The danger of being overly attached to the concrete:

The case of division by zero

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In this article, we use the example of division by zero to illustrate how reliance on concrete representations of mathematical concepts can become an obstacle to understanding. We describe some difficulties encountered by prospective elementary teachers in dealing with division by zero and we show how these difficulties could be explained by the students' desire for a (non-existent) physical interpretation of the mathematical problem.

For some years now, considerable attention has been paid in mathematics education to contextualizing mathematical notions and to using manipulatives to represent these notions. There is no doubting the usefulness of such an approach. Occasionally, however, concrete models can become a trap for «would-be users» who are unaware of the limitations of the models they are using. Clements and McMillen (1996) offer a thoughtful discussion of what manipulatives are and how they might be used effectively in the mathematics classroom. They stress that "although manipulatives have an important place in learning, they do not carry the meaning of the mathematical idea" and that "manipulatives alone are not sufficient - they must be used to actively engage children's thinking with teacher guidance" (Clements and McMillen, 1996, p. 271 and p. 276). Pallascio (1991) brings out the idea that any model is but an imperfect representation of a mathematical idea. The author uses two examples, one dealing with the properties of multiplication on relative numbers, the other with the notion of dimension. In the present article, we will show how reliance on concrete representations may hinder understanding in the case of division by zero. We shall then use this example to make a few remarks on the more general problem encountered in explaining mathematical outcomes using physical models.

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Why is it one can't divide by zero? In order to answer this question, a person must first conceive of division as the reverse of multiplication, and then perform a *reductio ad absurdum* — in other words, provide indirect proof of this proposition. When division is conceived of as the reverse of multiplication, dividing seven by five, to take but this example, will be interpreted as the search for a number which, when multiplied by five, will produce seven: "7 ÷ 5 =?" becomes "? x 5 = 7". In this instance, the answer, or quotient, which is sought after is 1.4. From this point of view, dividing seven by zero amounts to looking for a number which multiplied by zero produces seven: "? x 0 = 7". Since any number which is multiplied by zero always produces zero, the problem offers no solution whatsoever.[1] The simplicity of this method of reasoning must surely be deceptive, for, as we shall see; many a person hesitates over the result of division by zero, or at least is unsure of how to justify his or her answer.

In this article, we present a number of reactions on this subject which we observed among preservice elementary school teachers. It should be pointed out from the outset that the lack of understanding manifested by these persons concerning division by zero is far from being exceptional. In the United States, for example, Wheeler and Feghali (1983) administered a test to 52 preservice elementary teachers which included six cases of division by zero. They noted that 33 subjects (63 % of the sample) answered these six questions incorrectly, another 7 erred at least once, and only 12 (23 %) answered all six questions correctly. Ball (1990) has reported similar results upon questioning 10 preservice elementary teachers. Two were unable to state the result of 7 ÷ 0; 5 answered incorrectly; two came up with the right answer but were unable to justify their response; and only one student was able to find the right answer and justify it. (The prospective secondary teachers whom she interviewed fared better, but although all gave the correct answer, only 4 out of 9 could explain its meaning.) In the two aforementioned articles, it was not stated if, how and when survey participants had been taught division by zero.

The purpose of this article is to contribute to better understanding the nature of the difficulty caused by division by zero. In particular, we shall highlight the role played by (implicit) beliefs about the relationship between the mathematical operation of division and its physical models. The data we collected fall into two categories: 1) the written responses turned in by 47 prospective elementary teachers; and 2) a videotaped interview conducted with 2 other students. All students were enrolled in the Bachelor's program in preschool and elementary education at a large university in Quebec.

The Written Test

The question of division by zero is dealt with in a mathematics content course that is a first-year requirement. After observing that many second-year students taking their first methods course still had difficulty handling this subject in spite of having completed the prerequisite content course, we decided to devote class time to guiding the group in reconstructing the line of reasoning outlined in the introductory portion of this article.

