Open ended problem solving in geometry

Hans Erik Borgersen

This article is a documentation of a project on open ended problem solving in geometry at Agder College. The project is part of a geometry course and is based on co-operation in small groups. Why we did it, how we did it, how it developed, and what came out of it is the main core of the article. Our conclusions are:

- *It is possible to have students experience the whole process of doing mathematics.*
- *Geometry is a rich resource for finding problems to do so.*
- *It is possible to create learning environments where students co-operate and give each other support to go through the process together.*
- *It is possible to do this within ordinary courses in the first year at college/university.*
- *There has been observable progress on process writing and evaluation. The students' attempt to generalize and formulate new problems are not satisfactory. The emphasis on process writing and evaluation seems to have an effect on the students' motivation and understanding of doing mathematics.*
- *The students experience a great need for encouragement and support in the process of doing mathematics.*
- *Co-operation in small groups is highly recommended by the students.*

Introduction

During my sabbatical year in 1982/83 at the University of California, Santa Barbara, I worked closely with professor Paul J. Kelly in geometry and professor Julian Weissglass in mathematics education. Kelly's close friend and colleague professor Ernst G. Straus had given a nice analytical solution to an elementary geometry problem. We tried to give a geometric proof. This was the beginning of an exciting period of investigation and discovery, of generalization and connection to other problems and applications. In short, it was amazing how much real mathematics came out of it. It also gave us personal pleasure and inspiration (Borgersen, 1984a, 1984b, 1987, 1993).

I will call this activity, starting with an elementary problem in geometry and going through the whole process of doing mathematics, open ended problem solving in geometry. In my opinion there are

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two basic challenges in mathematics teaching at all levels. One is to have students experience what mathematics is (that is to give a true picture of mathematics and of doing mathematics) including to experience mathematics as a process giving full respect to the students way of thinking and need for time. These factors create pleasure, inspiration and confidence in doing mathematics. The other challenge is to create a learning environment of acceptance, safety and trust, which encourages the students to take responsibility for their own learning, for their common learning environment and for them to use their full potential in the learning process.

In school, geometry has traditionally been presented deductively. Our experience is that open ended problems in geometry are excellent for exploring the creative and intuitive part as well. Doing open ended problem solving in elementary geometry requires lots of work with concrete materials (models, drawings and calculations), and with trying, testing and qualified guessing. It is possible to reach interesting and deep results at all levels of knowledge without a battery of tools. Starting with an open ended problem and ending with a written paper, you are forced to go through the *whole* process and to use yourself and the mathematics you have assimilated.

Both Kelly and Weissglass encouraged me to try out this open ended problem solving method in geometry in my own teaching, which I have done since 1986 through a project as part of a geometry course at Agder College. Kelly, Weissglass and I have corresponded frequently since 1982/83 on various topics related to this work. Both of them have been an important inspiration. Kelly has contributed with lots of problems to the project. The problems we have used in the projects are part of the mathematical folklore. Beside Kelly, our main sources have been Coxeter & Greitzer (1967), and Honsberger (1978).

At Agder College we organize our teaching of mathematics as a blend of lecturing and problem solving in small groups (Dahl, in press). The students are responsible for the work and progress in the small groups and the teacher is available for questions. Small group teaching promotes the students to get to know a core of fellow students very well, to learn by co-operation, and to get into better contact with the teacher. For the teacher, it is a way to get to know the individual students and to get ongoing feedback. It is our experience that small group teaching is a good instrument to create learning environments of acceptance, safety, and trust.

In order to fit our mathematics program (how it is organized) and how the students are encouraged to learn mathematics, we based our open ended problem solving project on co-operation in small groups.

In several countries, through the eighties, problem solving has become an important part of mathematics curriculum in public schools. (Cockcroft, 1982; California State Department of Education, 1985, 1992; Kirke- og undervisningsdepartementet, 1987; NCTM, 1980). In addition, co-operative learning in small groups has become a very much advocated way of learning mathematics (Davidson, 1990; Johnson & Johnson, 1987; Johnson et al., 1988; Slavin, 1985; Weissglass, 1993). Our way of organizing co-operative work in small groups differs from most of the studies described in the American literature (Dahl, in press). Some main differences are related to how the groups are established, the number of students in the groups, the responsibility of the students for the work and progress in the groups, the grading of the groupwork, and the fact that the groups have separate rooms and are not in a classroom situation. Both in preservice and inservice programs, there is a great need in mathematics teacher education to emphasize open ended problem solving and to experience co-operative learning in small groups (Bekken et al., 1978; NCTM, 1989, 1991; Schoenfeld, 1990). This article is a contribution to this need.

