Mathematics education as theoretical knowledge

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Didactics of mathematics (DM) is the theoretical part of our knowledge in mathematics education. Its connections with the mathematical sciences and with school mathematics determine the independent character of DM as a scientific discipline. Features of school mathematics such as its unique aims, the highly abstract nature and hierarchical construction of the material to be studied, and the varied kinds of educational activities lead to its specific character, situating it among the school subjects and making accepted theoretical conclusions exclusively applicable to mathematics education. The social character of DM generates the approaches to constructing categories within the discipline (inexact and "inaccurate" conceptions, exclusion principles in formulating absolute statements, plausible reasoning, diversity of proofs) and promotes the selection of adequate methods of research that are not typical of positivistic science (expert review, discussion, pedagogical experiment). The applied character of DM determines an appropriate methodology of research and efficient ways of overcoming contradictions. The concrete practice of education (partly in the form of experimental tests) gives teachers an opportunity to use "inaccurate" empirical methods, reasonable considerations, intuitive choice, and so forth. Some practical advice is given for discussing and conducting research projects.

Mathematics education as a field of social activity can be considered from various perspectives: as a creative art of education in mathematics, as appropriate mastery (know how), or as a science of the laws of such education. The choice of one of these perspectives predetermines a model within whose framework we search for the answers to urgent questions of the field. In the concrete practice of education, however, these positions are not absolute. Instead, they are mutually supportive: Wherever authentic scientific knowledge is not present, intuition and experience help out. Nevertheless, the everyday use of

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intuition and experience does not serve to establish the non-necessity or impossibility of theoretical knowledge about mathematics education, usually termed *Didactics of Mathematics* (DM).

DM as a scientific discipline lies at the intersection of various sciences. It investigates the process of educating children of a certain age and consequently is compelled to use the results and methods of developmental physiology, psychology, and general pedagogy. The study of that process occurs within a specific social and cultural environment that determines general philosophical approaches to education as a whole (the paradigm for transfer of social experience, the aims and goals of education) as well as the conditions for carrying it out (opportunities in society and in school). It requires a connection to such sciences as philosophy, theory of culture, sociology, and theory of schooling. DM does not lose its independent scientific status because it uses the methods of related sciences, any more than the use of numbers in physical calculations makes mathematics a part of physics. Moreover, the statements of DM could not be the direct consequence of any specific discipline without taking into account the influence of other disciplines and the features of mathematics as an object of study. One could not deduce general didactics from psychology, for example, or DM from general didactics.

The specific character of mathematics education

The question of the *specific character of mathematics education* in its connection to features of the subject of mathematics plays a principal role in determining the independent scientific status of DM. We limit ourselves here to analyzing mathematics education in general schools as the most conventional area for the application of DM.

The major appearance of the specific character of general mathematics education is in its *unique aims,* in which the formal goals of education (like the upbringing and development of a child) act on a par with the actual ones (mastering the purely mathematical content and the skills needed to apply mathematics to the solution of practical problems) (Khinchin, 1961, 1963/1968). Speaking figuratively, we teach mathematics in school not only, and probably not so much, for the sake of mathematics. The cultural impact of school mathematics education becomes comparable to the cultural significance of the mathematical science.

School mathematics is characterized by *highly abstract study material* that does not possess close analogues to other school subjects. The objects to be learned have indirect connections to reality. Paradoxically, the initial concepts of mathematics (like number, set, point, and space) are often more abstract than many of the derivative concepts. The majority of the mathematical models studied in school do not represent first-level abstractions, and the correspondence of such abstractions to reality frequently looks artificial. Moreover, the mathematical material resists attempts to give it some form of primitive life, just as an explanation of the rules for moving a knight in chess by making reference to real horse jumps is of little help to one's understanding of chess.