Two weeks later, the following question was placed on the final exam:

> *Françoise says that $7 \div 0 = 7$, "because if I have 7 cookies and no friend to share them with, I keep all 7 cookies." Denise says $7 \div 0 = 0$, "because if I have 7 cookies and no friend to share them with, then I don't give away any cookies at all." Evaluate the answers of these two fifth-grade students and state how you would approach this problem with them.*

In order to answer this question correctly, students had to recognize that both Françoise and Denise were mistaken and that the situation which they described did not correspond to an instance of division by zero (indeed, the situation presents no problem which would require use of an arithmetical operation). Of the 47 students who were questioned, 17 (36 %) answered incorrectly. Replying either that they agreed with Denise (15) or with Françoise (2), with 6 others (13 %) providing only a vague, confused, illogical or incomplete answer, which made it hard to establish whether they had understood the impossibility of dividing seven by zero.

The remaining 24 students (51 %) knew that it was impossible to divide seven by zero, but only 7 among this group (15 %) gave evidence of using an approach to this problem which focused on the relationship of division to multiplication and which drew on a valid form of reasoning. As for the other 17 persons in this group, 6 made do with stating the rule or relying on a calculator, whereas 11 argued within the limits provided by the situational context or by an analogous situation which drew on counters or other manipulatives. We should like to focus here on the latter set of responses. The thrust of their argument consisted in imputing the impossibility or the absurdity of dividing by zero to a physical situation associated with such a division. A number of respondents adduced a situation of partitive division[2], which offered parallels to that which was suggested in the exam question. The following is a sample of their statements to that effect:

[After having argued that the situation put forward by the two children corresponds to 7 ÷ 1 and not 7 ÷ 0.] That way, I would make them understand that there is no situation which can illustrate "7 ÷ 0" and that as a result there is no answer to this division problem.

What's the point of dividing 7 cookies if there's no one I can give them to, not even myself? The very notion of division loses all meaning. It no longer serves any purpose.

In order to share [candies], there has to be at least one person you can give them to. If I don't have anyone, then I can't share. Hence, I can't divide. Hence, division by 0 is impossible.

[Françoise] doesn't understand that there's no way 7 cookies can be divided into 0 set(s).

Other students preferred drawing on a measurement (or quotitive) type of division[3]:

So that the girls get a clear grasp, the teacher should use materials (scissors, ruler, fabric) and ask them a number of questions, for example: "Take the fabric and cut it into pieces of 0 m." Then they will see right away that it is impossible to perform this operation. [Underlined in the original]

I would tell them that it's impossible because when you divide, that means that you're creating bundles or packets of objects. If I divide by five, I make packets of five. But if I divide by zero, it's impossible to separate seven objects into packets of zero object(s).

I would explain to them that if you have 7 elements and you want to divide by 0, it couldn't be done because sets of 0 element(s) do not exist.

We would suggest that whereas the errors which were described in the exam question derive from establishing an invalid connection between a concrete situation and an arithmetical operation, the responses which have been quoted above originate in the belief that a valid connection must always obtain and that arithmetical operations are necessarily bound up with corresponding physical situations. If such were the case, then arguments similar to those which were evoked in these answers would also serve to demonstrate the impossibility of dividing by a fractional number or a negative number, since dividing seven cookies into ½ a set or into -3 sets scarcely makes more sense than dividing them into 0 set(s)![4] Weshall return to this question below.

This experience convinced us that division by zero represents a serious teaching problem and prompted us to investigate it further by interviewing two students.

The Interview

The students who participated in the interview and whom we shall call Nathalie and Annie in order to preserve anonymity were, at that time, enrolled in the first year of the Bachelor's program in preschool and elementary education. They were taking the first of two required mathematics content courses. At the time the interview took place, they had covered only the properties of addition, subtraction and multiplication. Hence, we could expect that in an interview dealing with division by zero, we would be able to obtain indications of their own representations and not the reflection of recently learned information.

The interview was based on the following three questions:

What goes through your mind when I ask you to solve $12 \div 1$? $12 \div 4$? $12 \div 12$? $12 \div 0$?

Imagine that a sixth-grade student named Julie asks you: "What do I get if I divide 12 by 0?" How would you answer her question?

Comment on each of the following two explanations which two of your colleagues, Micheline and Jeannine, gave Julie. Micheline: "$7 \div 0 = 7$, because if I have seven pieces of candy and no friend to share them with, then I keep all the candy." Jeannine: "$7 \div 0 = 0$, because if I have seven pieces of candy and no friend to share them with, I don't give out any candy at all."

