Semiotics as an analytic tool for the didactics of mathematics

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This paper is a theoretical analysis of (what the author perceives to be) one of the most exciting and promising directions in research on the didactics of mathematics: studying the learning of mathematics as the initiation to, and internalisation of, certain semiotic systems. Three principal ways in which this point of view can contribute crucially to didactical research are presented and exemplified; they concern the cognitive, social and cultural aspects of mathematics education. Finally, as a topic transcending the three aspects, we consider the use of digital semiotic appliances in mathematics teaching; some results from research in this new area are outlined.

Semiotics[1] is, according to the classic definition, the 'science of signs and their life in society' (de Saussure, 1966/1916, p. 33). A sign is, basically, an asymmetric *relation* between a material *expression* (the signifier) and the *content* to which it is meant to refer (the signified). The intentional nature of the sign (i.e. that the signifier is *meant* to refer) is important because it implies human agency and excludes 'natural signs' (like smoke as a 'signifier' of fire[2]). This means that semiosis – creation of signs – is understood, in semiotics, as an *act of communication*, in which some expression is deliberately used to *represent* some intended meaning. In particular the sign implies a *sender* with *intentions* to indicate something to a *receiver* (who may be identical to the sender, as in simple book-keeping for personal purposes). The fact that signs are not only *intended* but also *interpreted* (potentially, in different ways) is explicitly emphasised in Peirce's triadic model of the sign[3].

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The usefulness of signs stems especially from the fact that they may relate (e.g. refer) to each other, often in highly structured *systems* of signs. The most important of these systems are the natural languages, but the main point of semiotics is not that it includes these (the subject of ordinary linguistics) in its study. The main idea of semiotics is *to study other systems from the same, more global, point of view* (originating in, but not limited to, the study of natural language). Among the systems studied by semioticians are the varieties of scientific, cultural and literary codes, from gestures and cooking to writing and speaking. One may say that semiotics is the study, from a structural linguist point of view, of this broader field of human sign systems.

In the most immediate way, semiotics is of relevance to mathematics education because mathematics itself may be described from the point of view of the sign systems it uses (which include, but are not restricted to, natural language). This point of view is discussed in the first section. And from this basic recognition of the 'semiotic' nature of the subject, we may proceed – for research on the didactics of mathematics – to various levels of inquiry (dealt with in the next three sections):

– the 'cognitive approach' studying the cognitive requirements, implied for the individual learner, to acquire and participate in mathematical activity (in particular, semiosis);

– the 'social approach' of the interaction between mathematical sign systems and the educational environment in which mathematics is taught and learned, dealing in particular with communication functions in learning;

– the 'cultural approach' in which we explore the importance, for learners, of relations and interactions between mathematical and other cultural semiotic systems that form part of the learners' field of experience.

Finally, to illustrate how these three approaches may be combined in a concrete problem area of didactics, we shall consider briefly the semiotic analysis of the use of computer technology in mathematics education.

The objects of mathematics

One need not subscribe to the general form of the Sapir-Whorf thesis, stating that reality is inconceivable without language, in order to acknowledge that some form of it holds for the special case of *mathematical* reality. Duval (1998) explains how mathematicians' and philosophers' views of the relationship between mathematical signs and mathematical objects have developed historically, from assuming a causal primacy of *a priori* objects, as envisaged in different ways e.g. by Descartes and Kant[4], to the view that free signs (or, rather, sign structures) are primary – with classical mathematical objects being merely interpretations or models. Indeed, according to most contemporary views, the objects of mathematics – ellipses, functions, matrices and so on – come into being by *signification* and *abstraction*, and the results (objects) of abstraction are shared (i.e. social) entities as a result of *signification* being shared. Notice that such a view is not at all equivalent to nominalism[5]. In fact, recent versions of philosophical realism[6] (in particular, Resnik 1997) consider the possibility of 'abstraction' from 'template' to 'pattern' as the central motor of creating mathematical objects; roughly speaking, 'the signifier creates the object' (*per se,* and this is *not* identical to its interpretations). From a quite different position, the so-called 'social theories' of mathematical knowledge (some of which are frequently quoted in Anglo-Saxon mathematics education) also affirm this rather general view on the status of mathematical objects: *the objects of mathematics are constituted by mathematical signs in use* (Ernest, 1998, p. 193).

Mathematical signs always appear in both a *local* and a *global* context. The global context may be thought of as roughly speaking the 'area' of mathematics (arithmetic, calculus, homological algebra etc.); formally it is constituted by the 'preceding' corpus of text, that is the background sign structure potentially referred to by the signifier (cf. Duval, 1995, p. 225). This is what makes us recognise, without explanation, the signifieds of the signifiers x and y (as abscissa and ordinate) in a text (based) on Cartesian geometry. However, other signs are determined by the local context, that is, information visible in the text where the sign appears, e.g. the reference of p could be defined by a declaration like 'Let p be a prime number'. Only when we consider the sign within a determined context can it refer to objects (like 'prime numbers'). In that case we say that the signifier *represents* the object.

