disposed to make systematic errors or build misconceptions, suggesting that prior knowledge interferes with the acquisition of the new concept. This kind of knowledge acquisition is typical in specific domains of science (Posner, Strike, Hewson & Gertzog, 1982).

Researchers (e.g. Vosniadou, 1994; 1999; Hatano 1996) make a distinction between different qualities of learning processes targeted at conceptual change: continuous growth and discontinuous change. The easier level of learning is called *enrichment*, suggesting continuous growth, improving the existing knowledge structure. Discontinuity of learning occurs typically in a situation where prior knowledge is incompatible with the new information and needs *reconstruction*; where significant reorganization – not merely enrichment – of existing knowledge structures is needed. In the theory presented by Vosniadou (1994; 1999), the concept formation from naive assumptions to scientific understanding of the concepts is described as a progress through different levels of synthetic models. There, the synthetic mental model refers to situations where students attempt to synthesize the currently accepted scientific information with the system of their initial concept, without making any radical changes in the prior thinking. Synthetic models are constructed when students are presented with scientific explanations which are highly inconsistent with the intuitive explanations they have constructed on the basis of their everyday experience (Vosniadou, 1999). For example, a student's number concept could be called synthetic, if he/she is working reasonably well with rational numbers on the operational level (see Sfard 1991), while her/his explanations of numbers are still based on discrete natural numbers (e.g. Merenluoto & Lehtinen, 2002; Neuman, 1998).

There is plenty of empirical research from the point of view of conceptual change in the field of biology (e.g. Carey, 1985; Hatano & Inagaki, 1998), physics (e.g. Ioannides & Vosniadou, 2002; Vosniadou, 1994) and also to some extent of mathematics (e.g. Merenluoto & Lehtinen, 2002; Vamvakoussi & Vosniadou, 2002). In the majority of these empirical studies, the large cross-sectional empirical data have been categorized into different levels representing the development of the conceptualization of understanding, starting from the very initial ones.

Table 2. *The theoretical framework combined from the theory of reification and theories of conceptual change in explaining the different levels of understanding in the process from discrete numbers to the concept of limit at the secondary level*

Theories of conceptual change[1]	**Theory of reification**[2]	**Description of the level**
Initial models totally based on naïve thinking.		**Primitive level** The students' written answer is totally based on the rules of (”add one”) natural numbers or everyday experiences.
Different level of synthetic models where the student has enriched his/her prior thinking with aspects of the scientific concept	**Interiorization** Familiarizing with the object through operations, recognition of some characteristic feature in different representations.	**Partial identification** The student identifies at least one substantial feature of the concept – the prior thinking of whole numbers and everyday experiences still dominates the explanations.
	Generalization of a general feature Transfer between different representations. *Transitional stage from prior thinking to a more theoretical description.*	**Operational understanding** The student explains his/her answer with operational explanations, such as adding decimals.
	Condensation Complicated processes are condensed for easier manipulation Recognition of the relations between objects – paying attention to the structure of the concept.	**Beginning of structural understanding** Identification of some structural feature of the concept, such as that there is infinity of numbers between any two rational numbers.
Understanding/explaining a scientific concept as a result of conceptual change (revision).	**Transition level from the level of condensation to reification** The concept is detached from the operations that gave rise to it. The new concept is ”officially” born. Identification of the relations between the objects and generalization.	**Level of structural awareness** Indication of the structure of the number domains and of comparing them. Continuity is based on the concept of limit Real numbers represent the points of the number line, where there are no gaps.

Notes. [1] Vosniadou, 1994; 1999. [2] Sfard, 1991; Sfard & Linchevski, 1994; Goodson-Espy, 1998.

In this article, we use the theoretical framework described above to explain the difficulties students have in learning the density of numbers on the number line and the concept of limit. Further, our aim is to point out the most difficult transitions where support in teaching is particularly needed.

Method

Participants

A number concept questionnaire was given to 538 students (mean age 17.2 years, boys 62.4% and girls 37.6%) of advanced mathematics from 24 randomly selected Finnish upper secondary schools after their first calculus course.