At the time we were designing the framework for the interview, the second question[5] appeared to us to be the most important of the three because, in our view, it should serve as the main source of indicators as to Nathalie and Annie's representations of division by zero. The first question was included as a way of starting off the interview, while the third question was meant to verify if the answer and justification provided for the second question would stand up to two different representations of division by zero which made use of concrete situations. As will be seen, the last question gave rise to the most interesting discussions, particularly with respect to the role, which the student interviewees ascribed to concrete models in mathematics.

Nathalie and Annie were interviewed together, as we believed that they would be more verbal if working collaboratively and hence that they would allow us to get a more accurate picture of their thought processes. During the interview, which lasted about one hour, they adopted a number of different strategies for solving the problem of division by zero, and changed their answers and justifications in the process.

At the start of the interview, when we put the first question to them, Nathalie and Annie explained that they referred mentally to their multiplication table to solve small, simple division problems. They immediately acknowledged that dividing twelve by zero posed a problem and admitted they did not know the answer to this question. Then, Nathalie timidly suggested that the quotient of twelve divided by zero might possibly be zero, but Annie refuted this at once, basing her disagreement on the following justification: "But if you look at your proof, twelve times zero gives you zero, that's where the problem is!" (Actually, the justification should have been that zero times zero does not produce twelve.) At that point, Nathalie also recognized that that quotient couldn't be zero. She used the same argument to show that the quotient couldn't be twelve either. The two students thus came to the conclusion that dividing twelve by zero, as with any other division by zero, was an impossibility.

When they were asked the second question — i. e., how they would explain this answer to a child —, they drew on the proof via multiplication once again. However, despite this explanation, for a short while Nathalie attempted to provide a concrete image of division by zero by adducing a situation involving partitive division: "You've got twelve and you want to divide it into zero packet(s)...». But since Nathalie was unable to produce a satisfying correspondence between such a situation and division by zero, she came to the conclusion, with the help of Annie, that there was some unexplainable difficulty with zero. In order to justify their position that division by zero was impossible, Nathalie and Annie came back to their initial strategy — i.e., proof via multiplication.

It is interesting to note that when forced to explain their result to a child, the student interviewees attempted to make use of a strategy other than that of proof via multiplication, as though they considered this argument to be too difficult or too abstract for children. Maybe they thought that "good pedagogy" required that they produce a concrete representation of the division of twelve by zero. However, having failed to find one, Nathalie and Annie had to fall back on their original explanation. They added, though, that there must be some other reason why division by zero was impossible, but that this reason escaped them on account of their "lack of mathematical knowledge."

Turning to the last question, when we asked Nathalie and Annie to comment on the explanation provided by Micheline (according to whom the quotient of seven divided by zero is seven), we immediately sensed that they found this explanation appealing. Annie, however, quickly noted that the situation so described corresponded more to division by one than to division by zero:

> *But what are you going to do with seven divided by one [if seven divided by zero gives you seven]? [...] If she is all alone and she alone takes all the candy [...], then this is dividing by [...] one, by herself.*

This explanation nevertheless caught her off balance because for a moment she wondered if dividing by one and dividing by zero were equivalent. After using the proof via multiplication, however, she realized that one times seven makes seven whereas zero times seven does not make seven. As for Nathalie, she voiced her own reservations as follows: *I find what Annie said really interesting, except maybe we should find out what the real answer is.* It is clear that at that moment, Nathalie was no longer convinced that division by zero was impossible. However, lacking some other answer and accompanying explanation, she could not deny that division by zero was impossible. Thus, when the time came to deal with Jeannine's explanation, Nathalie and Annie still believed that division by zero was impossible and that the most satisfactory justification for this position was the indirect proof via multiplication, but they seemed to be less convinced of this than at the outset. Moreover, they also made a point of explaining to me that they were entitled to be mistaken!

