Quadratics in Japanese

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Starting from a description of two lessons on quadratic equations in a junior high school of Tokyo, this paper attempts to throw new light on the principles and philosophies underlying secondary mathematics teaching in Japan. In particular, the paper concludes by discussing their relation to constructivism and structuralism in the Western sense. To put this description in perspective, some examples of analogous Danish conditions are mentioned.

1 Introduction

Following the appearance of international quantitative studies of mathematics achievement (see Robitaille et al., 1994; Beaton et al., 1996), there has been an increasing interest in East Asian ideas and practices in mathematics education. Gradually, it appeared that East Asian students – especially those from Singapore, Korea and Japan – tend to be superior to Western students at the same level. This applies not only to the context of solving routine tasks based on "traditional skills", but also to more subtle and advanced forms of mathematical competency and practice (loosely speaking substantial non-routine problem solving in pure as well as applied contexts) which are often emphasised as the main learning goals of modern mathematics instruction in the West. If the difference had been restricted to routine skills that have been deliberately given lower priority in many Western countries in recent years, there would be little reason for surprise; we would simply be seeing some reasonable consequences of different priorities. We could have kept with the comfortable standard image of Asian mathematics teaching as based on the traditional discipline-training-test schema which has

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since long been abandoned in the modernised West. But because of the above-mentioned outcomes of the quantitative surveys, and not least the qualitative follow-up studies like (Stigler et al., 1999), there is no such easy comfort. Of course, we must keep in mind that teaching methods and priorities are only one of the factors that influences the effectiveness of student learning. The impact of cultural factors – such as the high value put on academic education by society at large, and not least by parents – is a well-known supplementary model of explanation that is clearly important[1]. The work of Stevenson and Stigler (1992) directly addresses these issues. This paper aims to point out some more philosophical aspects of teaching principles and methods found in Japan.

My own interest in the area originates from my stay in Japan as a Ph.D.-student in mathematics (1992-1994) [2]. In the following years, I began research on mathematics education. Then, in the summer of 2000, I went back to Japan to study secondary level mathematics teaching. In particular, I observed four different classes of junior and senior high school over three weeks, with the basic aim of getting better knowledge of the discursive patterns of mathematics classrooms at this level.

This paper is centred on two lessons observed in a third grade class of junior high school (ninth year of compulsory schooling), where the topic was an introduction to *quadratic equations and their solutions.* I have chosen this particular piece of evidence for its capability of clearly illustrating some aspects of Japanese classroom culture that, in my opinion, are both characteristic and illuminating as regards the overall educational philosophies underlying Japanese mathematics teaching at this level.

2 Two lessons on quadratics.

The observations resumed in this section took place on July 3 and July 5, 2000 in the class 3M (third grade) of the Junior High School attached to Ochanomizu Women's University in central Tokyo. Although the university admits only female students, the junior high school is mixed. This class had about 30 students, with about the same numbers of boys and girls. The students sit at individual tables, all facing the blackboard and the teacher's desk[3].

Lesson 1. July 3, 2000, 13.10-13.50.

The class is very noisy as the teacher enters. No initial "salutation", the teacher initially talks through the noise. While she starts writing on the board, students take out their notebooks and the class becomes quieter. As introduction, the teacher writes the quadratic equation $x^2+2x-8=0$, then the factorisation $x^2+2x-8=(x+4)(x-2)$, and the conclusion that the equation has solutions -4 and 2. No explanation is given as to how the factorisation is obtained, and as the teacher explained later, this is meant as an 'appetiser', something to 'wonder about' (and, we might add, to create attention).

The teacher then asks (writing the question on the board): "How do we solve the equation $x^2-3=0$? Any ideas?" No-one reacts, the class is silent. The teacher reformulates the equation on the board to: $x^2=3$. "Do you remember how to solve this?"[4] One student suggests the answer $\sqrt{3}$, the teacher writes this suggestion on the board, and then explains why the full answer is $\pm\sqrt{3}$ (and what the "plus-minus" sign means). She then writes the following equation: $9x^2-2=0$, and asks if someone will solve it on the blackboard. No one volunteers. A male student is pointed out by the teacher, goes to the board and writes the following equation below the initial one: $3x=\pm\sqrt{2}$, and below this he writes $x=\frac{\pm\sqrt{2}}{3}$ Without having said a word, he is about to return to his place, as the teacher prompts him: "It is correct. Can you put some words on that to explain what you did?" The boy mumbles "No" and returns to his seat. As no one in the class seems to be able or willing to provide an explanation, the teacher then writes very explicit clarifications of each step (next to the equations, with another colour of chalk, and inside "cloudy bubbles" as used in cartoons to indicate a person's thoughts). She then presents an alternative set of equations leading to the result: $9x^2=2$; $x^2=\frac{2}{9}$; $x=\pm\frac{\sqrt{2}}{3}$. This is also equipped with explaining "bubbles", and the teacher returns to the meaning of the "plus-minus" sign when explaining that $x=\frac{\pm\sqrt{2}}{3}$ and $x=\pm\frac{\sqrt{2}}{3}$ are really equivalent.

We are now 10 minutes into the lesson. The next task is set by the teacher and written on the board: solve $(x-3)^2=16$. The male student from before is leaning over the table in ostensive "sleeping position"[5] – the rest of the class is attentive and taking notes. A girl suggests that this reduces to $x-3=4$. Another girl raises her hand and says: "No, it should be ± 4 there." The teacher invites her to explain, and writes on the board the following symbolic outline of what the student says (one equation per line): "$A=x-3$; $A^2=16$; $A=\pm 4$; $x-3=\pm 4$." The student then goes on saying that the last equation corresponds to the two cases $x-3=4$ and $x-3=-4$, hence to $x=7$ and $x=-1$. The teacher returns to the solution of the first student ($x-3=4$) which is still on the board, explaining why it is not complete. She then goes back to the equations which were taken from the suggestions of the second student, and adds short verbal explanations, such as "We define" ($A=x-3$), "then" ($A^2=16$), etc.