The logical organization of the mathematical sciences entails a *highly hierarchical construction* of school mathematics courses characterized by a strong development of links among subjects. To reach any particular branch, it is necessary to go a long way along the whole mathematical tree: It is impossible to encounter differentiation without mastering transformations of algebraic expressions, transformations without mastering the arithmetic of fractions, and fractions without a knowledge of the multiplication table. The design of other school subjects reminds one of a bush whose various elements have few interconnections (it is not necessary to know anything about the climate of Brazil to remember what the capital of Great Britain is).

Finally, school mathematics is marked by *a variety of kinds of activity,* all of which are necessary for mastering the material under study. This feature becomes especially evident when comparing mathematics with such subjects as history or geography, in whose study one kind of activity predominates, namely storing (memorizing). One should note that mathematical activity is highly instrumental. It permits one easily to transmit *patterns of activity* to students by presenting mathematical problems whose solutions involve these patterns (Firsov, 1982).

Thus, the field of application of our efforts – the school subject of mathematics – has a notably specific character, distinguishing it from the other school subjects and thereby making accepted theoretical conclusions applicable exclusively to mathematics education, but not to other disciplines. In other words, besides the facts, statements, and theories of general didactics, we can consider pedagogical facts, statements, and theories relating only to education in mathematics (but not to physics, history, the mother tongue, etc.). In particular, it is enough to recognize that DM claims an independent scientific status.

Mathematics and mathematics education

Unfortunately, it is difficult to find the mathematics in some articles in the field of mathematics education: The content of such articles might be related to any subject. In the two-word collocation *mathematics education,* the authors take into account the word *education* only. *Mathematics* seems to be something like a figure of speech or an add-on. That practice immediately removes such articles from this scientific field. Moreover, I believe that in specified cases even very valuable ideas cannot be realized in practice: General pedagogy and psychology penetrate into school through a subject with subject didactics as mediator and creator.

An opposite error can be found to no small degree in DM research. Proximity to the highest standard of theoretical knowledge – mathematics – frequently disorients beginners engaged in research into DM problems. Developing questions of DM, they attempt to copy the methods and design of mathematics. This proximity frequently forces even experienced mathematicians, the professionals, away from the right path, causing them to neglect any theoretical knowledge that has been produced in a way they find unusual. Using their own experience, many of them sincerely believe that pedagogical decisions should be accepted exclusively on the basis of intuition, experience, tradition, stable opinions, and not least, common sense.

A similar position appears to be connected to the misunderstanding that DM is constructed as an area of social and applied knowledge. The *humanistic character* of DM is connected to the fact that its object is typically the social process of a child's mastery of complex scientific knowledge. The person in society is one of the most complex objects for investigation. It is well known that the so-called hard sciences (mathematics, physics, and so on) do not have an adequate apparatus for describing and investigating this phenomenon: The person will not fit into a formal scheme. In the field of humanities, in contrast, appropriate procedures have been developed and certain methods elaborated that enable us to investigate and understand social objects and processes. The social character of DM generates approaches not typical of the hard sciences to constructing the categories of this discipline and to selecting adequate research methods.

Unlike mathematics, therefore, DM uses *inexact and "fuzzy" concepts* such as *development, creativity, understanding, problem,* and *spatial imagination.* The attempts to construct rigorous definitions for similar concepts that one finds in some papers generally create a poor impression. The introduction of concepts through informal description or examples oriented toward the context of their use is more natural for the social disciplines. Such an approach in DM is not at all a weakness; it is applied quite deliberately. The fact is that the use of inexact definitions allows one to operate with them in ill-determined "widespread" contexts that are typical of the social sphere. On the other hand, any refinement kills them insofar as it reduces the area of their possible application. Interestingly, one can encounter a similar situation in the most classical mathematics: The evolution of the concept of polyhedron in geometry is a brilliant illustration (Lakatos, 1967).

Negative principles

The absolute statements of social theories are formulated, as a rule, as negative sentences of the "do not" type; that is, they have the character of *interdiction principles.* The tradition dates back to the ethical systems of ancient times: It is enough to mention Biblical commandments like "Thou shalt not kill," or "Thou shalt not steal." However, a similar approach is sometimes characteristic of the most modern areas of science: The terminology, introduced by Pauli, is borrowed from modern physics. Moreover, what is the law of conservation of energy but an exclusion principle: This law does not say how a physical process in a closed system will continue, but it requires agreement between the quantity of energy the beginning of the process and at the end, which bars any processes not satisfying the requirement.