Procedure

In the questionnaire students' preliminary conceptions of real numbers were tested, such as: (1) the hierarchical nature of real numbers, where irrational and rational numbers, integers and natural numbers are understood as sub sets; (2) the density of rational and real numbers on the number line, and (3) the concepts of limit and continuity, which are traditionally taught in the context of functions. The students were asked to estimate their certainty while answering the questions using a scale from 1 to 5, where 1 meant that their answer was a wild guess, and 5 that they were absolutely sure, as sure as they know that 1+1=2.

Qualitative analysis and scoring of answers

In the critical questions, where students were asked to explain their answers in their own words, the qualitative analysis of students' written answers and explanations was based on the theories of conceptual change together with the theory of mathematical concept formation as explained in Table 2. Twenty-two percent of the qualitative data (students' explanations on number line, continuity and limit of a function) was independently analyzed by a teacher of mathematics, teaching upper secondary courses, and 92.2% unanimity with the researcher was reached. After the qualitative analysis, the answers were scored from 1 to 5 so that the lowest level of answers, which was totally based on thinking of whole numbers, scored 1 and the highest level of answers 5. Further combined variables were calculated to represent the domains measured in the test:

the hierarchy of number domains, density of numbers on the number line, and continuity and limit of a function (Table 3).

Table 3. *Combined variables of the different aspects of real numbers measured in the test.*

Combined variables	Number of items	Examples of the items	Alpha	
			Scores	Certainty
Numbers and hierarchy	6	Write the names of all the domains of numbers the given number belongs to (like the numbers -5; 8√3; 12; 22/7) Write an example of an irrational number in a decimal form How many real numbers of form *a*√2 are there on the number line between -4 and 10, where *a* is an integer?	.640	.791
The density of numbers on the number line	4	How many fractions are there between numbers 3/5 and 5/6 on the number line? Which fraction is the "next" after 3/5 and which real number is the closest to 1.00?	.761	.792
Function, limit and continuity	5	Identify the functions (in picture) which are continuous/ have a limit at $x = 1$. Sketch a function which is discontinuous at $x=1$ and has a limit $f(x) = 0$. Explain in your own words what is meant by the continuity and limit of a function.	.684	.790

Results

It was typical of the students' answers that the answers were better in the domain of numbers than on the density of number line (Table 4); and that only a few (fewer than 10 of the high achieving students) found any connections between the density of numbers on the number line and the concept of limit. Further, there was a significant association with gender both in the scores, $F(1, 535) = 4.82$; $p = .029$, and in the certainty estimations, $F(1, 535) = 23.57$; $p = .000$; the boys had both better scores and higher certainty estimations than the girls. The scores and certainty estimations were highest on the domain of numbers and the lowest on the domain of functions, in the ANOVA for repeated measures there were significant difference in the scores, F (1, 522) = 5.06; p = .025, which had an interaction with the achievement level of students in mathematics, F (1, 522) =5.06; p = .025, and to the gender F (1, 522) = 1. 522) = 0.49; p = .002. All the students had the lowest scores in the questions pertaining to the density of numbers, where also the differences between the different levels of achievers were the smallest. This tendency was most obvious in the girls' scores, where there were no statistical difference between the high and low achieving girls (Table 4). Respectively in the certainty in the ANOVA for repeated measures, F (1, 524) = 128.5; p = .000, and a significant interaction with achievement level, F (1, 524) = 6.43; p = .002, but no interaction with gender. Both the boys and the girls gave the highest certainty estimations in questions pertaining to the numbers, and the high achieving girls gave their lowest certainty estimations on questions about the density of numbers.

It is important to notice the general tendency to overestimate the certainty in the domains of numbers and density of number line, and under estimate it in the domain of the function. The rather high scores on the tasks pertaining to the functions come mainly from the identification tasks, where the continuity was identified mostly correctly (no gaps).