The most basic observation concerning mathematical representations is that they are, contrary to popular assumptions, *polysemic,* i.e. there is no one-one correspondence between object and representation[7]. For example, in an appropriate global context, all of the signifiers in Figure 1 could represent the same object. The main difference between the polysemic nature of mathematical representations and those of, for example, daily conversation, is that the object indicated by a mathematical representation is determined *entirely* by the discursive context (local and global). So, we can have polysemy in the sense of different representations of the same object, but not in the opposite sense[8].

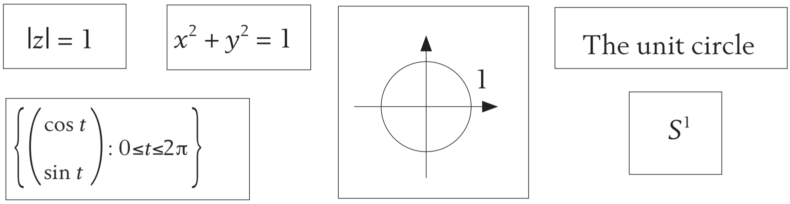

Figure 1.

The signifiers in Figure 1 do not all belong to the same semiotic *register* (as defined in Duval, 1995, p. 21): the three to the left are algebraic symbol strings, one is a Cartesian graph, one is a noun phrase in natural language, and the last one is a conventional symbol derived from algebraic symbolism. Yet going from one of them to another can be viewed as a transformation which preserves the reference to a common object. These changes of representation may either occur *within*, or *between*, registers of representation (processings and conversions, respectively, cf. Duval, 1995, p. 39-44). Notice that the pertinent registers are usually defined or given by the global context. Some processings are almost automatic, like the change between algebraic symbol representations by simple calculation – for instance, the possibility in many contexts to multiply the two sides of an equation by the same non-zero number – while conversion is almost never automatic. Of course, different representations may allow or incite the subject to 'see' different aspects of the object; for instance, the representation S^1 suggests the position of the object in a hierarchy of 'spheres' determined by dimension, while the representation '$|z| = 1$' makes the *radius* (of the circle) the salient characteristic.

At any rate, there are very strict rules – depending on the involved registers *and* on the global context – as to what processings and conversions may be carried out while preserving the *object* represented. We shall call these rules of *object preserving transformations*. Once these are fixed (by the global context and the associated registers) we may think of objects as *representations modulo object preserving transformations* (in symbolic shorthand, Rep./OPT). This gives flesh to the statement, quoted above, that 'the objects of mathematics are constituted by mathematical signs in use': they can, in a given context and with appropriate precaution, be regarded as equivalence classes of representations under object preserving transformations (see Figure 2). Obviously, this is only metaphorically a 'formal definition', but notice that there is no circularity in appealing to

’OPT’ in the definition of ’objects’, as the rules determining the object preserving transformations are defined by the local and global context. So far, almost nothing has said about how individuals relate to mathematical objects, a significant problem both in the philosophy and in the didactics of mathematics. However, it should be clear from the above that the interaction happens at the level of *representations* and so is indirect (cf. Figure 2); we shall return to this point in the next two sections[9].

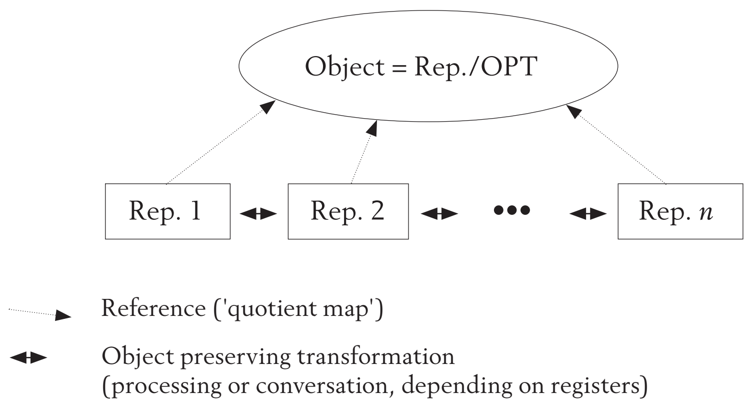

Figure 2.

Before closing this short outline of the semiotics of mathematics, we should notice that the register of *natural language* plays a very special role in almost any mathematical discourse, formal or informal. Just as many mathematical entities and relationships are difficult or impossible to express using only the register of natural language, mathematical discourse *without* natural language is in practice impossible[10]. While mathematical *objects* (including relations and transformations among them) are often signified in other registers, the actual *status* and *role* of such signs are almost always indicated and developed by framing elements in natural language, and by systematic forms of substitution in natural language syntax. Here, ’frames’ and ’signs’ together form the complete texts which are typical of mathematical discourse (cf. Winsløw, 2000a); for instance, in the phrase ’Hence p is a prime number’, the symbol p fills the place of a noun phrase.