Research on problem solving

There is a large amount of research literature on *problem solving* in general. If we restrict ourselves to problem solving in mathematics, and more specifically to geometry at the college/university level, a different picture emerges.

In 1945 George Polya published his famous book "How To Solve It" (Polya, 1945), in which he introduces the four-stage model of the problem solving process: *understanding the problem, devising a plan, carrying out the plan,* and *looking back.* These and more detailed strategies are rules of thumb for making progress on difficult problems. Polya called it "modern heuristics". The ideas were explored at much greater length and depth in the two volumes of "Mathematics and Plausible Reasoning" (Polya, 1954). Polya claims:

> Mathematical facts are first guessed and then proved. ... If the learning of mathematics has anything to do with the discovery of mathematics the student must be given some opportunity to do problems in which he first guesses and then proves some mathematical fact on an appropriate level. (Polya, 1954, Vol. II, p. 160)

This book concentrates on developing the students' problem solving ability. In the two volumes of "Mathematical Discovery" (Polya, 1962, 1965), Polya focuses on the concerned teachers and students of high school or early undergraduate mathematics. He offers problems from all areas of mathematics and theoretical descriptions of the problem solving process.

From late 1970's there has been a renewed interest and continuation of Polya's work on problem solving. The state of the art in research on mathematical problem solving for all ages and all levels of mathematics is well reflected in Silver (1985). For problem solving at the university level Schoenfeld (1985), is a main book of reference today. The two major questions Schoenfeld asks are:

> What does it mean to "think mathematically"?
> How can we help students to do it? (Schoenfeld, 1985, Preface)

Schoenfeld's work is based on Polyas's analysis but goes well beyond his ideas and methods. While Polya mainly used himself (his own knowledge and reflection), Schoenfeld has developed research methods for observing students during problem solving sessions, and he has collected a series of empirical studies as a documentation for his analysis. Schoenfeld's theory is a framework for the analysis of complex problem solving behaviour:

> It consists of four qualitatively different aspects of complex intellectual activity: *cognitive resources*, the body of facts and procedures at one's disposal; *heuristics*, "rules of thumb" for making progress in difficult situations; *control*, having to do with the efficiency with which individuals utilize the knowledge at their disposal; and *belief system*, one's perspectives regarding the nature of a discipline and how one goes about working in it.
> ...My interests in understanding and teaching mathematical-thinking skills go hand in hand. My research indicates that when instruction focuses almost exclusively on mastering of facts and procedures, students are not likely to develop some of the higher-order skills necessary for using mathematics. It also indicates that when teaching focuses on these skills, students can learn them. (Schoenfeld, 1985, Preface)

Both Polya and Schoenfeld developed specific courses and programs on individual problem solving at college/university level.

Our project is based on the same philosophy as expressed in the quotations of Polya and Schoenfeld. We have expanded Polya's four-stage model into seven steps, and in addition we emphasize cooperative work in small groups, and we organize our problem solving as part of a geometry course. Our experiences agree with Schoenfeld's statement that when teaching focuses on higher-order skills necessary for using mathematics, students can learn them.

This project may be looked at as integrating elements from different fields of research in Mathematics Education. It emphasizes *discovery and investigation* (open ended *problem solving*) in *geometry* as *part of a geometry course* at *college/university level* based on *co-operation in small groups.* This article is a documentation of the project. Why we did it, how we did it, how it developed and what came out of it, is the main core of the article.

The project framework

Geometry (MAT 6) is a course in the second semester of a one-year mathematics program at Agder College, Kristiansand, Norway. The course has three main parts:

1. The history of geometry with emphasis on Euclid's "Elements".
2. Plane, real projective geometry.
3. Modern Euclidean geometry (transformation geometry) with applications on symmetry.