At this point, after 16 minutes of whole-class teaching, the teacher asks the students to go to page 64 in their textbook and solve exercise 1–2 there (containing a total of 7 tasks of the type considered before, in addition to one already solved on the board). She clears the board and divides it in seven numbered fields, where students are expected to write their solutions (see below). Then she comes down to the class to offer help to the students, who work one by one at their desks. Only a few students ask for help. After 6 minutes of working silence, the teacher asks for volunteers to present their solutions. Gradually, as the students finish and volunteer, a total of seven students are selected and, more or less simultaneously, write their solutions in the numbered fields on the blackboard; this takes a few minutes, during which the class is somewhat noisy. Then the teacher goes through the solutions – all essentially correct – while adding necessary comments and clarifications with red chalk. The students (except a few "sleepers") listen and take note of these comments, as the teacher stresses the importance of writing solutions "the right way". She commits a small mistake with one of the solutions, which – although it is immediately spotted by herself – she seems to find quite embarrassing.

As 35 minutes have now passed, 5 minutes are left. The board is cleared again, and the teacher poses the following problem: "Now, can you solve this equation: $x^2+6x-1=0$?" After a while, several students reply that they cannot. The teacher then resumes the work so far thus: "We can now solve equations like $(x+p)^2=q$ by using square roots like before. Think about that…" Then, goes on: "Consider the following. If we have (*writes, with the dots being actually underlining in different colours on the board*): $x^2+2x\ldots=(x+\ldots)^2$, then what is missing?" After a few seconds, a boy raises his hand and responds: "It should be 1." The teacher adds two 1's and explains why this is right. Then, the cases $x^2+4x\ldots=(x+\ldots)^2$ and $x^2+6x\ldots=(x+\ldots)^2$ are dealt with the same way. The teacher points out that what is added on the left hand side is always the square of the half of the coefficient of x (9 in the latter case). As the bell announces the end of the lesson, she point to the last expression on the board – which now reads, "$x^2+6x+9=(x+3)^2$" – and says: "Now, think about how this may be used to solve the original equation" (points at $x^2+6x-1=0$). The class is dismissed without further formalities.

During a short conversation after the lesson, the teacher told me that, in her preparation, she had counted on at least finishing this example, but that the treatment of student blackboard solutions took longer time than she had expected, in part due to her "mistake".

Lesson 2. July 5, 2000, 10.50-11.40.

Two days after the preceding lesson, the teacher goes to the board and says “Let’s begin”. The students settle down, but there is still quite some unrest for the first few minutes. She then says: “Last time, we solved the following kinds of equations by using square roots (*writes*): $x^2=q$; $(x+p)^2=q$.” She reminds the class of the unsettled equation, $x^2+6x-1=0$. She then writes down the formula $(x+p)^2=x^2+2px+p^2$, which the students studied some weeks before, and poses the following problem for the class: fill out the missing parts of $(x+\ldots)^2=x^2+10x+\ldots$ Immediately, a student says that 5 and 25 are the missing numbers. The teacher insists that this must be *explained*. She develops, using very suggestive writing on the board (around the general formula as well as the previous example) how the missing parts are filled out first by inserting half the right hand coefficient of x in the parenthesis on the left, then taking the square of this number as the missing element on the right. She then goes back to the problem $x^2+6x-1=0$, and says: “We will now see how this can be solved”. All students, including the usual sleepers, are attentive during the following explanation, centred on the transformations necessary to produce, successively: $x^2+6x-1=0$; $x^2+6x=1$; $x^2+6x+9=1+9$ (*here, particularly, is written: ’half of 6 is 3, and the square of 3 is 9*) ; $(x+3)^2=10$; $x+3=\pm\sqrt{10}$; $x+3=\sqrt{10}$ or $x+3=-\sqrt{10}$; $x=-3+\sqrt{10}$ or $x=-3-\sqrt{10}$. The teacher then returns to the third transformation (“here, particularly…”) and emphasises the importance of this step. We are 10 minutes into the lesson, as she asks them to look at their notes for a moment to check if they have understood or if they have questions. Also, they are asked to explain that the two numbers obtained are really solutions. The teacher moves around in the class and talks to several students; it is clearly crucial for her to check that everyone has caught the previous explanation. During this, the class is not at all silent, with groups of students turning to each other and talking (mathematics at first; after a while some turn to more hilarious topics).

As much as 11 minutes pass this way, before a student is called to the blackboard to explain the validity of the solutions. The student writes down the whole equation ($x^2+6x-1=0$) with $-3+\sqrt{10}$ in the place of x, and repeats the equation while calculating the left hand side until he has “$0=0$”. The teacher criticises the “way of writing”, as it apparently assumes what is to be shown; but also, after making the necessary corrections, she lets the student understand that his calculation is essentially correct.

Now, 24 minutes into the lesson, the students are put to work on two exercises in the book (solve $x^2-4x=3$ and $x^2+8x-14$). After only four minutes, two students are called to the blackboard. The solutions are evaluated by the teacher (and are both correct!).

The last 5 minutes are spent on yet another example ($x^2+3x+1=0$), which the students are solving in their notebooks while the teacher checks that everyone is doing all right. At the end of the lesson, a boy and a girl are giving oral explanations (from their seat, the teacher taking notes on the blackboard) on how they solved this final problem, in particular how they completed the square.

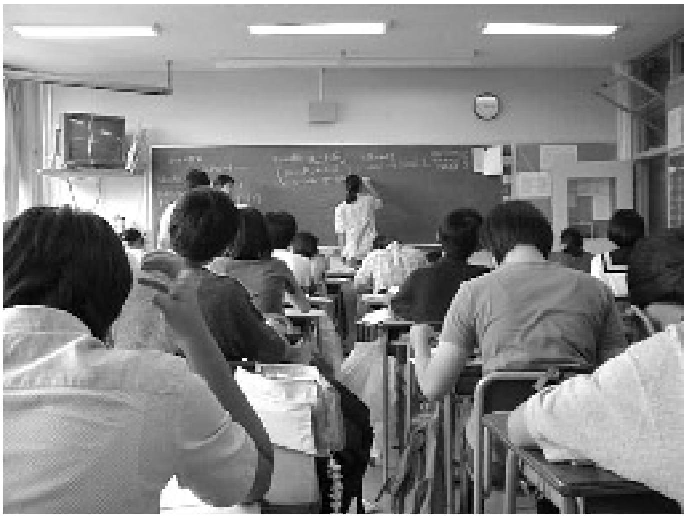

Figure 1. A typical scene from Lesson 2: students at the blackboard.