The cognitive approach

Since the classical studies of Piaget on the cognitive development of children, including very detailed studies of the development of numerical and logical concepts, the study of *cognitive mechanisms and constraints* pertaining to the individual mathematics learner has been among the central and fundamental areas of research in the didactics of mathematics. It is, in some sense, the 'micro-level' of didactical research, asking: what are the requirements, in terms of cognitive action and resources, for the mathematics learner to succeed? Clearly many other factors are important or even decisive (in particular, I shall touch upon social and cultural ones in the following sections); but Piaget and his followers have demonstrated quite clearly the pertinence of the cognitive approach for understanding some very specific and fundamental *conditions* for the elementary practices of mathematics, which – and this is essential – are *common* for *all* learners, as they derive from common features of human cognition.

There is no need to repeat here the classical framework of conceptual schemata and the cognitive processes involved in constructing them. What is of interest here is the specific ways in which mathematical signs serve as *tools or prostheses of the mind to accomplish actions as required by the contextual activities in which the individual engages* (Radford, 2000, p. 241); and in particular how *the mediation between mathematical signs and reference objects is dependent on the activities* (Steinbring, 2002, p. 9). At the cognitive level, two related questions arise:

(1) How does the individual manage to build conceptual schemata corresponding to mathematical objects (in the sense of Figure 2), having access only to some of the underlying semiotic representations?

(2) At a more fundamental level, what allows the learner to conceive of the context (as 'meaningful') and hence to see signs as representations in it?

As for question (2), it is interesting to compare the 'epistemological triangle' of *reference context*, *sign* and *concept*, explained in Steinbring (2002), with the three notions of (*mathematical*) *context*, (in French: *cadre*), *register* and *conception* discussed and interrelated in Balacheff (2002). They are clearly parallel, yet in Steinbring there is (due to the triadic relationship envisaged) an apparent possibility of *direct* interaction between reference context and concept, while for Balacheff, registers serve solely as *mediators* between context and conception, with learning being the process of the latter converging to the former, as the subject interacts with a didactical *milieu*. However, the difference seems to arise from what Steinbring calls the ”exchangeability” of sign and reference context, which arises temporarily as the signs themselves ”embody” structures and relations, as in the development of the number concept. While this may partially ruin the symmetry implicit in a triadic model (as there is certainly no possibility of interchange sign/concept or concept/context) it is certainly consistent with Balacheff's (2002, p. 9) concession that *the force of representations is sometimes so great that ... they may seem to replace the context or to merge with it* (my translation). And especially in the absence of a context which (to the learner) is well established, how could it be otherwise?

Hence, one possible answer to how contexts are established for representations to make sense is, paradoxically, that the representations may partially *create* the context – not by themselves, but through the individual's experiences in activities where they are present. This alchemy of 'symbolizing context into being'[11] is what is at stake for the learner being introduced to 'new numbers' in the form of symbols (like fractions or imaginary unit): the context is, at least formally, mainly established by new symbols which do not refer in any existing context.

But even in these special cases where no appropriate context exists for new representations to be introduced, it is only a partially correct answer. Whether or not a context is available, it is common and may indeed be highly conducive to comprehension if *more than one representation is given and claimed to 'refer' to the same object*. For instance, simple fractional symbols like ½ and ¾ will typically be introduced together with geometric representations ('cake diagrams') suggesting their share of a unit. More general fractions (not corresponding to parts of a unit) may then be introduced in a second appeal to analogy, now at the level of fraction symbols and their operations. The advantage of this common practice of establishing a new context by simple analogy is indirectly demonstrated by the 'example of Dan', presented by Sfard (2000). In a series of interviews, the author introduced Dan, a gifted student, to working with ordered pairs of integers, modulo an equivalence relation, and equipped with certain operations. Although this was not said to Dan, the mathematical set-up was that used in defining the rationals as a field of fractions over the integers. A main point of this experiment is the difficulty, even for a very gifted student, to make sense of (and eventually recognise) the context of rational numbers, when these are given in the abstract symbolic form – that is, with no familiar analogies present. In fact, the practice of using naïve analogy to establish just some 'touch' of context for unfamiliar representations is not only of importance at the elementary levels. For instance, two-dimensional diagrams are frequently used as illustrations in introductions to metric space. Even professional mathematicians often have recourse to naïve 'pictures' (Sfard, 1994).

This means that the *use* of representations against a partially flawed context may be not only a tolerable exception in the process of learning of mathematics, but that it is a rather common and unavoidable condition to make sense of new representations, even elementary ones like those of rational numbers. In some sense this represents a cognitive requirement that is far from the public image of mathematics: you need to jump without seeing the ground first. However, refusing to do so (e.g. relying entirely on an axiomatic development) has even higher costs. Of course, for the interiorisation of the register of representation, some practice with *processing* may also be needed; in particular to establish rules of object preserving processings. In the example of rational numbers, these would include rules of cancellation of common factors in nominator and denominator, which (in the simple cases) may still be supported by the diagrammatic analogy. Notice that being able to handle fractional symbolism (or other systems of representations) with reasonable confidence does not in itself guarantee that a working conception of *rational numbers* has been formed! And it certainly does not guarantee that the 'fraction symbols' are related to other relevant registers, like that of decimal numbers.