After 2-3 weeks of recalling the classical geometry, the students start their project work in groups of about 5. The project lasts for one month, and the groups work in parallel to the teaching in class, which is now on projective geometry.

In the first meeting the students are supposed to make a schedule for at least one meeting a week and how to work in between meetings. They read through the problems and make decisions on which problem(s) to choose. The first meeting is at the same time for all groups and I'm present for consultations. Later, the groups have to make an appointment if they want to discuss the problems with me. The groups are expected to write a log on their meetings and their way of organizing the groupwork. During each meeting the groups are supposed to write about their work and how the problem solving develops, obstacles and openings, and every idea and suggestion that comes up. This is called *process writing.* At the end, the log and process writing as well as *proof writing* are put together in a report, summing up the work and results. The students are asked to comment on how they feel about doing the project and if it made sense to them. It is emphasized that describing the whole process is the important thing and that the students should try to reflect on that. All students are expected to participate, that is all students have to belong to a group, and all participating students in a group are to sign the final group report.

After the reports are submitted I read all the reports and give each group my written feedback. The feedback is mainly encouraging, focusing on the positive aspects of the report. If there are serious mistakes or misunderstandings, the group is invited for additional discussions with me. In entire class lectures I give more general feedback on the groupwork (collaboration, process writing, proof writing, evaluation, ...) and on the mathematics (nice ideas and proofs, standard mistakes and comments on unsolved problems). The students are also encouraged to give their feedback.

The groupwork is not graded. The whole project is looked upon as part of the learning process in the course. It is an attempt to have the students experience a more true picture of doing mathematics and to motivate further (hard) work in the course. During the final exam the students are tested individually, and one part of the exam is on problems of the type given in the project.

Open ended problem solving

In preparing the students for the project I select different problems that we work through on our own and together in class. One such example is the following problem.

> ***Best place on Stadium***
> Given a soccer field or handball field.
> You have a ticket for "the long side" of the field.
> Which place is the best for watching the goal of your home team?

To give you an idea of what I expect the students to ask and look for, lets study this problem briefly by going through seven main steps which are an expansion of Polya's four-stage model.

Analyzing and defining

What is the situation and what is the problem? Do we understand the meaning of the words? How do we understand (define) the situation and the problem?

This problem is given in a context that students easily recognize. It is very important to be explicit and for the students to agree on how the problem is defined. In teaching mathematics it's important to make connection to previous (and assimilated) knowledge. In problem solving it's important to put a given "pure" problem in a context that makes it familiar ("applied"). Does the problem make sense to me? Is it possible to find a context where it does? These are important questions in analyzing a problem.

A critical question in the given problem is: What do we mean by "*best* for watching ..."? If we don't agree on that we will probably

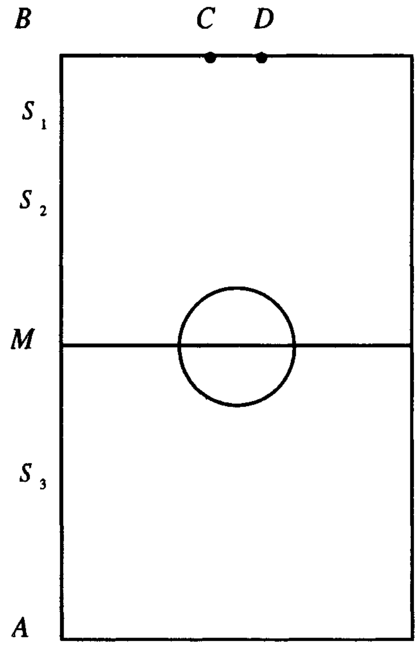


Figure 1. Which place on AB is the best for watching the goal CD?