When they were asked to comment on Jeannine's explanation, according to which seven divided by zero equals zero, Annie immediately identified the same problem as in the explanation put forward by Micheline:

> *It [the second teacher's explanation] makes a certain amount of sense, too. But if she doesn't give out candy, then she keeps it all for herself. It's the same thing [the same type of reasoning as in the first explanation]. She still keeps seven. It's seven divided by one [equals] seven.*

Nathalie, on the other hand, came back to her first idea; in which seven divided by zero makes zero. In the following excerpt, she explains her change of mind:

> *Forget the zero. Assume I have twelve pieces of candy and I hand them out to three [friends]; I give each of them four. Now I use other numbers. [If I have twelve pieces of candy and three friends] to share with, then each will get four pieces. Now if I have seven pieces of candy and I have zero friend(s) to share them with, then the friends won't get any. That's the right answer [because] it involves the same steps as in the other.*

Nathalie mistook one situation for the other by making only a surface comparison. She claimed that the two situations were alike. However, in the first case, three multiplied by four makes twelve, whereas in the second, zero times zero does not equal seven. Why then did she choose zero for her answer? Probably because it was her first idea. She herself admitted as much: *Originally, I agreed with the fact that, in my opinion dividing by zero gives you zero. That's why this explanation [Jeannine's] suited me better.* Nathalie's reaction provides a clear illustration of how the mechanical application of the partitive model of division may become an obstacle to understanding division by zero.

Annie disagreed with Nathalie. She stuck to her idea that the situation touched on in Jeannine's explanation corresponded more to division by one than to division by zero. She explained that by giving out candies to no one, a person kept them all to him- or herself, thus entailing division by one. After listening to Annie's arguments, Nathalie reconsidered the line of reasoning which had led her to believe that division by zero produced zero and admitted that it contained a major flaw:

> *It doesn't really work, because if I take my equation of twelve divided by three equals four, well four times three gives you twelve. Whereas if I go seven divided by zero equals zero, according to the same line of reasoning, zero times zero doesn't give me seven, it gives me zero. My line of reasoning doesn't work.*

Having repeatedly failed to picture division by zero using a concrete situation, Nathalie and Annie finally concluded that the problem was beyond their powers of understanding, even though they seemed to have grasped the argument by *reduction ad absurdum* at the outset of the interview! The following are examples of their comments:

> *I think, well, I'm starting to think that the whole thing is beyond me. You know, maybe it's just a property that division by zero is impossible, period. (Nathalie)*

[You can't divide by zero because] it's part of...what holds true...of a theory. (Nathalie)

A sort of theorem that everyone takes as given that you don't divide by zero because it just isn't done. (Annie)

They nevertheless continued to assert that, for lack of anything better, they would explain to a child that division by zero is impossible by drawing on proof via multiplication. This proof would have to suffice for themselves and for children, but a more convincing sort of argument must exist somewhere! The following excerpts were taken from the last remaining minutes of the interview:

And I think that by doing a short proof like that, you could get by. Perhaps there are other proofs that are more formal. (Annie)

[The proof using multiplication] is formal enough for me and for a child, too. But maybe it's not what you were after. (Nathalie)

Conclusion

Although the observations, which we have recounted here, do not constitute systematic, in-depth research into the difficulties presented by division by zero, they nevertheless allow us to formulate a hypothesis as to one of the causes of these difficulties. Namely the conflict between the need for physical models to aid in understanding mathematical concepts and the absence of such a model in the case of division by zero. We wish to take up each of these contributing factors in turn.

The absence of a physical model for division by zero

The concept of zero itself does not have any satisfactory physical model. The common belief that zero is 'nothing' is an ambiguity at best. As to the operation of division, its physical models.[6] In a representation of multiplication, one has merely to reinterpret the product as the dividend, one of the two factors as the divisor and the other as the quotient in order to come up with a representation of division. For example, the area of a rectangle is the product of the length of one side times the length of the other side. Hence, the length of one side equals the area of the rectangle divided by the length of the other side. In the case of division by zero, this representation would make us look for a "rectangle" one of whose sides measured zero!

The most frequently encountered model of multiplication is the union of a number of disjoint, equipotent sets. Owing to the different roles played by both factors — the number of sets being united and the number of elements in each set — this model of multiplication gives rise to two distinct models of division: partitive division and measurement (or quotitive) division. From the first perspective, dividing by zero amounts to trying to picture the dividing of a set into 'zero equal subset(s)', or the cutting of a string into 'zero equal piece(s)'. From the second perspective, dividing by zero brings into play the splitting of the set into 'subsets containing zero element(s)' (actually, we should write 'subset' in the singular because the empty set is unique) or the cutting of a string into pieces measuring '0 meter(s)' in length. We are dealing here with situations that defy not only logic and the imagination, but grammar as well!