In the qualitative analysis of students' answers (see Table 5), the discrete nature of the students' explanations of the number line was obvious

Table 4. *The means and standard deviations of scores and certainty estimations, where these are calculated as per cents of maximum.*

	Achievement level in mathematics	*n*	NUMBERS		DENSITY		FUNCTION	
			Scores[1]	Certainty estim.[1]	Scores[1]	Certainty estim.[1]	Scores[1]	Certainty estim.[1]
Boys	Low	106	38 (18)	45 (21)	27 (18)	36 (24)	36 (16)	39 (21)
	Average	135	44 (16)	52 (19)	35 (20)	46 (28)	42 (17)	41 (22)
	High	88	55 (17)	62 (20)	42 (22)	53 (31)	48 (20)	44 (24)
Girls	Low	45	37 (15)	40 (19)	21 (16)	31 (24)	38 (14)	31 (17)
	Average	114	42 (18)	47 (20)	27 (20)	33 (26)	41 (16)	35 (17)
	High	40	53 (17)	57 (22)	25 (19)	27 (24)	50 (17)	39 (21)

Note. [1] Mean (Std.)

in the primitive level of answers, where they used the spontaneous logic of whole numbers, such as that there are six or seven numbers between the numbers 3/5 and 5/6. The answers classified on the partial identification level were indefinite, such as ”many”, ”infinite” without any explanations. However, a clear difference was seen on the operational understanding level, where students clearly referred to the use of operations for adding the density or explained their answer by stating that it is not possible to define the next number with arguments such as ”... because it is always possible to make them more exact”, ”... add decimals”. These answers indicate operational kind of understanding. Very few of the students gave any signs of structural abstraction of the number line in their explanations, for example, ”It is not possible to define the next one, because the principle of the next number is valid only in the domain of natural numbers or integers”.

In the respective analysis of the answers and explanations in the domain of functions, the answers classified as being on the primitive level described the continuity as a cause of something that ”does not stop” or ”continues pretty long and terribly far”. In the respective explanations of the limit, the students gave explanations where the continuity was bounded by the limit ”it’s the place where the graph ends”, or they described it as the maximum or minimum of a function where ”it is the value where the function has its maximum or minimum”, or even a poetic answer ”the place of the hills and valleys of a function”. The explanations classified on the partial identification level were clearly based on the dynamic of continuity caused by motion ”the function does not jump over the numbers”, by the operation of drawing, ”it is possible to draw without lifting the pen”. In all these answers there was either a clear intuition of motion as a cause of continuity.

Although the explanations were clearly based on an everyday intuition of continuity, in these answers there was a different dimension of continuity compared to the previous level. These answers were closer to the mathematical continuity that is defined at one point. In the case of the concept of limit these were the answers where the students identified the abstraction of successive division of the limit concept, and explained this abstraction as dynamic approaching. These answers were still very indefinite, and vague, but there were a clear change compared to the previous level; ”... it approaches a value ”, ”the graphs approach the point but do not hit it”, ”the value the function approaches but never reaches”. Although the students had recognized the abstraction of successive division, the answers were tied to the dynamics of motion. The intuition of a limit as a boundary (see Figure 1) was still there, but instead of explaining the limit as a 'stop' sign, the boundary was explained as something that is forever possible to get closer, but never reach.

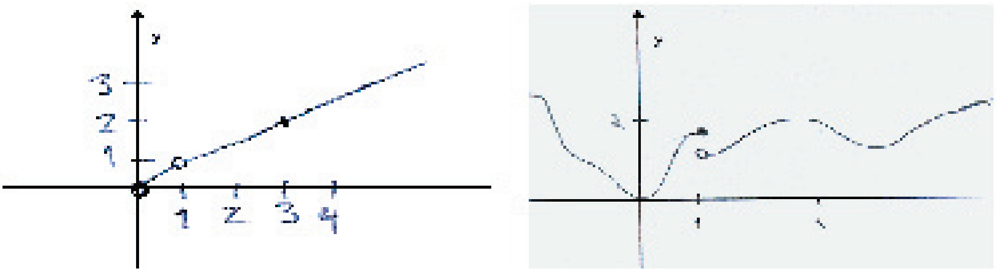

Figure 1. *Limit of a function as a bound; the most popular drawings (42 %) of the students in the task where the students were asked to sketch a function which is discontinuous at* $x = 1$ *and has a limit* $f(0) = 0$.

The answers classified on operational understanding levels were those where the students had got away from motion as the cause of continuity and tried to get closer to the mathematical continuity. They tried to describe the unbroken continuity as a graph; "a function has a value *y* for every value of *x*", "a function is continuous everywhere and can have all the values of the number line". The responses of these students were not mathematically correct, although, the change of explanations from the dynamics of motion to a more mathematical description of the continuity at one point was seen.