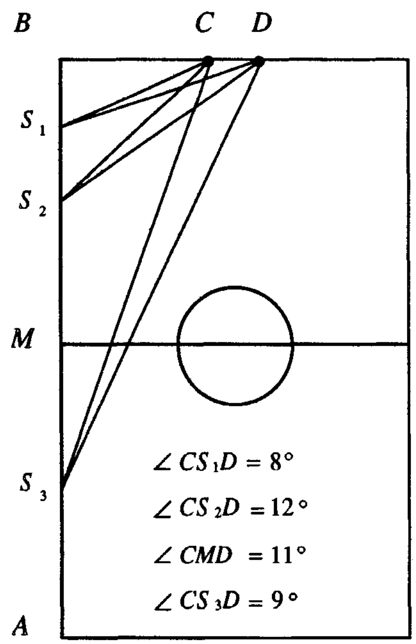


Figure 2. From which place on AB do you see the goal CD under the largest angle?

work on different problems. In the following, I define "best for watching ..." to mean "seeing the goal under the largest angle". (To be precise, the seeing is by one eye.)

Modelling or drawing

The students are encouraged to find or make a model of the situation. They can visit an actual playing field or make any illustration of a field e.g. a drawing on small scale. To make a drawing seems very important at all levels of problem solving in mathematics. It may be part of the analysis, and it's an important way to "get started" and to watch for "trails" to investigate.

Qualified guessing by trying and failing

Having a model we can investigate by trying and failing. We choose some points on the long side and measure the angles towards the goal. Doing so we probably get a better understanding of the problem. We start guessing on the best position, doing more measuring, reguessing and so on. We are building up an intuition on the problem. And our next guess is based on our experience and intuition so far, which we call qualified guessing.

Finding hypothesis

In the process of qualified guessing we start looking for patterns and ideas that may give us an argument for our stand. We draw and investigate different "fields", "long sides" and "goals", and we apply Polya's advice: "Go to the extremes". At this level we look behind the context and ask for the essential parts and concepts in the problem. We make the problem a pure mathematical problem (taking away the "noise" from the context). While doing this we try to formulate (postulate) a general solution, which we call a hypothesis. This is still a guess and requires more testing, which may conduct us to moderate, reformulate or reject the hypothesis.

Development of a proof

In the process of formulating a hypothesis we may or may not have seen a complete argument, but hopefully we have got some ideas of how to prove it. If we succeed, the hypothesis becomes a theorem. In my experience the choice of method and tool for a proof are very personal matters. For some of us it is natural to use the language and methods from analysis and algebra, and for others it is more natural and appealing to find geometric proofs. In many cases it is a mixture of those. It is interesting to experience ones own preferences, and it is important to know that there are many different ways of proving a theorem. It is ones own understanding of the problem and ones own operational mathematics that may give a proof. Sometimes we don't come up with any proof. It may be too difficult, our hypothesis may be wrong or we simply need more time. It's common in doing mathematics, to put problems aside. We just give it a rest or ripening. It's still in our mind. We are looking for new perspectives, new methods, deeper understanding of the concepts involved and growth in our own ability to do mathematics. Now and then we make a new study of the problem. This time we may succeed or we have to put it aside again. This process is part of doing mathematics. And most often you are grappling with several problems at the same time.

It's important that students understand and experience these procedures as normal. It's normal *not* to solve every problem. Now and then you succeed and those are great moments! Moments to appreciate and enjoy. But the students also need to learn to appreciate and enjoy the whole process of having problems and processing them. How to do this is one big issue in teaching mathematics.

Going back to the given problem it may help the experienced student to ask: "What type of problem is it?" We are trying to maximize an angle given that the legs go through two fixed points and the vertex varies along a line. Maybe we can express the angle-measure analytically as a function and then use standard techniques from analysis to reach an optimum. Well, this is an interesting idea/route to follow.

I will now follow another track. From classical plane Euclidean geometry (Theorem of Thales, – 600) we know that a circle through two fixed points is the locus of the vertices of equal angles with legs going through the two points. Angles with vertices inside (outside) the circle are larger (smaller) than those with vertices on the circle. Now, given a circle through the given points that intersects the line in two points, Figure 3. The angles with vertices on the line inside the circle are larger than those with vertices on or outside the circle.