The establishment of a context sufficient for operating with pertinent representations – or, the interiorisation of pertinent semiotic *registers* – is, indeed, a central condition for mathematics learning. Even if many learning situations take place in a context already established – using semiotic registers previously used by the students – any learning situation will *develop* the context (making it richer, adding to familiarity with registers, and so on).

Let us now consider question (1) above: what are the cognitive conditions for learners to develop appropriate schemata of conceptual objects and their relations from representations of them? Duval (e.g., 2000) has repeatedly emphasised the necessity of more than one representation as a condition for conceiving of mathematical objects. Notice that, as we just saw, this need for interaction among representations is already present in the way *new* registers of representations are interiorised using the contexts provided by informal analogy[12]. But even when the pertinent registers of representation are available, the possibility of changing between them remains an important operational condition for the conception and handling of mathematical objects. Duval identifies the *coordination* of representations of objects *in different semiotic registers* as a crucial condition for obtaining a functional conceptual schema of the object that is not based on one particular representation. It is clearly necessary, as the failure of learning in the presence of just one (by necessity, isolated) register of representation[13] shows: having just one 'signifier' and, in a sense, no signified, how could the learner avoid the trap of identifying the object with the signifier? The condition is, of course, not sufficient in the strict sense; the mathematical context does more to determine its object than merely providing a frame of reference for single representations. However, if one does not focus exclusively on the individual mathematical object – after all, mathematical concepts are not learned 'one by one' but as coherent patterns or structures – the ability to coordinate different representations is decisive because *each representation entails its own semantic qualities and relations* (cf. the different meanings and relations of 'circle' contained in the representations in Figure 1). It is the ability to access and combine these different qualities which enables the individual to conceive of the objects, not only (if at all) *per se*, but as part of conceptual structures.

The social approach

The main point of the preceding section is to formulate a set of *conditions* for the individual's understanding of mathematics, related to the availability and coordination of informal and formal semiotic representations. From a didactical viewpoint, this may locate the problem but not solve it. For by what means can the individual be motivated and enabled to acquire such mastery of semiotic representations and the conceptual landscape based on them? Or, more broadly: what are the forms of mathematical semiosis occurring in educational contexts and how are they to be analysed? Clearly, no satisfactory answer to these questions can be given without taking into account the social context of learning. Indeed, communication in mathematics classrooms is much more than the formal, written code exemplified in Figure 1; it may include informal oral dialogue and instructions, various forms of 'body language' such as pointing, nodding, etc., as well as plane or three dimensional models and other artefacts. These forms of communication all contribute to the students' construction of mathematical signs. And, even if they may often refer to formal mathematical signs (such as pointing to geometrical figure), they are only meaningful and conceivable when understood within the social context.

Of course, concepts are still constructed by the individual, even if this is made possible by the interaction with others. In the conclusions of a study where he analyses the signification of students working on a problem on number patterns, Radford (2001, p. 260) writes: *we see the emergence of algebraic thinking as resulting from the encounter between individual's subjectivity and the social means of semiotic objectification*. At a very general level, this corresponds of course to the classical focus of didactics on the *interaction* between a student and a socio-mathematical environment or *milieu*, with conceptions emerging from this interaction (see e.g. Balacheff, 2002 for references). However, a much finer analysis of the 'social means of semiotic objectification' is possible from careful analysis of the discourse of students, including non-verbal parts.

In fact, the *theoretical necessity* of considering signs as representations within a partial or incomplete context, discussed in the previous section, can only team up with *pragmatic possibility* due to very delicate forms of social interaction, the common point of which is the *tacit consensus to equate different levels of signification*. The simplest forms are *synecdochical signification*, as when using a particular signifier (e.g. a triangle) to indicate something more general (e.g. 'any triangle'), and *metonymies*, e.g. representing an infinite dimensional space by a plane figure. Another form, which is so common in non-written mathematical communication that the associated conventions may easily be overlooked, is the use of deictic signs alone (e.g. using the right hand to explain the direction of the crossed product of spatial vectors) or in combination with written signs (such as pointing at them). Often special 'ad hoc' codes are developed by students to annotate or otherwise keep track of discourse pertaining to written signs (cf. Steinbring, 2002, p. 16f).

But there are other, more subtle forms. In the study just mentioned, Radford exemplifies what he calls a *generating action function* in the student dialogue considered. This function consists of *linguistic mechanisms expressing an action whose particularity is that of being repeatedly undertaken in thought* (opus cit., p. 248), such as saying "then you just keep adding up the terms". The importance of this function lies in its potential of generating an abstract or 'general' object from referring to a particular instance – which may, in elementary settings, replace formal procedures like mathematical induction, limit operations etc. In fact, as explained by Rotman (1988), similar functions (related to imaginary agents) are crucial to certain forms of objectification even in research mathematics; and whether the imaginary action is left, by the conventions of written mathematics, to an imaginary "Agent"[14], or (as in student discourse) by appeal to what 'you' or 'one' could do, the object comes into being by considering *potential* or *partial* signification at the same level as *'real'* or *complete*. This is possible only by (tacit) social consensus.