In the social sphere we never authentically know "how it ought to be"- theories laying claim to such knowledge are considered fairly well connected with violence and totalitarianism. But in a number of important cases we authentically know "how it ought not to be," and we express that knowledge in a form of a corresponding statement. Such a statement does not always have the external form of a prohibition: The general didactical principle "Education should be within children's capabilities," expresses the requirement, it seems, in positive form. Actually, however, it forbids one to make pedagogical decisions that do not satisfy the requirement.

Interdiction principles find a significant application in normative documents: study plans, curricula, standards, and so forth. All these documents impose some limitations such as "Allocate no fewer hours to mathematics than those specified in the study plan," "Teach an amount of mathematics no less than that designated in the curriculum," and "Achieve a level of mathematics preparation of students not below that given by the standards" (Firsov, 1995).

Logical reasoning in the didactics of mathematics

DM operates, of course, not only with interdiction principles, but also with positive statements. Such statements have the character of more-or-less reasonable hypotheses, which require appropriate proof. Naturally, the statements, formulated with the help of "inaccurate" concepts, appear inexact, which complicates the application of a strict logical proof within a social context. Correct figures of syllogisms, when applied to inexact concepts and statements, result in greater "inaccuracy," and in many cases distort the meaning.

Logical reasoning in the field of DM reminds one of building a wall out of bricks having amorphous and indistinct sides. Following the rules of logic, we lay one brick on another. Nevertheless, because of the "inaccuracy" of the sides, bricks are sometimes placed with a significant displacement of their centers of gravity, and as a result the wall collapses. In DM, therefore, logical proofs can be replenished by *plausible reasoning,* so well described by Polya (1954). Accordingly, purely logical criteria are supplemented by other criteria for verifying the statements.

No way of verifying a hypothesis used in DM is a proof in the strictly logical sense of the word. When one shifts to a logical proof, consistently constructed systems of reasonable conclusions usually emerge that lead one to an explanation of the hypothesis or a demonstration of its agreement with known facts. Such a verification certainly gives one a positive answer with some (most often an unknown) measure of reliability that is less than 1. Therefore, *a variety of proofs* of stated hypotheses becomes a major methodological requirement, enabling one to increase the reliability with which they are verified.

Expert review

Social practice frequently resorts also to the *expert examination* of a resulting hypothesis or intermediate statements used in drawing a conclusion. Formalized procedures for selecting experts are sometimes applied. The explicit formulation of statements, being subject to examination by experts and, on occasion, of appropriate criteria for evaluation increases the quality of that examination and, hence, the reliability of any conclusion drawn from it. Experience shows that the organizers of examinations too frequently neglect this work, proposing that the experts evaluate the result as a whole. As a consequence, the results of the experts' work appear not very comparable.

Discourse

Discourse is an interesting method of research, having no close analogue in classical and modern mathematics. Here speech occurs not as it arises spontaneously but as it yields a *consciously organized discussion* as an element of the methodology of social knowledge. The history of discourse ascended to the walks of Socrates and the dialogues of Plato, passed through the impressive period of theological disputes during medieval times, and can currently be seen in lively duels in the pages of newspapers and journals and on the platforms of congresses and conferences.

Over a long period, therefore, considerable experience has been accumulated in organizing discourse and in increasing its productivity. I note only some ways in which these purposes can be accomplished:

- Consciously include discourse in the plan for conducting research work instead of attempting to avoid discussion and reach consensus under the pretext of increasing the work's efficiency.
- Indicate clearly one's positions ("I declare the following") and intentions ("I am going to challenge the following") for discussion and critique.
- Analyze the initial assumptions and "shake" the statements so as to locate the weak spots in a piece of work.
- Encourage generous, constructive critiques. The discussant's opponent is not the enemy but rather his or her assistant. The adherents of a position frequently do not notice "lacunas" in their conclusions and justifications. Neglecting discussion while the work is underway, they encounter unexpected failure after its completion (the New Mathematics?).
- Analyze both the positive and the negative sides of a proposed decision, as well as its consequences. So frequently we see discussions in which the proponents of Approach A discuss the advantages of A and the disadvantages of B, while the proponents of Approach B simultaneously discuss the advantages of B and the disadvantages of C.