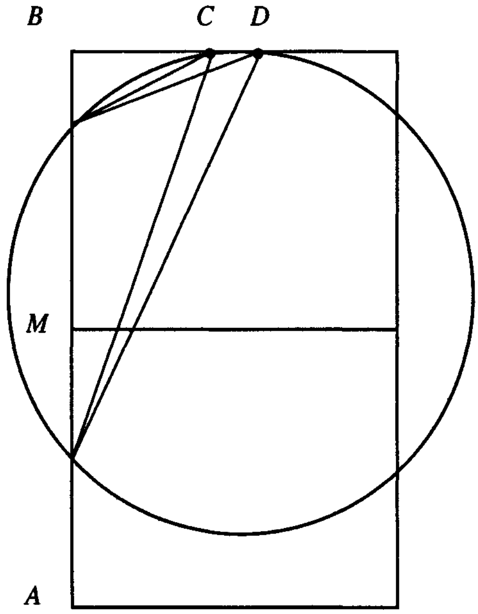


Figure 3. There are two places on AB where you can see the goal under the same angle.

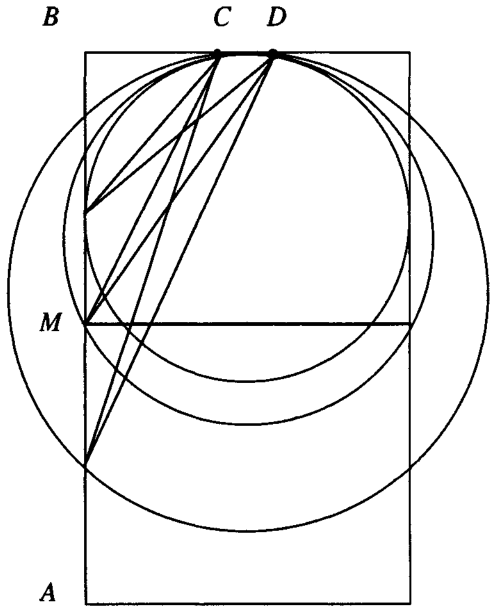


Figure 4. Different circles – different angles.

This argument shows that the point of tangency (that is, where a circle through the two given points is tangent to the line) is the one that gives the largest angle, Figure 4.

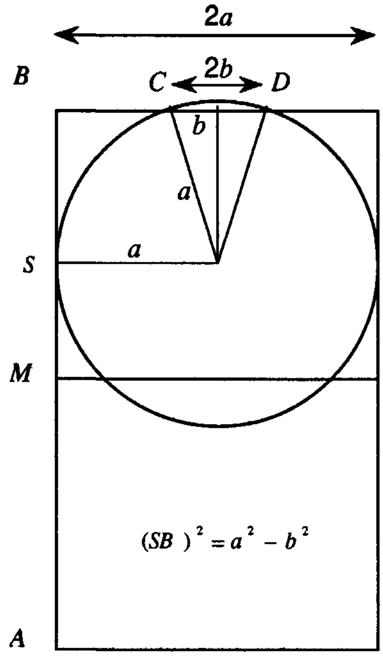
At this point the students are expected to write a proof as part of their report. But they are encouraged to work on the proof throughout the project, both on the method chosen and on the written presentation.

Characterization of the solution and interpretation

It's a good feeling to see and be able to give a proof. The method chosen here gives us a geometric characterization of the solution. And it is usually a happy time to find algebraic expressions for the solution and, as in this case, to make a nice drawing and construct the circle that gives the point of tangency. Doing this you are also focused on looking for other (equivalent) characterizations. At the end we go back to the given problem and interpret the solution in the language of the given problem.

Unifying ideas, generalizations and applications

For students who have worked on calculus in several variables it's satisfying to see the geometric proof as nothing but a geometric



$(SB)^2 = a^2 - b^2$

Figure 5. An algebraic expression for the solution.

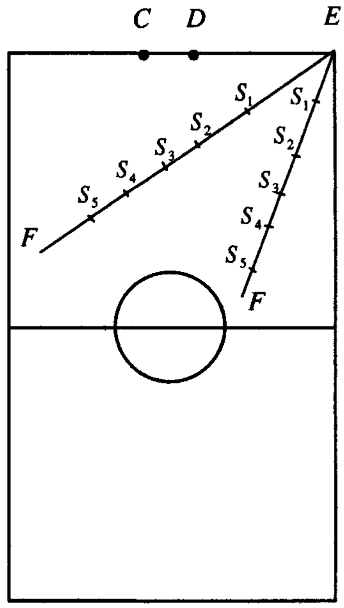


Figure 6. Varying the given problem. Which position is the best for the player to shoot?

solution to the problem of maximizing a function in two variables (the angle measure as a function of the vertex) constrained by a linear condition on the variables (the vertex of the angle is on a given line).