A more classical discourse analytic approach, drawing on Halliday's theory of language as social semiotics, is presented in Morgan (2001). In a systematic analysis of protocols of student dialogue, exemplified in the paper, three main functions are argued to be crucial in order to describe interpersonal aspects of constructing meaning in and of mathematical activity in the educational setting: *ideational functions*, concerning choices that determine the context of activity; *interpersonal functions*, establishing the roles, status and identity of participating agents; and *textual functions*, realising the mode of discourse (narrative, giving instructions, reasoning etc.). In particular, the study of the first function – including reflecting on possible alternatives – is argued to provide a more direct way of assessing the nature and construction of students' beliefs and images regarding mathematics than more traditional methods of belief research like interviews or questionnaires.

There is no doubt that the proliferation of metaphorical and otherwise indirect signification in mathematics classrooms is both unavoidable and at the root of the alienation of many students with respect to what happens there. There are students in every classroom to whom the teachers' and the other students' talk of mathematical objects makes little or no sense, and to whom 'the vicious circle of reification' (Sfard, 1991) makes mathematics education an increasingly painful experience. A semiotic analysis of these subtle forms of shared objectification – and of students' difficulties with them – will certainly not in itself provide a cure for these problems. But it may at least improve our understanding of those complex patterns of socially situated signification that are crucial to the individuals' formation of mathematical concepts – both in the sense of identifying them (among a host of irrelevant significations) and of analysing their role in learning.

The cultural approach

Besides its application to analyse the patterns of signification employed in science and technology, semiotics is widely (and indeed mainly) used to study 'cultural' codes, such as indigenous crafts and rituals, workplace etiquette, novels or *haute couture*[15]. The relevance of such studies to mathematics education – and indeed, a main advantage of the semiotic approach to the didactics of mathematics – is due to the fact that mathematical and other forms of signification interact in multiple and complex forms in the educational context, as well as in society at large, and that semiotics offers a privileged unified viewpoint for analysing this interaction and its impact and importance for learning mathematics. This may be done in two complementary, but in practice rather different ways.

The first, which is manifestly the most developed, departs from the view that mathematics itself is intertwined with the rest of human culture, as a kind of 'subculture(s)'. From the last half of the 20th century, we have a number of anthropological and socio-historical studies of mathematics as a cultural phenomenon, such as the works of Wilder, Bishop, ethnomathematics, and others (see Radford, 2001 for an overview). Inspired by Foucault's epistemic archaeology', we may draw on these while searching 'the supporting semiotic configurations, the nature of the symbol and the representations it allows to form' (ibid., p. 293). It means that signs are seen as tools used (and, indeed, required) in culturally situated activities of 'meaning making'. In particular, mathematical sign systems are developed and cultivated in certain historical and societal contexts with certain (often quite practical) purposes. Sfard (2000) distinguishes the discourses of 'actual' and 'virtual' reality, and notices that *if a sign exists and functions in both AR and VR discourses, it is likely to be meaningful in the latter mainly due to the perceptual-world connotations it brings with it from the former* (p. 87). Although a main point of her paper is to show that such a possibility of 'meaning transfer' is not available even in quite elementary contexts, she also claims that *for many people a signifier ... must find its place in AR discourse in order to be meaningful* (p. 86). Whether or not this is really so, no one would disagree that real world interpretations are often extremely helpful (cf. our discussion of informal analogy above).

Indeed, in education, especially at elementary levels, it is a common strategy to strive for a kind of continuity between the students' world of experience and the problem settings that motivate the introduction of mathematical forms of representation. Here, chains of signs (each signifying the preceding one) may be argued to provide 'bridges' between cultural practice and abstract mathematics. For example, Presmeg (1997) develops such a 'bridge' between a certain structure of tribal kinship relations, and the dihedral group D_4. The idea of making such (rather surprising) connections explicit is to point to potential ways of coherent semiosis which might 'facilitate students' constructions of mathematical ideas' (ibid., p. 7). Of course, the use of daily life problems provides less surprising, but probably more common ways in which this idea may be used as a tool for didactical design and analysis. In any case, the point is to achieve semiotic mediation between familiar and unfamiliar (target mathematical) cultural practices[16].

Two key-words in this first 'cultural' approach are *situatedness* and *diachrony*. The contingency of mathematical signification, in terms of *activity*, *purpose* and *historical development*, is emphasised. Often, the boundaries between mathematical signs and other forms of signification are open to discussion (cf. Presmeg, 1997, p. 5).

