Litteraturanmeldelse

Approaches to Algebra
Perspectives for Research and Teaching

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Bednarz, N., Kieran, C. & Lesley, L. (eds.) (1996) Approaches to Algebra. Perspectives for Research and Teaching. *(Mathematics Education Library, Volume 18). Dordrecht, NL: Kluwer Academic Publishers, 345 p.*

Teaching and learning algebra has received great interest within general research on teaching and learning in recent years. This is probably connected to the difficulties encountered by students in school algebra, but also to the discussion of what is the core of algebra. The students' misunderstandings and problems in finding the meaning of school algebra raise important questions concerning the transition to algebraic thinking. How does algebraic reasoning function? What characterizes the algebraic mode of work, and what situations may be conducive to its development?

This book addresses these issues. It is a collection of 21 articles written by distinguished persons in the area, and refers to a rich body of research and trials of material. We find substantial discussions, analyses and research reports, and we find some critical comments on articles which can enliven issues and stimulate progress.

The main part of the book looks more closely at some approaches to algebra which are aimed at giving meaning and understanding, and which are considered in contemporary research. These approaches are given as 1) generalization especially of numerical or geometric patterns, 2) problem solving by forming and solving equations, 3) modeling real situations, and 4) the study and use of functions.

These different main approaches to algebra in education are investigated, and the outcome from these approaches are analyzed. Research on these four perspectives have addressed the question of the development of algebraic thinking by a dual focus on epistemological and didactical concerns. The history of algebra enlightens the issue of what is algebra, what are characteristics of algebraic reasoning, what have been the core problems and how are

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they solved. Further didactical theoretical issues are raised and discussed, like the students' meeting with algebra, and situations are indicated which can promote development of algebraic thinking.

Those four perspectives – generalization, problem solving, modeling and functions – are the four main parts of the book. The first part, however, is a historical perspective on algebra, giving a background which focuses on its roots. The book ends with part six on synthesis and directions for future research.

Historical perspectives

There are several reasons for including a part on historical perspectives. It is important to reflect upon what algebra is, and what constitutes its characteristic features. History might correct and balance an educational view of algebra, for example by showing the close historical connection between symbols and their meaning, and the role of geometry in the historical development. The historical development may even give ideas for teaching. Further, the study of phenomena observable in the history of mathematics and phenomena involving a rupture in models and languages, are important for our understanding of students' difficulties.

Louis Carbonneau discusses relations between geometrical and algebraic aspects in a historical perspective - from Euclid to Descartes. Equality of magnitudes and equalities of ratios play a significant role in Greek mathematics, and they have a different nature. Thus there is an interaction between analysis, the core of which is "suppose the problem is solved, what do we know about the solution..", and synthesis, which is activated in verification and proof. "Algebra", Carbonneau concludes: "is not only an extension of numerical domain, not only a question of symbolism, it is a way of expressing and manipulating relations, and analysis is at the heart of algebra. The school curriculum has something missing: high school students know arithmetic of numbers, without knowing arithmetic of magnitudes. How can they understand the beauty of algebra?" he asks.

Luis Radford continues by showing some historical roots of our main modern elementary algebraic concepts, like that of the significant ideas of the unknown and of the variable. Algebraic procedures developed slowly: in Babylonian algebra, an informal "cut-and-paste"-geometry has been the basis of solutions of equations, so have the methods of false position and proportional thinking. Radford gives premises for the question of the role geometry and arithmetic could play in the teaching of basic concepts of algebra, to what degree "cut-and-paste"-geometry, false position, proportional thinking and the distinction between the unknown and the variable may be used as approaches in the classroom.

Teresa Rojano points out the separation students tend to make between manipulations of algebraic expressions and their use in modeling and problem solving. History emphasizes how the creation of symbols was connected to their meaning. This shows the didactical risk of teaching symbol manipulations as an object of knowledge in itself, which hides the significance of its origins and the semantic background of its grammar. The potential of the algebraic symbolism and syntax should be appreciated. Students cannot do that until they have reached the limits of their prior mathematical knowledge. One area where they may reach such a limit is in the search for generalizations and patterns.

A generalization perspective

John Mason points out the richness of the generalization process as an approach to algebra: to construct a formula may rest on visualisation, manipulation of the figure on which the study is based, formulation of a recursive rule showing how the next term follows from the preceding ones, finding a pattern that leads to the formula, comparing and discussing different equivalent formulas. Generalizing is closely linked to justification and proving.