Right after the lesson, I interviewed three girls from the class to see what level of understanding they had really attained in this fairly short span of time. One thing I wondered about was whether the performance of the students in exercise solving was 'just' due to good homework on the examples to be studied in class, but the girls were in fact able to solve, without preparation, other quadratics of the kind that had been worked on in the class. Also, quadratics with just one solution were within their capacity. I then proposed another one: $x^2+x+1=0$. Using the method from before, they arrived surely at $(x+½)^2=-¾$ – and then were bewildered for a long time, until one of the girls exclaimed: "I am at a loss, this is not possible." Of course, I told her that (and why) this was the best answer she could give!

3 Initial discussion of the two sessions.

I suspect the reader is, as I were, quite impressed with the progress of the students within two 40 minutes lessons: from acquaintance with the formula for the square of a sum together with a rough understanding of the use of square roots to solve an equation of type $x^2 = q$, to a working knowledge of the solution of quadratics[6]. It is worth noticing that this new knowledge is ostensively based on previous knowledge, not just a "magic formula" – the students are not simply 'programmed' e.g. to use the general solution formula (which, in fact, is not mentioned). Also, the reader is likely to be surprised by the intensity of the work of the class, combined with the overall informal and relaxed atmosphere. The teacher clearly retains complete control of the structure of the lesson, yet there are long periods of noisy student activity, not all related to the subject – and, not a single time does the teacher make disciplinary remarks[7].

One way to analyse the structure of these carefully prepared lessons is to start by observing the distribution of "plenary class work" and "individual/group work" according to the following classification of discursive patterns (which, indeed, applies well to all of my observations in Japan, and bears some similarity to the categories used in Stigler et al, 1999).

1. *Plenary class work*: the teacher is, in principle, controlling all communication, which is "plenary" (in principle, directed to and heard by everyone in the class). There are three main forms, which, however, are often "mixed" (e.g. when the teacher injects question-response dialogues into the presentation of new material at points where this is based on principles familiar to the students):

   a *Presentation by the teacher.* The teacher is speaking and writing on the blackboard; the students take notes.

   b *Presentation by students.* This typically means students writing exercise solutions on the blackboard, often simultaneously with other students; sometimes, they are also giving oral explanations, mainly after or instead of writing.

   c *Whole class discussion.* Students may be very active in discourse, typically reacting to questions by the teacher, suggestions from other students etc.; and all dialogue is meant for/heard by the entire class.

2. *Other class work*: the teacher suspends his control of communication for some time (always "inside" the lesson, that is, after and followed by a plenary class work segment). This puts the class in one or both of the following modes:

a *Individual student work*. The students work, on their own, on some exercise or other task set by the teacher. The teacher usually circulates in the class to help individuals.

b *Group work*[8]. In groups, the students discuss some exercise or other task set by the teacher. The teacher usually circulates in the class to help individual groups.

With this rough classification, we can resume the discursive structure of the two lessons as follows (indicating also the duration of each segment and a keyword to its content).

1a *4 mins.* Intro: $x^2+2x-8=0$	**1a+1c** *12 mins.* Work on equations of type: $(x+p)^2=9$ Idea: put $A=x+p$	**2a** *6 mins.* Exercise solving	**1b** 4 mins. Blackboard writing	**1a** *5 mins.* Teacher reviews solutions	**1a** *6 mins.* Completion of squares: examples

Figure 2. Lesson 1 (Starts 3 minutes late).

1a *3 mins.* Intro: "Last time ..."	**1a (+1c)** *7 mins.* Generic example: $x^2+6x-1=0$	**2a, (2b)** *11 mins.* Students work on their notes and discuss.	**1a + 1b** 3 mins. Check $-3+\sqrt{10}$	**2a** *4 mins.* Work on two equations.	**1b, 1a** *5 mins.* Blackboard writing and review	**2a, 1b** *5 mins.* Final example: $x^2+3x+1=0$

Figure 3. Lesson 2 (Starts 2 minutes late).

The frequent shifts in mode are eye-catching and remarkable, and they are not arbitrary at all; I was shown, by the teacher, the written plan[9] of the first lesson, which was quite similar in style to the figure above (although not specifying exact duration, and of course much more detailed on specific contents). However, the plan is not rigidly followed; in the case of lesson 1, the teacher had really planned to do what corresponds to the second segment of lesson 2 at the end of lesson 1 (cf. the note at the end of the résumé of lesson 1).

We note that although most of both lessons take place in mode 1, this is much less so in the second lesson, where the students work by themselves almost half of the time. In both lessons, the segments in mode 2 are central in the sense that they are prepared and reviewed by segments in mode 1. One may say that the lessons are basically organised to further student activity as much as possible while maintaining a continuous progression in subject matter[10]. The centrality of the student activity phases is also confirmed when analysing the actual classroom discourse rather than just these overall modes; the introductory presentation of the teacher is focused on explaining concisely those elements of knowledge (new or formerly acquired) that the students will need to solve the tasks, and especially the reviewing segments after the student's presentations attend with extreme care to the difficulties and pitfalls that are implicit in their work. This does not only apply to the purely technical or conceptual levels, but also to more "formal" matters – it seems to be strongly emphasised and valued by teachers that the students write a clear and "well formed" explanation of each step in the problem solving. Perhaps these "reviewing segments" constitute the most original and remarkable single element of the Japanese mathematics lesson. Indeed, in the terminology of (Winsløw, 2000b), they demonstrate that the entire mathematical register is to be acquired, rather than just fragments of symbolic inventory syntax and transformational structure. In elementary arithmetic, the symbolic inventory of the mathematical register is typically restricted to single strings ("computations"), which can be produced from rather procedural algorithms (with more or less understanding). At this stage, framing explanations will often (have to) be kept informal. There is a crucial alternative when moving on towards more complex transformations: to develop the full transformational power of the register, or to stay with informal language and procedural computation. The solution of simple algebraic equations is one of the crossroads where this alternative has to be decided. This is further highlighted in the next section.

4 Quadratic equations in two textbooks.

The lessons described above were based upon exercises and style of exposition from the textbook used (Hitotsumatsu et al., 2000, Chapter 3, Sec. 1). The treatment of quadratic equations occupies 10 pages[11], including a large number of exercises. The first page has a large drawing of a rectangular hen run with four hens inside, seen from above (Fig. 4, left).