But youngsters familiar with reading maps can also see the same unifying idea: The family of circles through the two given points is "the curves of height", which expresses the levels of equal angles. In this terrain you are allowed to move along a track which on the map looks like a straight line. Walking along this track you are ascending (descending) as long as the line is crossing a circle, and you are on the top of the track when the line is tangent to a circle.

Having seen this, you have understood the underlying idea of attacking a whole class of optimization problems, which applies to many different situations.

And it's worthwhile to go back, varying the given problem, for example ask for the best position for a player to shoot if he runs along a line directed towards the corner flag. What about other curves than straight lines? Are there other situations where this method could be applied?

Questions follow questions which are fascinating outcomes of doing open ended problem solving. And most important: it is the students themselves that ask and create new problems.

Remarks

In our problem "Best place on Stadium" no solution was given. It had an open end. Other problems are given as theorems. Still, the students are expected to go through the different steps in their investigation of the problem searching for a proof. The process will generate lots of questions and new problems, which the students are encouraged to investigate. Even if the students don't succeed in proving the theorem they can have good understanding of it and apply it in other problems or in real life situations.

It should also be mentioned that most of the problems are given as pure problems without an applied interpretation.

The 1986 and 1990 projects

In this section we present the problems given in 1986 and 1990. For each year the number of students, the number of groups, and how long the project lasted are presented. We also give short characteristics by placing a code on the final group reports, namely if it includes a log (L), analysis and drawing (A), process writing (P), proof writing (C), generalization or asking new questions (G) and evaluation of the group work (E). By log (L) we mean a short description of the time schedule the students followed. Comments are made whether the project problems were appropriate, as well as on the final group reports.

The coding system is not the same as the seven steps in our problem solving model. The four middle letters (A, P, C, G) are the ones that primarily involve the model:

- A represents the first two steps,
- P represents the third and the fourth steps,
- C represents the fifth step,
- G represents the last two steps.

These four codes are less detailed descriptions of the seven steps. L and E are codes that involve the co-operating small group work and the students' own view of their outcome of the project.

- L represents the working plans of the groups and how they have been carried out
- E represents the students' evaluation of the group work and the students' reflection on their own learning

From 1986 to 1990 there have been two important changes. We started by giving one problem, the same to each group. And the project lasted for two months. For the students to have a choice we started in 1988 to give more than one problem, but still the same problems to each group. The students are supposed to choose a certain number among the given problems. From 1989 we reduced the duration of the project to one month, which seems to be more appropriate.

The 1986 – project

Twenty-seven students in 6 groups co-operated on the following problem in two months:

The Chord problem
You are supposed to study two arbitrary, intersecting circles. A line through one of the intersecting points meets the circles in two other points, which determine a line segment. The length of the segment varies as the line varies (rotates about the given point of intersection). Investigate what happens with the length of the segment as the line varies. Has the segment a smallest or largest length? If so, find the positions and the lengths of the segments which give these extremes.

Table 1. The group reports from the Chord problem.

Group	Number of students	Characteristics of the final group reports *					
1	7		A	P	C		E
2	5		A	p	c		
3	5	L	A	P	C	G	
4	4		a	p	c		
5	3	L	A	P	C		
6	3	L	a		C		
Sum	27						

* *log (L), analysis and drawing (A), process writing (P), prewriting (C), generalization or asking new questions (G), evaluation of the group work (E). Small written letters have the same meaning as capital letters with the appendix "a little of".*

Comments
This problem worked very well for open ended problem solving. It turns out to be necessary to define precisely what is meant by "the segment" as the line varies, especially when the endpoints collapse.

The problem is excellent for investigation by drawing, measuring, and going to the extremes. It's not too difficult to make interesting hypothesis and to prove them. It's more difficult to generalize the problem or to find applications without any guidance from the teacher. But the problem is challenging and highly motivating in itself.

All group reports contain analysis (A) and proof writing (C), all except one had process writing (P), but only one group evaluated its own groupwork (E), saying:

> The group functioned well and it was a fine and reflective way of working. As future teachers we are concerned about this, not only to be served, but to investigate and explore by ourselves.