The researcher in the field of DM gains considerably from understanding the basic *dialectical character* of social statements. In this case, the researcher could easily resolve the paradox, concluding that many confirmed statements seem self-evident and even trivial (for example: "It is necessary to develop the interest of students in studying mathematics"). If one stops to think, however, the opposite statement will be obvious too. An analysis of the specified contradiction results in the need to determine the boundaries of didactical statements. Many also believe that a rather useful component of research work is the researcher's sense of humor (or awareness of its absence).

Pedagogical experiment

A pedagogical experiment usually serves as major method of verifying hypotheses, permitting the test of an appropriate hypothesis or its components. A pedagogical experiment places a theory in a practical field, in which positive and negative features of the theory are uncovered. Being an organic part of research, an *exploratory experiment* is especially valuable for these purposes. The qualitative analysis of the results of an exploratory experiment permits one essentially to refine preliminary research hypotheses and in a number of cases to put forward new hypotheses and claims.

In conducting a pedagogical experiment and analyzing its results, one should take into account a kind of pedagogical "Heisenberg effect," like that in quantum mechanics: The performance of an experiment changes the conditions of the associated educational process that "shifts" the conclusions drawn. One should also take into account the impossibility of completely reproducing the conditions of an experiment at another time: It is impossible to repeat the lesson anew in the same class.

To counter these phenomena, researchers resort to *randomizing* the conditions of experimentation (to control initial fluctuations), trying by this means to "quench" the influence of the experimental conditions or the changes in those conditions. The expansion of the scale of experiments (in my country's practice termed experimental implementation) aims at the same goal of eliminating the factor of experimental influence.

At the same time, the value of the quantitative analysis of experimental results is, in my opinion (the view of a professional mathematician), considerably exaggerated. The sophisticated apparatus of modern mathematical statistics is too frequently applied incorrectly. This practice, in particular, is promoted by the uncritical use of computer programs for statistical data processing that were developed for use in statistically homogeneous fields with known laws of distribution (for example, to process data from missile firings or the results of technical measurements). In DM, in contrast, statistical homogeneity and normal distributions are the exception rather than the rule. I do not reject at all, however, the necessity and utility of statistical processing of the data from pedagogical experiments. I merely consider it to be just one method of verification among many.

Didactics of mathematics as an applied discipline

Another important aspect of DM arises from its being an *applied discipline,* directed at the making of decisions suitable for use in the practice of mathematics education. The practice serves as a source of problems for DM: Its needs and its contradictions justify the existence of research on appropriate problems and the possession of research hypotheses. Practice in the form of confirmatory and exploratory pedagogical experiments becomes one of the main tools of research. Finally, practice represents the main criterion for the validity of the theoretical knowledge received.

The applied character of DM is displayed most clearly in the methodology of research typical of applied disciplines (Myshkis, 1971). The applied orientation of school mathematics education (Firsov, 1977) is revealed, in particular, through the use of this methodology to identify effective ways of overcoming contradictions arising in practice. A well-known joke contrasts the methodological differences of applied and theoretical disciplines:

> The theoretician shows how what is possible is necessary;
> the "applicatician" shows how what is necessary is possible.

The joke illustrates the obvious orientation of DM toward getting a result that is acceptable in practice. The potential claims of practice on the results of research serve as the main basis for formulating and conducting research. It is not accidental that in my country's experience of awarding scientific degrees in the field of DM, the topicality of the scientific problem of a thesis is justified, as a rule, by its potential contribution to eliminating defects or resolving conflicts in the real practice of mathematics education. Thus, the position of "art for art's sake," characteristic of much pure theoretical work, seems unnatural for DM.