The text poses the following problem: *We are to make a rectangular hen run as shown, while using the wall of the farm as one of the four sides, and a steel fence on the others. Now, we have 24 m of steel fence. We must* *use all of the fence and the area of the hen run must be 70 m². How long will the sides of the hen run have to be?* On the following page, it is explained how to translate this problem into the equation $x(24-2x)=70$, when deciding that the length of the wall side of the rectangular hen run is to be x meters. Then this equation is transformed to $x^2-12x+35=0$ and it is explained that this is an equation of the form $ax^2+bx+c=0$, called a quadratic equation. The text goes on with several examples of quadratic equations in the form of simple exercises, where one is to determine the coefficient and guess or check solutions. As for the equation from the hen run example, it is "solved" by guessing (the text suggests trying the numbers 3,4,5,6 and 7). After this, four pages are used on exposition and exercises on linear factorisation of simple quadratics (all with integer solutions). The last four pages contain essentially what is covered in the lessons referred to above, namely a thorough explanation of the method of completing the square, including a nice geometric illustration of the example x^2+6x-1 (which was also considered in the lessons). The last page, which makes up the fourth section of the chapter, contains two columns of equations. They display the parallel transformations of a *particular* and a *general* quadratic equation, using the method discussed before. The last equation of the 'general' column is, of course, the general formula for the solution of quadratics. This formula is repeated in a box at the bottom of the page, but the general solution is normally

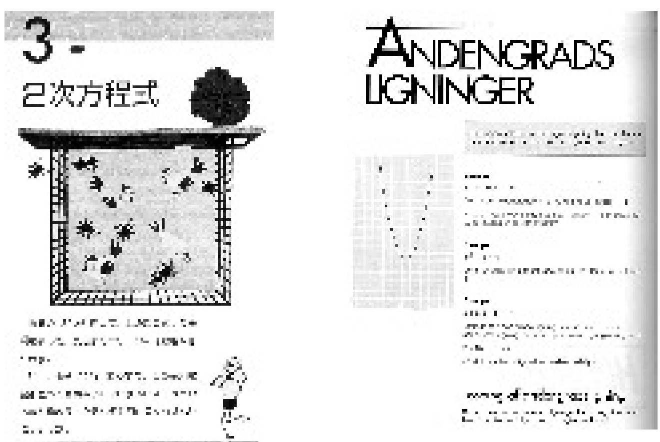

Figure 4. Left: First page on quadratics in the Japanese textbook. Right: First page on quadratics in the Danish textbook.

not used or covered at this level – according to the teacher, this more abstract approach is provided as an option for "strong" students. Finally, we notice that the chapter does not contain further examples of extra-mathematical applications, which in general seem to be considered by mathematics teachers as tools to motivate mathematics teaching rather than as relevant subject matter[12]. Likewise, no discussion or examples are provided for the case of quadratic equations which do not have real solutions[13].

To illustrate that the approach to quadratics sketched above is by no means the only option, let me briefly describe how the subject is treated in a Danish textbook (Hessing et al., 1992) for the same level (grade 9)[14]. The final chapter of the book has the title "Quadratic equations", and spans 7 pages with no exercises[15] but with several worked examples. The text begins with a framed box containing the following (clearly incorrect[16]) definition: *An equation is called a quadratic equation, if there is at least one x in the equation which is to be multiplied by itself.* As an example, the text gives the equation $x \cdot x+6=x \cdot 5$: *This is a quadratic equation because of the term* $x \cdot x$. It goes on to explain: $x \cdot x$ *is normally written as* x^2 *(reads x to the second), so that's why it is called a quadratic equation*[17]. Next to the text, there is a graph, which is neither explained nor related to the text (it represents the parabola $y=x^2$, cf. Fig. 4, right). Two more examples are given of how to use the definition to establish that a given equation is quadratic. Then, three examples are given of how to solve quadratics with no term of degree 1 (one example being the equation $x^2=-4$, which is observed to have no solution), and one example of how to check that given numbers are solutions (namely, the numbers 1 and 2 for the equation $x^2-3x+2=0$). It is observed that it is usually difficult to guess solutions. The final section, "Algebraic solution", takes up the remaining five pages of the text. It is introduced as follows: *There is a sure method by which one may always solve a quadratic equation. In order to use it, one must first arrange the terms of the equation*. The following pages contains 10 examples (and two "rule boxes") on how to find the coefficients *a*, *b* and *c* in the standard form $ax^2+bx+c=0$ of a given quadratic (not necessarily given in this form). Then, the text says: *once you have found a, b and c, they "just" have to be inserted in the following formula!* Follows the solution formula with a note on the meaning of ±: *it just means that one first does the computation with + and then with* −. The text also says: *Why it looks like that will be covered later (e.g. in 10th grade)*[18]. After this, two examples are treated in meticulous detail (both of the equations have two rational solutions). At the end of the chapter, one finds two short remarks, which I translate in their full length: (1) *Be careful about the signs, especially under the root signs. It is usually here* *that most errors are made. It surely will take you some training to become proficient in solving quadratics*. (2) *You probably can't find any examples of the quadratic equation being of significance in our "everyday lives". So, why learn about it? Because it is a piece of "mathematical culture", and because it has great significance within slightly more advanced mathematics, physics, biology and technology.*

The two texts are exemplary of two very different directions that may be taken from the 'crossroad' described in the preceding section. The texts proceed almost in opposite order. While the Japanese text takes an inductive approach, starting from a practical (although clearly constructed) example, the Danish text starts out with an "abstract descriptive" (albeit wrong) definition of the general subject. The Danish texts proceeds to explain the operational identification of coefficients, then states the general formula for the roots with no justification (except for the vague promise that it may be taught later); no understanding beyond the procedural level is sought. The Japanese text builds up the method of solution from examples and bases it on previous knowledge (e.g., the meaning of square roots, the square of sums formula). Every step is carefully explained. Only at the end of the text is the general formula given – as the conclusion of the "proof ", which is really just a formalisation of the method as developed through concrete examples. In the Danish text, two crucial remarks finish the discussion: the first is a warning that there is a potential danger with signs, which (by carefulness) can be avoided. Thus, the reader will never find another explanation than his own carelessness, should he meet an equation like $x^2+x+1=0$ and get stuck, cf. the end of Sec. 3. Secondly, the meaning of the whole exercise is addressed, but is essentially left in the air. One may wonder if the allusion to "mathematical culture" or to advanced applications will serve, for adolescent readers, to remedy this apparent pointlessness. At best, they have learnt to do what a computer could do in moments: apply a ready-made formula with no demands on structured understanding. Clearly, in the presence of a competent teacher, they may still learn more.