A second form of analysis is *synchronic* in nature and regards mathematical and other sign systems as *parallel* (interacting but definitely distinct). It could be regarded as merely an alternative viewpoint in didactical analysis, which is certainly consistent with the experience of many students that 'mathematics forms a world apart' (although this situation may, conceivably, be due to a lack of displaying continuities in teaching, as discussed above). But it is the *only* option if (or at those levels where) mathematical signification is essentially different, or 'discontinuous', from other forms of signification. The existence of such discontinuities has been argued in great detail by Duval (1995), both at the level of conditions to establish 'meaning' or 'objects' in mathematics, and at the level of mathematical discourse. In particular, mathematical representations acquire meaning only 'in groups' (using several registers, cf. 'The objects of mathematics' above, and Duval, 1995, Chapter I) and mathematical reasoning achieves its conclusion in virtue of its structural rather than its thematic coherence (Duval, 1995, Chapter V).

An example of this second type of analysis is due to Emori and Winsløw (2002), where the analogies and interactions between cultural and mathematical codes are investigated in the context of the secondary mathematics classroom in Japan. The study draws explicitly on Barthes' (1970) semiotic essay on certain aspects of traditional Japanese culture. In particular, the paper analyses the affinity between certain 'traditional arts' (e.g. flower arrangement or calligraphy) and the conceptions of mathematics implicit in various aspects of semiosis enacted in the classroom, as well as the socio-cultural (guild-like) structures surrounding and constituted by those signs17. In this context, mathematics is clearly a separate semiotic domain (of Western origin), and what is studied is the similarity of the codes surrounding 'enculturation' or initiation to this domain and to the traditional art, as they are enacted in a common socio-cultural context. One point of such an analysis is to provide explanatory models (rather than merely evidence) for the manifest differences in mathematical performance of students in East-Asian countries and in the West, which have been noticed in recent international comparisons like TIMSS and PISA. Another point is to propose a framework in which the interaction of scientific and cultural paradigms may be studied systematically, as an alternative or complement to more classical methods of anthropology as mentioned above.

The use of semiotic appliances

The scope of semiotic methods in didactical analysis seems, indeed, to be quite wide, and the examples considered in this survey reveal only a small part of the potential. At the end of this paper, let me point to an area which, as suggested below, extends across the three aspects considered above, and where semiotic analysis is, in my view, particularly promising (see Winsløw, 2003b for a more comprehensive treatment). This is the didactical problems and potentials arising from the increasingly common use, in mathematics teaching, of programs meant to facilitate the production and treatment of mathematical representations, including professional software (e.g. *TeX*, *Maple*) and educational software (e.g. *Cabri*). We shall refer to such programs as *semiotic appliances*[18].

Given that the conception and handling of mathematical objects depend crucially on the availability and coordination of *representations*, it is clear that semiotic appliances may have quite fundamental potentials as supports for individual student learning. But the facility they often provide, especially for the treatment of semiotic representations, may also lead to substantial didactical problems, at least if they are not explicitly considered and controlled by teachers. Some of these issues, in the context of computer algebra systems (CAS) in university teaching, are discussed in (Winsløw, 2003a), using mainly the cognitive approach outlined above. In this setting, the potential – and pitfalls – of using CAS are related to their capability of *transforming signs* (mostly within the same register, but sometimes by conversion from one register into another).

As semiosis is fundamentally a communicative act, and semiotic appliances in various ways support collaborative activities, their introduction in teaching may also affect the patterns and potentials of social interaction in didactical situations, including interaction transmitted via the internet. Many semiotic appliances can act simultaneously as a *medium* (for communication with others), as a *tool* (for treatment) and as a kind of automatic *agent* (reacting[19] to mathematical signification, sometimes requiring specially encoded commands, by transforming input to output). For discussions of these issues in various contexts, and from viewpoints related to those presented in this article, see Balacheff (2002), Misfeldt (2003) and Winsløw (2000b).

One particular circle of research questions have to do with pattern changes in classroom discourse that are caused by the participants' use of semiotic appliances. A very simple example would be a dialogue where one of the participants makes use of a CAS, and both may see the inputs and outputs. In Winsløw (2003b), I have shown how to use the method of 'dynamic semiotics' developed by Andersen (2002) to analyse such situations. The general idea is to represent the dialogue in three horizontal bands, where the middle band represent the (shared) signifiers, spoken or written, and the two other represent the level of signifieds of the participants (this, of course, must be reconstructed from the dialogue). Then, at crucial points, we see a shift of focus in the dialogue from references to previous contributions by the other party, to interpretations of the transformations effected by the computer.

Finally, computers and software packages may, in themselves, be regarded as a part of our cultural environment which are shaped by, but (at least classically) are not a part of the 'culture' of academic or educational mathematics. The codes and conventions of computer environments are clearly influencing many aspects of human life. How do they change and shape the cultural assets of mathematics classrooms? For instance, how do they affect the criteria or norms of what is considered pertinent problems or valuable methods of solution? Can computers help to bring about 'bridges' between AR and VR discourses[20]; and, perhaps, shorten the distance from the educational to professional (pure, applied) mathematical activity?