From a concept change perspective, this level represents a transition level for the student. At this level the students still seem to be relying on their primitive intuition of continuity, but not of motion, rather of an unbroken line, but dynamics of motion are no more there. In the case of the concept of limit, in students' answers it was explained in more exact terms, the operation of approaching was described as dependence between a variable and the function; "A function approaches a value while *x* gets closer some value".

It is only on the level of beginning structural abstraction, when the students begin to explain the continuity based on the limit conception, that we can see an essential conceptual change occur or is beginning to occur. These were the few answers in which the students explain "a function is continuous if it has at that point a left hand and right hand limit and they are the same and the same as the value of the function" or that "the limit is the value that the value of the function gets closer to when

Table 5. *The quality of the conceptual change in learning the basic concepts of real numbers and the percentages of different levels of answers*

		Number line		Limit of the function	
	Numbers	**Density**	**Limit**	**Continuity**	**Limit**
Starting level	Natural numbers	Discrete number line or natural numbers, no last numbers	”Next” number, a successor	Continuity caused by motion	Limit stops the continuity
1. Primitive level- coherent use of the rules of discrete natural numbers	There are decimal numbers, fractions, whole numbers, square numbers … 15.2 %	You can count the numbers between two fractions if you multiply the fractions so that the denominators are the same 45.2 %	You get to the next by adding one…[4] 71.9 %	Continuous if there is no end 50.9 %	Limit as a boundary to continuity or a maximum or minimum value 74.2 %
2. Level of partial identification – enriched model-wavering use of rules	Identification of some of the number domains, crucial mistakes[1] 51.9 %	There is infinity of numbers between any two numbers[3] 32.2 %	There is no ”next” number / there is infinity of numbers 16.9 %	Continuous if there are no jumps or if the functions continues its motion 22.3 %	Getting closer but never reaching 16.0 %
3. Level of operational understanding – enriched model-wavering use of rules	Systematic models of the hierarchy: domain of rational numbers is missing, problems with representations of irrational numbers[2] 28.4 %	Explanations of the density of the numbers on the number line is based on potential repetition, making numbers more exact or adding decimals 20.6 %	The successor is not defined because of potential infinite divisibility or infinite adding of decimals 9.1 %	A function is continuous if there are no gaps in the graph / if it is defined at every point (based on continuity as something without gaps or as an unbroken line) 24.0 %	Value of a function gets closer when the value of the variable gets closer 7.8 %
4. Level of beginning structural understanding – enriched model-coherent use of rules of compact numbers	Correct presentation of hierarchy of numbers 4.5 %	Explanations of the density of the numbers on the number line is based on explanations of the limit 2.0 %	The successor is not defined –always possible to find a number which is closer/ a limit 2.0 %	Continuous at the point where the limit of the function is the same as the value of the function at the point 2.8 %	A function has a limit when the left and right hand limits are the same 2.0 %
Total	100 %	100 %	100 %	100 %	100 %

Notes. [1]By crucial misconceptions we mean for example identifying irrational numbers as rational numbers or treating integers and rational numbers as sub sets of natural numbers
[2] Presenting irrational numbers as rational numbers, complex or negative numbers
[3] Answers where the students only wrote that there are infinity of numbers between the two given ones, but did not explain their answer.
[4] For example, that the ”next number” after 3/5 is 4/5.

the variable approaches a certain value". What is noteworthy from the conceptual point of view is that, in the case of continuity, the students are able to give mathematically rather high level of answers just by relying on their primitive intuition of continuity as something that does not have "gaps". In the case of limit, there were also only a few students who showed some structural intuition of the limit concept in their answers.

Altogether, the quality of students' knowledge in written answers seemed to be composed of fragmented pieces of knowledge with hardly any connections or relations between the components. In the Table 5 this lack of connection is represented in the table as an unbroken line separating the different components and levels. A double line represents the places of more difficult change where a special support for the learning is needed (also indicated by the percentages of different kind of answers).