4 Constructivism: "To enable students to understand …"

The central role of *student activity* is one of the most striking features of the examples of Japanese practices in mathematics education which are presented above.

In the textbook, exercises take up well over half of the space; indeed, apart from the introductory hen run example, the text can be seen as a commented sequence of successively more advanced exercises, which are linked together by minimal explanations. Student activity in the form of exercise solving is clearly a condition for making sense of the explanations

– there is simply no room for armchair readers[19]. By contrast, exercises are not important for reading the Danish text. Most likely, they would serve as training in using the mechanical procedure exposed by the text; at best they would reveal its insufficient treatment.

In the lessons, the teacher's activity consists in preparing, requesting, monitoring, assisting and evaluating student activity according to a carefully planned sequence of such actions. Teacher exposition of new material is not absent, but seems to be kept to a minimum. This is in the spirit of what has been called the "example integration method" (e.g., Koizumi, 2000), where a new topic is approached through a few generic *examples*, while integrating the students' ideas and knowledge in the treatment of these. The idea is that the "general picture" of the topic at hand will emerge from working on these examples with a maximum involvement of the students' previous knowledge, so that the new "general picture" is really an extension of the previous one. In the two lessons observed, one such basic example was the quadratic equation $x^2+6x-1=0$. It appears towards the end of the first lesson, in which the students have been working on equations of the form $(x+p)^2=q$. The teacher asks them whether they can handle this example – knowing well that they most likely cannot. It is important that the students realise the present state of their competency. Even if the work of this lesson is a large part of what is needed to handle this example, they still miss the "linking idea" of completing the square, which the teacher then proceeds to develop – through examples, ending with the case $x^2+6x\ldots=(x+\ldots)^2$. The teacher's final remark returns to the example: "Now think about how this may be used to solve the original equation" (points at $x^2+6x-1=0$). The task is to connect the two pieces, clearly a non-trivial task, but in principle within the reach of the students' understanding. However, this example is worked out in the book, which the students will of course notice when studying the next part of the chapter before the following lesson. If they have not been able to make the connection at the end of the first lesson, or before studying the text, they will still be able to see how the pieces are put together. This way, gifted students are given a chance to make the connection by themselves, while other students are not left behind. And, for this reason, it is not so surprising that the teacher, at the beginning of Lesson 2, presents the solution of this example in expository style, without student involvement. The real test is not this particular example, but the ability to transfer the principles inferred from it to other examples.

The focus on student activity in this matter is also repeatedly emphasised in the national mathematics program (Nagasaki, 1990), which contains the basic principles and contents for mathematics teaching in Japan (from kindergarten through high school). For each grade, the objectives and contents are specified, and these specifications are all of the form "To help students to …" or "To enable students to …". The same wording is used in formulating the general objectives, which for lower secondary school reads: *The aims are to help students deepen their understanding of the basic concepts, principles and rules concerning numbers, quantities and figures, and acquire the way of mathematically representing and coping with, and to enhance their abilities of mathematically considering things, as well as to help them appreciate the mathematical way of viewing and thinking, and thereby foster their attitudes of willingly applying them*. This is, in my opinion, a profoundly constructivist program, in full consistence with the example-integration method as well as with the observations in the preceding sections on actual practices.

Constructivism is primarily concerned with epistemological and cognitive questions, and educational practices are implicitly rooted in more or less coherent assumptions regarding these questions. Von Glasersfeld (1991) describes several general principles of teaching practice which he finds to be particularly consistent with a constructivist theory of knowing: motivation, or "reinforcement", is to come from *having enjoyed the satisfaction of finding solutions to problems in the past* (p. 181); *problem solving as a powerful educational tool* (p. 183); the *orienting function* of the teacher in guiding the construction of concepts by students (p. 183f); the ideal that this guidance be explicit, and attentive to existing student conceptions (p. 185ff); and the need for the teacher to act as a "helper" (more than as an "instructor") in fostering student reflection and conceptual change. The practices and official regulations of Japanese mathematics teaching, which have been discussed above, seem to be much in the spirit of these general principles. In particular the frequent mode shifts (Fig. 2-3) in lesson structure are imposed by the teacher in order to enable *and* orient the fostering of reflection, expression and autonomous understanding of the students.

One exception to this agreement could be the first point: the ideal of motivation being "intrinsic" (coming from enjoyment) rather than externally imposed. Clearly, for most Japanese students, the drive for learning is at least partially external[20]. Even if this is probably always the case in institutionalised education, the Japanese case in particular makes it worthwhile to consider a less individualistic version of constructivism. The main focus of classical constructivism is on the individual cognising subject, and so may appear not to apply to the dynamics of a classroom within a curriculum driven system of schooling. Notice that, despite his overall adherence to the most subjectivist stances of constructivism, von Glasersfeld does point to *Piaget's repeated observation that the most frequent* *occasions for accommodation are provided by interaction with others. Insofar as these accommodations eliminate perturbations, they generate equilibrium not only among the conceptual structures of the individual, but also in the domain of social interaction* (von Glasersfeld, 1991, p. 67). The collective review of (mainly individual) problem solving segments, which was pointed out as a significant trait of the Japanese lesson structure, is a clear example of a powerful tool to realise this wider sense of equilibration in practice. The teacher's frequent insistence on "writing (in the) right (way)", when evaluating student presentations, also means that what is to be developed in mathematics teaching is not only a common "understanding" but also a common "language use" (a register, cf. Winsløw, 2000b). This is, in my opinion, a capital point of difference in (at least) emphasis between Western ideas of teaching in a constructivist "spirit", and the practices I observed in Japan. In fact, with the central role of *harmony* (within a group and within society) in the Japanese culture, discussed in Sekiguchi (2000) in relation to the role of proof in mathematics education, the communal accommodation and equilibration may be seen as an ultimate goal of schooling, rather than just a consequence of individual conceptual development. On the other hand, a "collective" understanding of mathematical structures is inseparable from a common usage of language in communication about them (Winsløw, 2000a).