References

Andersen, P. B. (2002). Dynamic semiotics. *Semiotica*, 139 (1/4), 161-210.

Balacheff, N. (2002). Cadre, registre et conception. Note sur les relations entre trois concepts clés de la didactique. *Cahiers du lab. Leibniz-IMAG*, 58.

Barthes, R. (1970). *L'Empire des Signes*. Genève: Editions d'Art Albert Skira.

De Saussure, F. (1966, 1916). *Course in general linguistics*. New York: McGraw Hill.

Duval, R. (1995). *Semiosis et pensée humaine. Registres sémiotiques et apprentissages intellectuels*. Bern: Peter Lang.

Duval, R. (1998). Signe et objet (I): Trois grandes étapes dans la problématique des rapports entre représentation et objet. *Annales de Didactique et de sciences cognitives*, 6, 139-163.

Duval, R. (2000). Basic issues for research in mathematics education. In T. Nakahara et al. (Eds.), *Proceedings of the 24th conference of the international group for the Psychology of Mathematics Education, vol. 1* (pp. 55-69). Hiroshima University.

Emori, H. & Winsløw, C. (2002). Elements of a semiotic analysis of the secondary level classroom in Japan. In *Pre-conference proceedings of ICMI comparative study conference* (pp. 87-96). University of Hong Kong. (An elaborated version will appear in the 13th ICMI study)

Ernest, P. (1998). *Social constructivism as a philosophy of mathematics*. New York: SUNY Press.

Guiraud, P. (1971). *La sémiologie* (Collection 'Que sais je' no. 1421). Paris: PUF. [English translation, '*Semiology*', published by Routledge, 1975, several reprints]

Hirabayashi, I. (2002). A traditional aspect of mathematics education in Japan: mathematics as gei (art), its jutsu (technique) and do (way). In *Pre-conference proceedings of ICMI comparative study conference*. University of Hong Kong.

Misfeldt, M. (2003). Mathematicians' writing. In Pateman et. al. (Eds.), *Proceedings of the 2003 joint meeting of PME and PMENA, vol. 3* (pp. 301-308). University of Hawaii.

Morgan, C. (2001, June). *What does social semiotics have to offer mathematics education research?* Paper presented at PME-26 in the discussion group on 'semiotics in mathematics education', Norwich.

Presmeg, N. (1997, October). *A semiotic framework for linking cultural practice and classroom mathematics*. Paper presented at the annual meeting of the North American chapter of the international group for the Psychology of Mathematics Education, Bloomington/Normal, IL.

Radford, L. (2000). Signs and meanings in students' emergent algebraic thinking: a semiotic analysis. *Educational Studies in Mathematics*, 42, 237-268.

Radford, L. (2001). Sur les modes du savoir. In P. Radelet-de Grave (Ed.), *Histoire et épistémologie dans l'éducation mathématique de la maternelle à l'université*. Université Catholique de Louvain.

Resnik, M. (1997). *Mathematics as a science of patterns*. Oxford: Clarendon Press.

Rotman, B. (1988). Toward a semiotics of mathematics. *Semiotica*, 72, 1-35.

Sfard, A.(1991). On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, 22, 1-36.

Sfard, A. (1994). Reification as the birth of metaphor. *For the learning of mathematics*, 14(1), 44-55.

Sfard, A. (2000). Symbolizing mathematical reality into being: How mathematical discourse and mathematical objects create each other. In P. Cobb et al. (Eds.), *Symbolizing and communicating in mathematics classrooms: perspectives on mathematical discourse, tools, and instructional design*. Mahwah: Lawrence Erlbaum.

Steinbring, H. (2002, June). *What makes a sign a mathematical sign? An epistemological perspective on mathematical interaction*. Paper presented at PME-26 in the discussion group on 'semiotics in mathematics education', Norwich.

Winsløw, C. (2000a). Transformations in mathematical discourse. In B. Barton (ed.), *Communication and language in mathematics education, the pre-conference publication of WGA 9 at the 9th International Congress for Mathematical Education* (pp. 93-102). University of Auckland.

Winsløw, C. (2000b). Linguistic aspects of computer algebra systems in higher mathematics education. In T. Nakahara et al. (Eds.), *Proceedings of the 24th conference of the international group for the Psychology of Mathematics Education, vol. 4* (pp. 281-288). Hiroshima University.

Winsløw, C. (2003a). Semiotic and discursive variables in CAS-based didactical engineering. *Educational Studies in Mathematics*, 52, 271-288.

Winsløw, C. (2003b, September). *L'usage des logiciels dans l'enseignement supérieur des mathématiques : un panorama des question du point de vue sémiotique.* Paper presented at an invited symposium of Réseau Éducation Formation, Genève. [a book collecting the papers from this symposium is under edition]

Notes.

1 In most of the 20th century, the European tradition, going back to Saussure's seminal work, was referred to as 'semiology', in contrast to 'semiotics', the American school emerging from the writings of Peirce (cf. note 3). However, as these perspectives have in many ways merged and mingled, it seems increasingly common to refer to the entire field as 'semiotics'. The reader is referred to Guiraud (1971) for a classical and very readable introduction.