Constraints on research

Practice-oriented research that answers a concrete practical need has to be conducted at a particular time. It is also limited by the financial, material, and personnel resources at the command of the researcher. Finally, an essential limitation in many cases is that many problems of DM and adjacent disciplines do not have a theoretical resolution. All this compels the researcher to move away from conventional methods of the "pure" theoretical disciplines, drawing instead upon empirical data, reasonable considerations, and in some cases, intuitive choice.

Researchers are often ashamed of these moments and shade them carefully in their research reports. Their discomfort arises from a misunderstanding of the special character of applied science, which cannot possibly wait until all questions that might arise are resolved theoretically.

Demands of practice

Schools need textbooks and appropriate teacher aids in mathematics. These materials must be ready at a specified time, certain means need to be allocated to creating them, there is a limited number of people in each country who can participate in the project, and so on. It is unreasonable to expect many of the theoretical problems of textbooks to be resolved in the near future. Nevertheless, projects to develop and produce textbooks can have a strong research character. For this purpose, scientifically approved methods and procedures that were developed in DM should be applied in carrying out these projects, and the empirical elements should be supervised.

It should be noted that similar schemes are characteristic of all applied disciplines, including applied mathematics. Many mathematical processes used in the calculations of, say, the path of a rocket or the design of a nuclear reactor do not have a complete theoretical substantiation (i.e., appropriate proof of convergence). Nevertheless, such calculations are made, the rockets fly, and nuclear reactors produce energy. When doing this work, experts in the field of applied mathematics do not have an inferiority complex before their colleagues in pure mathematics. The basic opportunity to use unproved statements and even divergent processes is connected to the fact that an abandonment of the absence of logical contradictions – the unique criterion of validity in pure mathematics – provides an opportunity to test experimentally the results of plausible reasoning. In DM, in a similar fashion, the *experimental test* (or more generally, practice), when included in the core of a study as its formative and supervisory component, permits one to operate with reasonable considerations and empirical data, to use the intuition of the researcher and the experience of the expert.

Some practical advice

Many centuries of experience in mathematics education give us numerous patterns, rules, and prescriptions that appear useful in carrying out research studies and introducing their results into practice. I mention only a few:

- Practice *reasonable pedagogical conservatism* (e.g., that of A. Lincoln), preferring the known and the approved over the new and the untested. We are surrounded by an ocean of ignorance about the processes of education (and of social life) that does not permit us to predict the consequences of our decisions with reasonable reliability. We accept them for the best reasons (remember where good intentions lead). Therefore it is reasonable for the researcher who offers her or his recommendations to schools to be careful and to follow the conventional slogan, "First, do no harm!"
- Understand the *evolutionary character* of school reforms and the inadmissibility of revolutions. A school system is highly inert. It is like a hundred-thousand-ton tanker crossing the ocean. If the captain turns the wheel sharply, the ship's speed will not change, but the steering system will probably break down. It is only possible to change the ship's course by making smooth and small turns of the wheel. Therefore, educational practice will reject highly revolutionary proposals coming from researchers.
- Appreciate the desirability of subsuming specific research under more *general didactical concepts,* which permits one to simplify the procedures for verifying its results. Moreover, the probability of making practical use of non-isolated research results is increased.
- Orient oneself to exploring *stable decisions* whose results may depend in part on fluctuations in parameters (conditions of education, teachers' qualifications, etc.). It is difficult and expensive to introduce into schools a recommendation based on unstable decisions. Experience also shows that didactical decisions that take into account "marginal effects," and therefore possess too narrow an area of applicability, do not "work," owing to their instability. At the same time, "rougher" recommendations may appear more efficient.
- *Harmonize the goals and means* in conducting research and in obtaining results. One should not use a micrometer for measur-

ing the size of the handle when manufacturing a hammer. It is also incorrect to select goals for whose achievement the researcher is not suitably equipped. In general, the widespread belief that the goals determine the means is dangerous: It results in a choice of inadequate means and in the making of unstable decisions in social realms. The essence of the opposite, "resource" approach is illustrated by the statement: Tell me what means you have, and I shall tell you what goals you can attain.