> It is essential to draw attention out of the manipulated and particular and into the general if mathematical thinking is to take place.

There are also educational risks. There is a temptation to rush from words to a single letter, also a temptation to see the generalization narrowly as making a table, seeing a pattern, trying it out in one or two cases. A profound mastery of a single example may, on the contrary, be useful, it may serve as a generic example from which the general can be read. Mason's helix model connecting the triad: *manipulating - getting a sense of - articulating* may be useful in research and teaching practice. It clarifies steps in the process of mathematical generalizations, and algebraic thinking is embedded in this area.

Lesley Lee is concerned with how to introduce people into the culture of algebra, with its language, thinking, activities and agreements. Work on rich starting tasks, in an active, communicating group, may lead to activities which motivate the use of algebraic symbolism. Even more, the algebraic symbolism may come as a real clarifying help in the task to generalize, prove and communicate. It may give power and courage to carry on the work to search for more relationships.

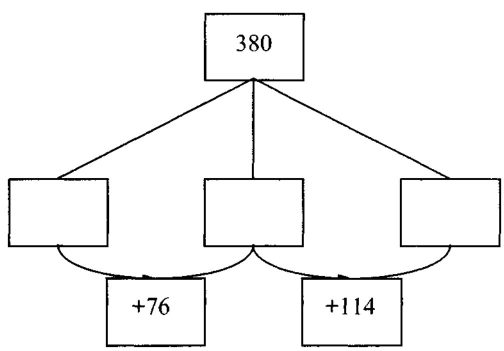
Why should students make generalizations? How can we evaluate generalizations in a didactical setting, what is relevant? The additional logical element in the classroom is crucial. When studying a pattern like 1, 4, 7, 10,... we might write the numbers as 1, $1+3\cdot 1$, $1+3\cdot 2$, ... leading to the goal, $1+ 3\cdot x$. Thus we will meet the issue of representation. We must also be conscious of what kind of algebra can be reached by generalization, as *Luis Radford* comments.

A problem solving perspective

Forming and solving equations out of a context demands certain abilities. Underlying the path from applications of rules to more specific skills there is a variety of conceptions. By analyzing historical work by Diophantus, al-Kwharizmi, Cardano and Viete, these conceptions may be revealed. It is also interesting to look into the solution strategies by students, thus getting a tool for describing the transition from arithmetic to algebra – what conflicts, adjustments, and new constructions are involved – related to different types of tasks. That is done by *Nadine Bednarz* and *Bernadette Janvier*. A look into their empirical research work reveals very significant issues. The problems they used may be exemplified by this:

> 380 students are organised in three sports groups. Basketball has 76 more students than skating, and swimming has 114 more than basketball. How many students are there in each of the three groups?

The problem, which is represented by the framework:



is easily varied. By changing the given relationship in the bottom boxes (using addition or multiplication) and alternating the known and the unknown (open) boxes, they are able to study the solution strategies by the students, the rate of success, and the way of passage to algebraic thinking in relation to the type of task. Some problems students will find more difficult to solve arithmetically, which may motivate a transition to algebra. This analysis gives a tool for studying the difference in strategy: 1) solutions going from the known to the unknown (arithmetic) and 2) solutions starting with the unknown and its relationships (algebraic). The use of a spreadsheet environment in the work with these kind of tasks seems to stimulate the development of algebraic strategies; the unknown then is represented by a cell, as documented by *Teresa Rojano.*

Most questions used in such research concerns daily life. They are, however, still artificial in the context. This fact seems to me to be a challenge also in research.

Is the transition to algebra a smooth evolution from arithmetic of whole numbers, or is it something new, a new world, with necessary intellectual conflicts everyone will meet? *David Wheeler* asks this basic question, and indicates that if it is, this doesn't contradict the imperative to take account of what the students already know.

Alan Bell find the categorizing of algebraic tasks given by Bednarz and Janvier interesting, but shows that it is necessary to have a fuller system. Bell also poses the crucial question: how should we, in teaching, combine manipulative skills (in solving equations) and authentic problem solving? There are indications that students are very reluctant to formulate equations until they have substantially experienced that such equations are solvable. However, the exploration of generic problems and their extensions may provide with a mathematically authentic mode of activity through which all of algebra can be learned. I would agree with Mason that Bell has great breadth and depth of experience of constructing tasks for pupils, particularly connected with the emergence of algebra, and a few of these are presented in this chapter, for the benefit of the reader.