Models based on the notion of an operator are scarcely of greater help. If division by n is represented by a 'function machine' that produces a token *each time it is fed n tokens,* division by 0 ought to be represented by a machine which produces one *token each time it is fed zero token(s).* But what does that mean? If feeding zero token(s) is construed as not doing anything, such a machine would have to produce tokens without end! At what rhythm or rate would it do so?[7]

The reader is free to pursue this exercise and apply other models to division by zero. In each case, what comes to mind is some kind of 'non-sense' reminiscent of zen koans (e.g., imagine the noise made by one hand clapping). These instances of non-sense are traps which drive the mind to hunt for meaning where none is to be found and which feed the suspicion that mathematics is an activity which departs from common-sense logic. The way out of such an impasse is to abandon representations of division, which have no effectiveness in the case of division by zero (nor in many other cases for that matter). To do so, however, requires awareness that one is dealing with mere representations having a limited range of effectiveness and not with complete, faithful images of division. We now turn to the difficulty involved in adopting such an attitude.

Attachment to physical models

A number of elements suggest that the formal demonstration of the impossibility of dividing by zero did not entirely convince Nathalie and Annie. The comments which they made during the interview, their turning to a mathematical law, and the hesitation over the result of dividing by zero which they manifested from beginning to end of the interview. All the same, these students seemed to have grasped the argument quite well. What, then, accounts for their dissatisfaction? And how is one to explain that only 15% of the students who answered the written test drew on this demonstration, despite having studied this topic (and not for the first time at that) only two weeks previously?

Obviously, there is the difficulty which is inherent to any indirect proof and the difficulty of accepting that the quotient of $7 \div 0$ does not exist in spite of the fact that similar operations — e.g., $0 \div 7$ or $7 \div 1$ — do indeed have results. Perhaps, too, these notions are only more or less integrated into the network of other knowledge and connections are too few or too weak, thus leading to what Simon (1993, p. 251) has called a very sparse "web of knowledge" and creating a feeling of insecurity. No doubt, like most of the prospective teachers interviewed by Ball (1990, p. 141), some of the students who participated in the written test showed evidence of a "rule-bound" understanding of mathematics. However, as we have shown, several others struggled to make sense of division by zero. We would like to suggest that these students' difficulties stem from the belief that the only, or the best, way to make sense of a mathematical concept is to relate it to a physical situation. Such a desire for physical images might well explain why 17 students were lured by the false arguments figuring in the exam question into opting for one of the incorrect answers, and why also another 11 students attempted to justify the right answer by arguments of the same type. Ball (1990, p. 142) made a similar observation. She wrote:

> *In answering the questions, many of the teacher candidates agonized over not having a concrete example or not knowing why something was true. One of the secondary candidates, for example, in answering the division by zero question, said she 'would hate to say it is one of those things that you have to accept in math' but that she might have to in this case if she couldn't think of a concrete example.*

It is not our intention to cast doubt on either the origins of arithmetical operations in the physical world or the usefulness of representing such operations by concrete or graphic means. We believe, however, that it is crucial to remain vigilant and avoid confusing arithmetical operations with their physical models, for the latter reflect only a portion of mathematical reality[8]. Concrete representations inevitably come bearing irrelevant limitations or elements and eventually become obstacles to understanding.

Division by zero is not the only case in which this happens. Attachment to the partitive and quotitive models of division, for instance, may explain a number of difficulties which arise in problems of division when the divisor is not a whole number or when the dividend is smaller than the divisor or the quotient (Fischbein *et al.*, 1985, Graeber *et al.*, 1989). According to Fischbein *et al.* (1985), such "primitive models" have become rooted in the mind and continue to unconsciously influence mental processes even after formal mathematical notions have been acquired. Our data suggest that a similar kind of influence may be at play in the case of division by zero as well, making prospective teachers uneasy with a purely formal approach.

Accepting that some mathematical concepts have no physical backing is a major step in the development of mathematical maturity, both individually and culturally. The search for physical interpretations is a search for meaning. Renouncing it requires a change in perspective concerning the nature of mathematical "objects" that is not easily accomplished. The historical argument surrounding the acceptance of negative integers and imaginary numbers is a good illustration of the extent of the difficulty involved in such a change.