2 Of course, natural signs and their interaction with (human) signs are of relevance in many other fields, such as the didactics of science.

3 Including, besides 'sign' and 'object', the 'interpretant'. Some authors, including Steinbring (2002), argue the merits of triadic models in relation to education. In this paper, the Peircian perspective is not emphasised, unlike what is the case in several mainstream German and American approaches.

4 Indeed, some form of this assumption has been held by most philosophers since Plato and Aristotle, the main exception being medieval nominalists.

5 Nominalism in general rejects the existence of abstract entities beyond their "name"; they are nothing but signifiers. For instance, there are no "numbers" beyond the symbols used; calculating is just a game with the symbols.

6 Realism, contrary to nominalism, holds that abstract entities exist, e.g. in the sense that they just are as "real" as physical objects.

7 Even Guiraud (1971) expresses this belief, at least if mathematical 'language forms' and 'codes' are included among 'scientific' ones. In fact, it is the controlled form of polysemy illustrated in Figure 2 here, which is at the root of mathematical signification and its power.

8 Here, *different* should not be confused with *several*. Clearly a 'variable symbol', like x, may refer to all numbers within some set, but this would be determined by the context; it would not be open if, for instance, it also represented a relation or a property. (This is also the most reasonable interpretation of Poincaré's famous statement that mathematics 'calls different things by the same name', as quoted in Sfard, 2000, note 17.)

9 Notice also that a delightful Peircian analysis of the 'agencies' involved in activity of the research mathematician was given in Rotmann (1988); it pertains exactly to the delicate relationship between subject and object, but considered from a viewpoint quite different from the one taken here.

10 The theoretic possibility of developing, for instance, the calculus in formal languages, does not seem at all pertinent to the didactics of mathematics; however, the *degree* to which logical structure is 'packed' in informal natural language equivalents *is*, of course, a variable of didactical interest.

11 This expression and the first example are derived from Sfard (2002).

12 In fact, as *all representations are cognitively partial with respect to what they represent* (Duval, 1995, p. 69) it may not be meaningful or necessary to maintain a sharp line between 'informal analogical representations' and 'formal' mathematical representations. However, I find it meaningful to distinguish the function of coordination in *acquiring new registers* (coordination formal–informal) and in *acquiring new concepts* (mainly coordination formal–formal) – even if the two may often go hand in hand.

13 Cf., for instance, the 'story of Dan' from Sfard (2002), as cited above.

14 In Rotmans article, the "Agent" is the implicit addressee of imperative utterances in mathematical texts such as "Suppose x is negative", "Sum this up to infinity, and notice that ..." etc.

15 In fact, the *vast majority* of semiotic studies concern cultural sign systems (in the terminology of Guiraud (1971), aesthetic and social codes).

16 Cf. also the notion of 'transitional language approach' by Radford (2000, sec. 1).

17 A similar affinity was independently pointed out by Hirabayashi (2002), although not from a semiotic point of view.

18 One might qualify this term, taken from Winsløw (2003), by an adjective like 'soft' in order to distinguish programs from 'hard' appliances like the abacus, a compass, or the computer itself. We prefer, at least here, to refer to the latter simply as 'media'. Of course, the medium function of the computer usually requires a semiotic appliance, and so strictly speaking it is the computer *together with* the appliance that functions as a medium, as mentioned later in the text.

19 Of course, in algorithmic and hence (in principle) predictable ways.

20 This is strongly suggested, for instance, by Sfard (2000): *It seems quite plausible that great parts of mathematical reality, which until now could only be imagined, will soon materialize on the computer screen* (p. 94).

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Carl Winsløw's area of research is the didactics of mathematics, and much of his research is drawing on methods from linguistics and semiotics. His background includes studies in French, general linguistics and mathematics. His doctoral thesis and his first 12 papers concerned pure mathematics. From about 1996 he then returned to linguistics but now applied to the epistemology and didactics of mathematics. In the latter field he has worked on comparative international studies, studies of ICT-tools in university education, and curriculum theoretical questions.

Since 2003, Carl Winsløw holds the first professor position in didactics at the University of Copenhagen, where parts of his work concern development of the teaching at the Faculty of Science. Some of this developmental work is closely linked to his research.

Sammanfattning

Denne artikel præsenterer en teoretisk analyse af (hvad forfatteren finder er) en af de mest udfordrende og lovende retninger i matematikdidaktisk forskning, nemlig studiet af matematiklæring som indføring i, og internalisering af, visse semiotiske systemer. Der præsenteres og eksemplificeres tre hovedområder hvor dette synspunkt kan bidrage afgørende til didaktisk forskning: kognitive, sociale og kulturelle aspekter af matematikundervisning. Afsluttende betragtes, som tværgående eksempel, forskning i brug af computeren som "semiotisk apparat" i matematikundervisning, hvor alle tre aspekter indgår eller kunne indgå.