"What is the historical role of problems in algebra?" is the question discussed by *Louis Charbonneau* and *Jaques Lefebvre,* looking at the main steps in the development - from the brilliant artist without any theory, Diophantus, to the artist and theoretician, Viète, and his calculations with letters. This again puts algebra in the sharp light of history, which is often, as here, a valuable light for reflections on curriculum, teaching and learning.

A modeling perspective

Ricardo Nemirovsky introduces the notion of mathematical narratives, the story-making students do, often from interpreting graphs or tables of numbers. He analyzes the work of high school students which exemplifies aspects of narrative construction and relates it to the introduction of algebra. For example, a height - time graph emerging on the screen may give the students the oportunity to be concerned about concepts like speed, distance, increasing speed, comparing growth, maxima and minima etc. Creating situations where a given mathematical relationship is a model is an open and rich activity, and fundamental to mathematical modeling. It also gives ideas for the introduction to algebra. Students may have problems with local and global features of the graph, but *Kathleen Heid* warns us not to see students' interpretations as misconceptions too quickly. A good reminder!

Claude Janvier is concerned with students' mental processes when working on school algebra. This work relates to modeling. Thus the formulation phase in the modeling process is crucial for giving meaning to basic formulas. Also the way a letter is interpreted: as an indeterminate value like O and d showing the relation in $O = \pi d$, as an unknown magnitude like r in an equation like $\pi r^2 = 68$, as a variable as the r if it varies in $A = \pi r^2$, as a polyvalent name or place holder, quantity or magnitude, will call for different mental processes.

A functional perspective

Substantial efforts have been made in designing an algebra curriculum built on a functional technology-supported approach, exemplifying functions, variables, equations, inequalities, equivalence in the context of mathematical modeling.

Kathleen Heid, in her report on the Computer-Intensive Algebra curriculum, refers to empirical research which studies the development of mathematical understanding in such a curriculum, and reflects upon the pedagogical context in which the learning occurs. This kind of curriculum seems to lead to changed classroom activities but also to different roles and responsibility for the teacher and students: more responsibility to the students, and the teacher more as a catalyst, formulating questions for group exploration. The students must develop deeper understanding of the different representations of functions, the ability to work smoothly between them, using different strategies, either tool based or based on reasoning about classes of functions. The outcome of such studies is clearly relevant for future technology-intensive functional approaches to algebra. But are the real life problems also real for the students? And are the formal definitions adopted in algebra natural and reasonable? we may ask like Nemirovsky.

The empirical experiences by *Carolyn Kieran, André Boileau* and *Maurice Garançon* with a different focus are also interesting. They are using the CARAPACE computer program, "an environment to help in algorithmically solving algebraic problems in an evolving framework". Although there is very sophisticated computer software available now (symbolic manipulations, solving systems of equations etc.), the representations this software uses may not necessarily be easily accessable to students. Therefore there is a place for software of more specifically pedagogical character, based on didactical analysis and empirical research, aimed at understanding. Also many students get little benefit from the current algebra curriculum; this fact challenges educators to envisage a new approach. Thus we find here an approach where all students would be required to acquire a basis of algebraic knowledge and where some could go further. This is why such a program is of special interest in many educational contexts. But, focusing upon the pointwise graph and discrete aspect of functions as the software does, will the students be able to reflect upon global properties, on continuous variation of a function? Does it hide some of the richness of the functional idea, as it is often met in daily life?

Conclusions

David Wheeler ends the book by observing that there is no full consensus on what characterizes algebraic thinking. He states that students have problems in the school algebra as it is presented. The difficulties - are they mainly caused by the complexity of the subject or by the way it is taught? He doesn't know how to answer, although research on students' conceptions and errors has given more insight recently, and it seems to be possible to redesign curriculum and teaching in a benificial way. The separation into four approaches to beginning algebra seems artificial, however, all of them are needed in any algebra program, and possibly more.

This book is very rich in points of view on the introduction of algebra and in historical, didactic and research perspectives. It includes interesting articles for people with a variety of interests, working in mathematics education – from the design of empirical research to the search for teaching ideas in algebra. The historical component, or maybe corrective, is a clear virtue of the book. Reviewing the book which introduces the complex issue of algebra into learning environments and real learning, generates a humble hope that more people should experience the beauty of the subject.