The pedagogical implications of the above observations are not straightforward. Fishbein *et al.* (1985, p. 15) consider that "teachers of arithmetics face a fundamental dilemma". On the one hand, if they introduce multiplication and division through the primitive models, they create an obstacle to the learning of the formal concepts of these operations. On the other hand, if they avoid the models and adopt a formal approach, they *violate the most elementary principles of psychology and didactics.* The problem is not specific to the teaching of multiplication and division. Current pedagogical wisdom suggests that, whenever possible, mathematical concepts must be introduced contextually, by means of examples, models and concrete or graphical representations. However, these early representations tend to assume an overpowering position in the students' concept images and risk being confused with the concept itself. Furthermore, we believe that the widespread use of models predisposes students to expect that *all* mathematical concepts have physical counterparts.

The solution dos not lie in avoiding models, but in making students aware of the models' limitations. Sometimes, new learning must be built against previous learning rather than on it. Ideally, teachers should be able to gauge their students' progress and to decide when the time has come to help them discard or revise previously learned ideas. In order to play this role; of course, the teachers themselves must be well acquainted with the path that must be followed and with the obstacles that must be overcome. In the case of division by zero - and of division in general - some of the prospective teachers who participated in our study did not yet have such a competence.

Following Simon (1993, p. 250), we believe that in order to teach division, it is not enough to have a concrete contextualized grasp of this operation. Elementary teachers must also be able to examine division as an abstract, mathematical object independent of all contexts. As teacher educators, we cannot take such a level of understanding for granted. Some of our students need to be helped through a change of perspective concerning elementary mathematical concepts, including the arithmetical operations. In particular, they must be made aware that no model of division fits all division problems and that some mathematical ideas, such as division by zero, have no physical representation.

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Notes

1 However, in the particular case of $0 \div 0$, any number multiplied by zero produces zero. Thus, there are an infinite number of solutions to this problem.

2 Partitive division consists in dividing up a set (or a length) into a given number of equal parts, with the quotient representing the cardinality (or the measurement) of each of the parts.

3 Like partitive division, measurement (or quotitive) division also consists in dividing up a set (or a length) into equal parts, with the difference, however, that the cardinality (or measurement) of each of the parts is given and the quotient is now represented by the number of parts.

4 It is always possible to draw on the measurement model of division to divide 7 cookies into packets of ½ a cookie, but the same does not hold for packets of -3 cookies.

5 This question is adapted from Ball (1990).

6 For a description of a number of different representations of multiplication, see for example Nantais, Francavilla and Biron (1994).

7 The image of a machine spewing tokens endlessly, the search for the number of pieces of 0 length that can be cut from a string. Or the study of the pattern formed by $7 \div x$ as x decreases, might lead one to think that the result of division by zero is infinity. Indeed, in some contexts this answer would be correct and among calculus students it would probably be a frequent response. None of the students who answered the written test mentioned the idea of infinity, possibly because of the elementary school setting in which the question was framed. Annie and Nathalie, on the other hand, came close to this idea toward the end of the interview when they attempted to compute $7 \div 0$ by repeated subtraction. They realized that they would have to subtract zero "an infinite number of times," but, lacking familiarity with the notion of infinity, they did not recognize it as a possible answer to their problem and gave up on this approach.

8 The opposite holds, too, of course, in that mathematical models represent only a portion of physical reality. Throughout his book, Pimm (1995) offers many provocative thoughts on the "uneasy links between the material world and mathematics" (p. 168).

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Farer ved å bli for nær knytt til det konkrete: Et tilfelle med divisjon med null.

Med utgangspunkt i eksemplet med å dividere med null tar denne artikkelen opp til diskusjon hvordan avhengighet til konkrete representasjonene av matematiske begreper kan bli en hindring for å nå denne matematiske forståelsen. Artikkelen beskriver noen vansker som lærerstudenter opplever i sitt arbeid med divisjon med null. Det blir understreket hvordan disse vanskene kan forklares ved studentenes ønske om en (ikke-eksisterende) fysisk modell for å tolke denne matematiske utfordringen.

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