The table does not suggest that for all the different components the developments of changes occur simultaneously. Instead the results suggest that it is possible that the development of different components is possible independently of the other components. A dotted line in the model represents the places where there seems to be a transition stage to the next level.

Discussion

The results of this study indicate that the majority of students had not restructured their prior system of beliefs to understand these concepts even at the preliminary level. Their real number concept seems to be a mixture of fragmented pieces of advanced logic and of powerful pieces of primitive logic based on finite processes and everyday experiences. The fundamental presumption of always having a next number is one of the basic experiences students have about numbers, and it is hard for them to imagine a situation where it can not be determined. Because the participants were students of advanced mathematics courses in upper secondary school they had a lot of experience with rational numbers and operations. It is therefore likely that most of them would have remembered that it is possible to divide the fractions infinitely if they had been reminded of it. The results, however, indicate a situation, in which this knowledge of fractions constitutes an isolated piece of knowledge. When the students read the word "next" they spontaneously used the logic with which they had more experience.

Adding to the density of the number line requires the enrichment of prior knowledge, and the students seem to be able to give correct operational level answers, although their thinking of numbers is still based on the discrete nature of numbers. Accepting the logic that it is impossible to define the ”next” number demands a radical but most of all a conscious reconstruction of one's prior thinking. This difficulty was clearly seen in some of the answers, where the students on one hand wrote ”it is not possible to define the next” but, on the other hand, continued to believe in the actual existence of this kind of number, ”... but it is the one which is the closest”, ”... the one which has most of the 9:s”, ”... I do not know but it has a lot of nines” (Merenluoto & Lehtinen, 2002, p.249).

From a conceptual change point of view, it seems that as long as the continuity is thought of as something without gaps, and the limit as something that it is possible to dynamically get closer but never reach, the enrichment of the prior knowledge is sufficient. Although the answers of the students are close to the mathematical continuity, the everyday intuition of an unbroken line supports the students but leads to restriction of understanding. Respectively, in all the answers where the students referred to approaching and not reaching, there remains the primitive based intuition of a restrictive nature of the limit. Therefore, these explanations still seem somewhat ”synthetic”. The student has preserved the essential feature of his/her prior thinking and enriched it with specific pieces of scientific knowledge.

The difference in students' certainty estimations in the domain of number line compared to the one in the context of function suggest that it would probably be wise to teach the difficult concept and abstraction of limit first in the context of the numbers on the number line. Later, the abstraction of limit could be transferred to the context of functions, for example, using a meta-concept of ”density” in the points of the graph. Although according to the mathematical viewpoint, it is logical first to teach the concept of limit in the context of discontinuous functions, this teaching order seems to strengthen students' everyday conceptions of limit as a boundary. The highly resistant nature of prior knowledge was very obvious in the students' answers, suggesting the necessity of explicitly highlighting the difference between prior thinking and the new concept in the teaching of these concepts. Moreover, it would be wise then to discuss the limit of function in the context of continuous functions where the function has a limit at every point where it is defined, and only later introduce the context of discontinuous functions.

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Notes

1 A function is continuous at a point x_0 if the right and left hand limits are the same as the value of the function at the same point x_0.

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Sammanfattning

I denna artikel beskrivs de viktigaste resultaten från en studie, där elevers förmåga att uppfatta matematiska tal testades. 538 elever (i medeltal 17,2 år) från 24 slumpmässigt valda finska gymnasier deltog i studien. Eleverna svarade på frågor som innehöll identifierings-, klassificerings- och konstruktionsproblem inom området för rationella och reella tal. Eleverna ombads också förklara sina problemlösningar med egna ord och uppskatta svarens säkerhet. Teorier om begreppsförändring (Vosniadou, 1999) och matematisk begreppsutveckling (Sfard, 1991) användes för att klassificera studenters förklaringar. Majoriteten av eleverna hade stora svårigheter med begreppen. Resultatet indikerar att elevernas tidigare föreställningar begränsade deras insikter. Utgående från resultatet hävdar vi att de problem som elever har med de svåra begreppen inte bara förorsakas av begreppens komplexa natur, utan också av kvaliteten på deras tidigare kunskap. Detta har inte tillräckligt beaktats inom den traditionella undervisningen.