6 Structuralism: "... appreciate the mathematical way of viewing and thinking ..."

In the analysis of the practice of Japanese mathematics education, I think it is worthwhile to invoke and involve another "grand idea" from the intellectual heritage of the 20th century, namely structuralism. Clearly, the stocks of *structuralism* in mathematics education suffered severely from the alleged failure of the "New Math" reform around 1970 and the (mistaken) identification of structuralism with deductivist teaching philosophies [21]. I neither will nor need to address here these deplorable historical events, which were a purely occidental adventure. Instead, I will explain my view that structuralism as a method for educational analysis is a useful key to understand the Japanese way of teaching, in particular the evidence described in this paper.
Structuralism can be viewed as a principle for intellectual inquiry: look for the basic structures of the subject, that is, the fundamental relations and rules of transformation that govern it [22]. Indeed, according to Piaget, structuralism is a method, not a doctrine (Piaget, 1968, p. 123), and as such it is theoretically compatible with – in fact "inseparable" from – constructivism (ibid., p. 13). As explained by Piaget (ibid., chap. II),

structuralism is particularly "at home" within modern mathematics, where it is hardly controversial to professionals of the discipline. But that is not the main issue here[23]; rather should we dwell a moment on the important structuralist contribution to the field of education due to Bruner (1960).

Bruner maintains that *any idea or problem or body of knowledge can be presented in a form simple enough so that any particular learner can understand it in a recognizable form* (p. 44). As an attempt to illustrate this, he describes an experiment in which four eight-year-old children are introduced to quadratic functions[24] over six weeks, by a scheme of activities (involving multiple representations) carefully sequenced and structured by a research mathematician in cooperation with a professor of psychology. Although the claim that "anyone can learn anything" is admittedly impossible to test and overly general, Bruner's point is subtler: *one must take into account the issues of predisposition, structure, sequence, and reinforcement* (p. 70) in any serious attempt to realise the claim in a specific instance. Since this task is ultimately about providing the representations of the subject matter that are best suited to the knowledge and abilities of the learner, *it is the enterprise par excellence where the line between subject matter and method grows necessarily indistinct* (p. 72).

Liping Ma (1999), in a very different context and time, draws an essentially similar conclusion in her study of elementary school teachers' knowledge of "fundamental mathematics": *teacher's subject matter knowledge of school mathematics is a product of the interaction between mathematical competence and concern about the teaching and learning of mathematics* (p. 146). Because the task of the teacher is to enable students to grasp, and to navigate in, a continuously growing territory of mathematical structure, it is crucial for the teacher to be able to represent and relate such structure in ways that are compatible with the structure of student knowledge. Moreover, as the teacher is usually not working exclusively with a single student, student knowledge is to be construed as the common, interactive knowledge of a group (a class) of students, which adds complexity but also potential for the task of supporting and directing the students' learning process. A truly structuralist approach to teaching mathematics recognises that the development of individual student conceptions of mathematical structure is embedded in social structures of interaction and communal development, which are much more complex than just "the union" of individual developments (cf. Gibson, 1984, p. 132). To grasp these structures of communal understanding, it is necessary to move the focus from the individual (psychological) process to the process of the group, while the object of understanding (mathematical structure) becomes a shared (rather than an individual) target. This is an essential part of the Japanese teachers' *problem of involving all of the students in their concept development* (Schmidt et al., 1996).

Such an understanding is latent[25] in the general objectives of mathematics teaching (cited above) when they talk about helping *students* (considered as a group rather than as individuals) to *appreciate the mathematical way of viewing and thinking, and thereby foster their attitudes of willingly applying them* (notice the singular form of "way", and the goal of common "attitudes" to willingly make use of this way). A corresponding, yet very different view is expressed in the Danish guidelines for mathematics teaching[26] (Undervisningsministeriet, 1995, p. 21): *The aim of the teaching of the school is not to make little mathematicians of the students. The subject must contribute to the personal development of the individual student, and the students must experience … how the subject provides them with potentials for action in practical situations.* Here, the focus is entirely on the aims and perceived needs of the individual student. Furthermore, while the Japanese regulations (and practice) see the students' coherent and communal appreciation of the mathematical way of viewing and thinking as a prerequisite for its application, the Danish guidelines suggest that application should be experienced without unnecessary exposition to mathematics as such. It may be hard to guess what is meant by "making little mathematicians of the students", but it seems to me that it is not far from what the Japanese standards set up as a positive aim (*appreciate the mathematical way of viewing and thinking*). Roughly speaking, the difference in aims here is between individually empowering bits of knowledge, and common, structurally coherent ways of viewing and thinking.

The Japanese classroom practice (in particular, the lessons described above) exhibits an ongoing concern for communal development of the students' "viewing and thinking", not least through the emphasis on "correct expression" and through the regular expositions and discussions of student work which were mentioned in Sec. 3. This development is not to be heading in an arbitrary direction, but it is in agreement with that of other classes because of the common goal (*the mathematical way of viewing and thinking*) and the close co-operation within the group of mathematics teachers (cf. note 9). Moreover, faithfulness to the structure of the subject[27] is also crucial for further learning: *The closer an idea is to the structure of the discipline, the more powerful it will be, consequently, the more topics it will be able to support* (Ma, 1999, p. 121). The entire curriculum (Nagasaki et al., 1990), from kindergarten to senior high school, reflects this concern for coherent development through careful sequencing and successive extension of central ideas.

As for the texts considered in Sec. 4, a crucial difference appears indeed to be the representation and relatedness of the mathematical structure involved. The method suggested by the Danish text is simple: put the equation in standard form, identify the coefficients a, b and c, and evaluate the general formula for the solution with these values. The structural knowledge needed to follow this receipt is restricted to the mastery of additive reordering of terms in a quadratic equation (according to their degree), reading off coefficients, and evaluating a given algebraic expression of type $f(a, b, c)$ with given values of its variables. No relation is established between this algebraic expression and the quadratic equation. However, *to learn structure, in short, is to learn how things are related* (Bruner, cited in Ma, 1999, p. 24)[28]. Instead, the text is a clear expression of *procedural* teaching strategies, as opposed to *conceptual* strategies (cf. Ma, 1999, pp. 33ff). The former do not seek *connectedness* (with other mathematical knowledge of the student, e.g. about the meaning of square roots), *multiple perspectives* (or representations), *basic ideas* (such as the completion of squares), or *longitudinal coherence* (invoking previously learned structure or preparing the future learning). By contrast, the Japanese text addresses all of these, in particular it develops the structure of the solution method from known structures, and it may support the learning of related structures (e.g. the case of quadratic equations with no real solution, polynomial equations of higher degree or the context of complex numbers). The central mathematical idea, completion of the square, is highlighted and is carefully explained as a new way to use the formula for the sum of a square. When existing knowledge is put to new uses, it is not only related to other knowledge but it is also transformed; the formula $(x+\frac{a}{2})^2=x^2+ax+(\frac{a}{2})^2$ is in itself (and in principle) part of previous knowledge, while in the above new context, the students discover something like "the point of using it from the right to the left".