- Use a *method of iteration* in planning and conducting research. Experience reveals that it is quite difficult to determine in advance all the actions needed for conducting research, especially in the case of large projects. Some of the completed actions may appear unnecessary, which is only discovered during subsequent stages of work. Therefore, the researcher ought not to "lick clean" the parts of the project separately. A more reasonable approach is to try to execute the project as a whole first and, having made a first rough approximation, to make a critical analysis of the resulting decision and on its basis to make a more precise iteration, and so on. This advice applies even more to the preparation of research reports.

References

Firsov, V. V. (1977). O prikladnoy orientatcii obucheniya matematike [On the applied orientation of mathematics education]. In *Uglublennoe izuchenie algebry i analiza.* Moscow: Prosveshchenie.

Firsov, V. V. (1982). The basic trends in the improvement of mathematics teaching in the present days of Soviet school. In B. Wilson (Ed.), *Second Anglo-Soviet Seminar on Mathematics Education* (pp. 22-27). London: British Council.

Firsov, V. V. (1995). K kontceptcii proekta standartov [To the concept of a standards project]. In *Moskovskie obrazovatelnye standarty. Obyazatelnyi minimum soderghaniya obrazovaniya. Proekt.* Moscow: Moscow Department of Education.

Khinchin, A. Y. (1961). O vospitatelnom effekte urokov matematiki [On the educative effect of mathematics lessons]. In *Matematicheskoye prosveshchenie* (No. 6). Moscow: Fizmatgiz.

Khinchin, A. Y. (1968). On the educative effect of mathematics lessons. In B. V. Gnedenko (Ed.), *The teaching of mathematics: Essays by A. Y. Khinchin* (W. Cochrane & D. Vere-Jones, Trans.; pp. 77-101). New York: American Elsevier. (Original work published 1963)

Lakatos, I. (1967). *Dokazatelstva i oprovergheniya* [Proofs and refutations]. Moscow: Nauka.

Myshkis, A. (1971). Chto takoe prikladnaya matematika [What is applied mathematics]. In *Problemy prepodavaniya matematiki vo vtuzakh* (No. 1). Moscow: Vysshaya shkola.

Polya, G. (1954). *Mathematics and plausible reasoning.* Princeton, NJ: Princeton University Press.

Matematikundervisning som teoeretiskt kunnande

Sammanfattning
Matematikdidaktik (MD) är den teoretiska delen av vårt kunnande om matematikundervisning. Dess samband med de matematiska vetenskaperna och med matematik som skolämne formar den självständiga karaktären hos MD som vetenskaplig disciplin. Utmärkande drag hos skolmatematiken som dess unika mål, den höga abstraktionsnivån, den hierarkiska uppbyggnaden av det stoff som skall studeras och de varierande slagen av aktiviteter ger ämnet dess speciella karaktär. Ämnets natur och struktur ger det dess plats bland andra skolämnen och gör godtagbara teoretiska slutsatser exklusivt användbara i matematikundervisning.

MD's sociala karaktär genererar ansatser som utvecklar kategorier inom disciplinen (oprecisa och "felaktiga" begrepp, uteslutningsprinciper för att formulera bestämda satser, trovärdiga resonemang, mångfald och bevis) och är vägledande för urval av lämpliga forskningsmetoder som är ovanliga i positivistisk vetenskap (expertbedömningar, diskussioner, pedagogiska experiment). Den karaktär av tillämpad vetenskap som kännetecknar MD kräver en speciell metodologi och effektiva vägar att hantera motsägelser. Den konkreta undervisningpraktiken (delvis i form av försöksverksamhet), ger bl a lärare tillfälle att använda "mindre formella" empiriska metoder, göra lämpliga överväganden och intuitiva val.

Artikeln avslutas med några praktiska råd att diskutera och synpunkter på hur man kan leda och genomföra ett vetenskapligt projektarbete.

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