The 1990 – project

Thirty-five students in 6 groups cooperated on at least two of the following problems for 1 month:

Problem 1

Referring to "the best place on Stadium-problem" you are supposed to investigate "the best position for a player to shoot as he moves on a line towards the cornerflag."

Find the position on this line for which the angle towards the goal is largest.

Find the distance from this position to the corner flag given that the width of the field is 60 meter $(2a)$ and the width of the goal is 7.32 meter $(2b)$.

Discuss how to construct (by compass and ruler) the optimal position.

Problem 2

Let D be the circumcircle of a triangle ABC. The incircle of triangle ABC has centre I. The line $\ell\ (C,I)$ intersects D in M.

a) Compare AM, IM and BM. Which is largest?

b) The perpendicular on AI through A and the perpendicular on BI through B intersect in J. Investigate the position of J.

c) Use your imagination and your knowledge from a) and b) to make more discoveries.

Problem 3

Given a circle C with centre O and radius R. Let P be a point in the plane and ℓ a line through P which intersects C in X and Y.

Prove that the product of the length of the segments PX and PY is constant (as ℓ varies). This constant is called the power of P with respect to C. Try to express this constant by means of R and d, when d is the distance from P to O.

Try especially to express this constant by means of the length of the segment PT, when P is an outer point (with respect to C) and the line $\ell\ (P,T)$ is tangent to C at T.

Problem 4

Given an isosceles triangle ABC with angle measure 80° at A and B. The point D on the side BC is given such that the angle measure of BAD is 50°. The point E on the side AC is given such that the angle measure of $\angle ABE$ is 60°. The line $\ell\,(E,D)$ intersects the line $\ell\,(A,B)$ in F.

Investigate this figure by finding angles and isosceles (sub-) triangles. Try to find the angle measure of $\angle BEF$ and $\angle AFE$.

Table 2. The group reports from the 1990-project.

Group	Number of students	Problem no chosen 1	2	3	4	Number of problems	Characteristics of the final group reports *			
1	7	x	x			2	L	A	P	C
2	7	x	x		x	3	L	A	P	c
3	7	x			x	2		A		C
4**	3			x	x	2		A	P	C
5	6	x	x	x		3	L	A	P	C
6	5		x	x	x	3	L	A	P	C
Sum	35	4	4	3	4	15				

* *log (L), analysis and drawing (A), process writing (P), proof writing (C), generalization or asking new questions (G), evaluation of the group work (E). Small written letters have the same meaning as capital letter with the appendix "a little of".*

** *Group 4 worked on Problem 1 the first week, but then moved to Problem 4. Their own comment was: "We shouldn't have done that".*

Comments

Problem 1 and 2 were good choices for open ended problem solving, as Problem 3 in combination with one of those above. Problem 4 attracted most of the students. They worked hard on it and got frustrated on the last question. None of the groups came up with a convincing argument. It may have been better to make Problem 4 optional after having chosen two of the three first problems.

All group reports contained analysis (A) and proof writing (C). All but one contained process writing (P) and all but two contained log (L). None had generalization (G) or evaluation (E). As a curiosity group 5, which only had male students chose ancient runes as symbols for the angles in Problem 2. Group 6, which only had female students, on the other hand used colours only to differentiate and describe angles in Problem 2. The group reports were of remarkable even and good quality. In previous years, there also have been good or even better reports, but the variation in quality was much greater.

The 1993 – project

During my sabbatical year 1991/92 at the University of California, Santa Barbara, the project, as given from 1986 to 1990, was presented in the Teacher Leader Program at the Department of Mathematics and in a seminar on Mathematics Education at the Graduate School of Education. In the succeeding discussion, I was recommended to continue developing the project and to emphasize processing in lecture sessions throughout the project period. That is, I should have the groups sharing their experiences and evaluating their work. The idea is to have the students learn from each other and to stimulate further work in the groups. Processing is highly recommended by the Reform Movement in California (California State Department of Education, 1992). I decided to follow this advice in the 1993-project.