7 A final remark on meaning.

The structuralist emphasis on *relations* and *transformations* of knowledge structures may seem overly internalist to some readers. What about "the meaning of the whole thing"' for a teenager in modern society? I have already suggested that the communal nature, the "harmony" of knowledge structure developed in mathematics teaching, may be one answer to this. However there is a basic danger of "meaning-itis"[29] related to this question, and even to the answer suggested, which is expressed by Roland Barthes (1970, p. 92f), in the context of the *haiku*[30]:

> L'Occident humecte toute chose de sens, à la manière d'une religion autoritaire qui impose le baptême par populations; les objets de langage (faits avec de la parole) sont évidemment des convertis de droit: le sens premier de la langue appelle, métonymiquement, le sens second du discours, et cet appel a valeur d'obligation universelle. Nous avons deux moyens d'éviter au discours l'infamie du non-sens, et nous sousmettons systématiquement l'énonciation (dans un colmatage éperdu de toute nullité qui porrait laisser voir le vide du langage) à l'une ou l'autre de ces **significations** (…): le symbole et le raisonnement, la métaphore et le syllogisme.

For the Barthesian analysis of Japan, a central insight is that in this "empire of signs", one finds an unproblematic admission and even primacy of sign systems with no external signifiers. Ultimately, mathematics is one such system, as *it is "about" – insofar as this locution makes sense – itself* (Rotman, 1988)[31]. Hence, ultimately, their admission and appreciation is a condition *sine qua non* for teaching and learning mathematics.

References.

Barthes, R. (1970). *L'empire des signes.* Genève: Editions d'Art Albert Skira.

Beaton, A., et al. (1996). *Mathematics achievement in the middle school years. IEA's third international mathematics and science study.* Boston, MA: CSTEEP, Boston College.

Bruner, J. (1966). *Toward a theory of instruction.* Cambridge, MA: Harvard Univ. Press.

Gandal, M. (1997). *What students abroad are expected to know about mathematics. Defining world class standards, vol. 4.* Washington, DC: American Federation of Teachers.

Gibson, R. (1984). *Structuralism and education.* London: Hodder and Stoughton.

von Glasersfeld, E. (1991). *Radical constructivism in mathematics education.* Dordrecht: Kluwer.

Hessing, S. & Holmer, M. (1992). *Faktor 8-9, Begrebsbog* (In Danish. English translation of the title: Factor 8-9, Concept book). Copenhagen: Malling Beck.

Hitotsumatsu, S., et al. (2000). *Chugakkou Suugaku 3.* (In Japanese. English translation of the title: Junior High School Mathematics 3). Tokyo: Gakkou Tosho Publ.

Knoll, S. (1998). Anforderungsgestaltung im Mathematikunterricht. *Mathematik lehren 90*, 47-51.

Koizumi, S. (2000). Modernization (New Math) and teaching method – example integration methods. *Journal of the Study Group on Secondary School Mathematics* (special issue for ICME-9), University of Tsukuba, 13-14.
Ma, Liping (1999). *Knowing and teaching elementary mathematics.* Mahwah NJ: Lawrence Erlbaum.
Nagasaki, E., et al. (Eds., 1990). *Mathematics program in Japan*. Tokyo: Japan Soc. of Math. Ed.
Neubrand, J. (1998). Japanischer Unterricht aus mathematik-didaktischer Sicht. *Mathematik lehren 90*, 52-55.
Nohda, N., et al. (Eds.) (2000). *School Mathematics in Japan*. Booklet produced for ICME-9 in Makuhari. Tokyo: Japan Society of Mathematics Education.
Piaget, J. (1968). *Le structuralisme.* Paris: Presses Universitaires de France.
Resnik, M. (1997). *Mathematics as a science of patterns.* Oxford: Clarendon.
Robitaille, D. and Nicol, C. (1994). Comparative international research in mathematics education. In R. Biehler et al. (Eds.), *Didactics of mathematics as a scientific discipline* (pp. 403-414). Dordrecht: Kluwer.
Rotman, B. (1988.) Toward a semiotics of mathematics. *Semiotica 72*, 1-35.
Schmidt, W., et al. (1996). *Characterizing pedagogical flow*. Dordrecht: Kluwer.
Sekiguchi, Y. (2000). Mathematical proof, argumentation, and classroom communication: a Japanese perspective. In P. Boero et al. (Eds.), *Proof and proving in mathematics education*. Retrieved from http://www-cabri.imag.fr/Preuve/ICME9TG12/index.html.
Stevenson, H. & Stigler, J. (1992). *The learning gap. Why our schools are falling behind and what we can learn from Japanese and Chinese education*. New York, NY: Touchstone.
Stigler, J., et al. (1999). *The TIMSS videotape classroom study. Methods and findings from an exploratory research project on eighth-grade mathematics instruction in Germany, Japan, and the United States.* U.S. Department of Education.
Undervisningsministeriet (1995). *Matematik, Faghæfte 12*.
Winsløw, C. (2000a). Between Platonism and constructivism: Is there a mathematics acquisition device? *For the learning of mathematics 20 (3)*, 12-22.
Winsløw, C. (2000b). Transformations in mathematical discourse. In B. Barton (Ed.), *Communication and language in mathematics education: The pre-conference publication of Working Group 9 [ICME-9, Tokyo, July 30-August 6, 2000]* (pp. 93-102). Auckland: Dept. of Mathematics, University of Auckland.

Notes.
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1 The priorities of society are, as most readers will know, made clear through the tough system of 'entrance examinations' in which students are, from an early age, competing to enter the better schools; for an informative description, (see Gandal et al., 1997, pp. 44-58).

2 During this period, I became well acquainted with the culture of Japanese university mathematics, besides learning the basics of the Japanese language.

3 In fact, the physical aspects (dimensions, inventory etc.) of public school classrooms are legally defined, so they look similar in all schools (Schmidt et al., 1996, p. 153).

4 This, in fact, was worked on in the class a few weeks ago.

5 The teacher later explained to me that this student reads "ahead" and does not take the classroom teaching very seriously. He does well at tests and when prompted by the teacher in class. His attitude is apparently rather typical for beyond-average students, but is considered problematic by teachers. About 10% of lower secondary school teachers points to "students who study before school lessons in cramming school" as one of their "worries" about mathematics lessons (Nohda et al, 2000, p. 62).

6 Notice that no claim is made about this reflecting typical or "average"' standards in Japan. In fact, the high school of Ochanomizu University is considered among the best in Japan, while the class 3M is (according to the teacher) an "average" class for this school. What is important here is that the principles on which the lessons are based may be considered typical.

7 In other lessons, disciplinary remarks occurred occasionally, but never with any display of anger on the part of the teacher.

8 During my observations, group work was always "spontaneous" and never prescribed or arranged by the teacher; frequently group work degenerated into "small talk". And usually there were at least some students working on their own. Apparently, group work is often a more important and more organised mode of work in Japanese classrooms (cf. e.g. Neubrand, 1998).

9 It is well known that the design and development of such lesson plans is a collective task for the mathematics teachers at a given school, (cf. Knoll, 1998, p. 51). At the high school I visited, mathematics teachers had weekly meetings on teaching design.

10 Clearly, these two factors have a complex interrelationship, and their balancing is a crucial task for the teacher in the Japanese system with its ideal of *everyone* learning essentially the same curriculum (cf. Sec. 5).

11 A5 size pages, with about 20 lines of text on each page; a large amount of the text is in fact formulas and exercises. The verbal parts are short and concise. There are only three illustrations (drawings).

12 One teacher formulated this viewpoint as follows: *The applications are mainly to be treated in other subjects such as physics.* This is in stark contrast with the Danish situation, as we shall see.

13 One may, however, say that the method of completing the square works for these as well, as was illustrated by the interview referred to in the note after the description of Lesson 2.

14 Quadratic equations are mentioned but are not mandatory in the standards for this grade. Indeed, many textbooks for this level omit the subject entirely. None, to my knowledge, takes an explanatory approach using completion of squares in any way. The textbook studied here is currently one of the most commonly used in Danish schools.

15 Exercises are found in a separate ”workbook”. They are all similar to the examples in the text.

16 In the 1999 edition of the book, this definition was changed to an equally flawed one. Japanese and other readers might wonder how this book passed official screening. The answer is that there is *no* such institution in Denmark. Schools are free to choose from what the market offers.

17 In Danish, the word for a quadratic equation literally means “second order equation”, so the relation to the power 2 in the term x^2 should indeed be obvious to the reader.

18 Denmark has 9 years of mandatory education. The “10th grade” is an option for those who do not proceed to senior high school (“Gymnasium”, in the Germanic tradition) right after the 9th grade.

19 A similar pattern is found is textbooks from Singapore, which are currently gaining popularity in the U.S.

20 Even if discounting the most obvious external force (assessment) with all its implications in the Japanese system, cf. note 1.

21 At the extreme, these would run in the following style: The work of Bourbaki demonstrates that set theory is fundamental to all of mathematics. Therefore all mathematics teaching must begin with a thorough treatment of set theory.

22 This has been expressed differently by different authors: as *langue* underlying *parole* by Saussure, or as *deep structure* in Chomsky's theory of syntax.

23 However, it is in a sense "embedded" in the main issue, as explained later.

24 In fact, some of the ideas involved in this experiment are very similar to the procedure of Keirin-kan's book, particularly the insistence on multiple representations of the idea of "completing the square"; on the other hand, given the age of the four children, it is not surprising that the use of concrete materials is also central in the experiment (while absent in secondary level teaching).

25 Notice that this paragraph is not to be understood as an analysis of the isolated text excerpts from official standards, but as an interpretation of how they articulate points of emphasis which are also found in practice – some of which are described in this paper.

26 Unofficial translation by the author. Note that the text cited is, at least formally, not part of an official curriculum, but belongs to what is called *an inspirational material for the teacher's planning of the teaching* – issued, that is, by the Ministry of Education.

27 As explained, in *teaching* these are to be considered objectives for communal development in social interaction structures. Objectives provide *directions* for such development, but they do not suffice to attain it.

28 This is clearly consistent with the basic tenet of structuralist philosophies of mathematics: *in mathematics, the primary subject matter is not the individual mathematical objects but rather the structures in which they are arranged. The objects of mathematics...are themselves atoms, structureless points* (Resnik, 1997, p. 201).

29 By this allusion to a disease, I want to signal that the unjustified assumption of external reference may be a culturally based hindrance to grasping the coherence of semantically closed systems.

30 A *haiku* is a special Japanese form of short poem (composed of only three verses). Here is a rough translation of the quote: *The West imposes meaning on everything, as an authoritarian religion which forces baptism on entire populations; the elements of language (made of parole) are obviously legitimate converts: the primary meaning of language implies, as a metonymy, the secondary meaning of discourse, and this implication has universal validity. We have two means of avoiding the infamy of senseless discourse, and in a strenuous effort to cover any nullity that may expose the emptiness of language, we systematically submit the utterance to one of these* ***significations*** *(...): the symbol and the reasoning, the metaphor and the syllogism.*

31 Thus, ultimately, mathematics is *un langage qui se signifie soi-même* (that is, *a language signifying itself* – the expression used by Jacobson in his analysis of music).

Brief bibliographical notes of the authors

Carl Winsløw's area of research is the didactics of mathematics, and much of his research is drawing on methods from linguistics and semiotics. The present work has since been extended to a contribution to the 13th ICMI-study on comparison of East-Asian and Western mathematics education, in a joint work with H. Emori, U. of Utsunomiya.

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Sammandrag

Hensigten med denne artikel er, med udgangspunkt i sammendrag af to lektioner om andengradsligninger i en 9. klasse i Tokyo, at kaste nyt lys over principper og forestillingssystemer som ligger til grund for japansk matematikundervisning på sekundært niveau. Artiklen munder specielt ud i en diskussion af deres relationer til konstruktivisme og strukturalisme i vestlig forstand.

Med henblik på at sætte denne beskrivelse i perspektiv, beskrives nogle eksempler på tilsvarende danske forhold.