After presenting the problems I describe the processing in more detail. First I comment on the outcome of the emphasis on processing. Then I refer to parts of the evaluation given by each group. In the following section we quote some of the students presentations in plenum (the whole class).

The problems

Thirty-four students in 10 groups co-operated on Problem 1 and on at least one of the remaining problems.

Problem 1

Given a triangle ABC. Let Φ be an excircle of ΔABC, i.e. Φ is exterior to ΔABC having all three sides for tangents. Let Φ be interior to $\angle BAC$ and $\ell(A,B)$ a tangent to Φ in M.

Compare the length of AM and the perimeter of ΔABC. Choose other triangles and make similar measurements, formulate a hypothesis and try to prove it.

Problem 2

Given an angle with vertex A and legs m and n. Let P be an (arbitrary) point interior to the angle. A line ℓ through P meets m in B and n in C. We are going to study different positions for ℓ. Find the position for ℓ such that

a) $|AB| = |AC|$

b) P is the midpoint of BC,

c) BC is tangent to the incircle of ΔABC in P,

d) ΔABC has minimal perimeter,

e) ....

Problem 3

A (one eyed) person, standing 30 meters from a straight lined road, is observing that the distance between the front lights of the passing cars seems to vary as the cars approach.

Investigate this by trying to find a position such that the measure of the angle from the persons eye to the front lights of a car has a maximum.

Table 3. The group reports from the 1993 – project.

Group	Number of students	Problem no chosen 1	2	3	Number of problems	Characteristics of the final group reports *					
1**	2	x		x	2	l	A		c	g	E
2	6	x	x		2	L	A	P	C		E
3	6	x	x	x	3	L	A	P	C	G	E
4	5	x	x		2	l	A	P	C		E
5	5	x	x		2	L	A	P	c		E
6	5	x	x	x	3	L	A	p	C	G	e
7**	2	x		x	2	L	A	P	C		E
8**	1	x	x		2	l	A	P	c		E
9**	1	x	x	x	3		A	P	C	G	E
10**	1	x	x		2		A	P	C		E
Sum	34	10	8	5	23						

* *log (L), analysis and drawing (A), process writing (P), proof writing (C), generalization or asking new questions (G), evaluation of the group work (E). Small written letters have the same meaning as capital letters with the appendix "a little of".*

** *Three students worked on their own due to practical reasons. They are listed as group 8, 9 and 10. Group 1 and 7 had only 2 members each because they started the project out of order.*

Comments

All the problems functioned very well for open ended problem solving. Some groups had difficulties understanding Problem 1, i.e. to read and analyze it. By discussion all but one group (an individual) reached a correct understanding of the problem. All but two groups had good process writing. Four groups had no proof. One group (an individual) formulated new problems related to or stimulated by the problems given.

Problem 2 was chosen by eight groups. It has gradually tougher subproblems and an open end for generating new problems. All groups had good process writing or proof writing in a) and b). This was also true for five groups in c). Two groups formulated a hypothesis with proof in d). One group (an individual) used the computer program GeomeTriks to investigate the problem trying to formulate a hypothesis. (This was also true for Problem 1.) Two groups formulated new problems:

Find the position for ℓ such that

a) $|AB| : |AC| = 1 : t$ for a given positive rational number t,

b) P divides BC such that $|BP| : |PC| = 1 : t$ for a given positive rational number t,

c) The incircle of ΔABC is maximal,

d) ΔABC has minimal area

e) $\dfrac{1}{|AB|} + \dfrac{1}{|AC|}$ is maximal.

The group proposing e) proved that

$\dfrac{1}{|AB|} + \dfrac{1}{|AC|}$ is constant when P is on the angle bisector of $\angle BAC$.

Problem 3 was chosen by five groups. Four groups gave a nice proof by recognizing the problem as "best place on Stadium" (by letting the position of the car be fixed and varying the position of the person). Two groups formulated new problems solvable by the same method. One group formulated a hypothesis (which was incorrect) without testing it at all. All group reports but two contained logs, and all groups evaluated their work. Only three group reports contained generalization or formulation of new problems.

Processing

Throughout the project we did processing in plenum, once a week. A representative from each group gave a short report on the group's work that week, followed by a discourse. In the following, we present a summary of the outcome of the four processing sessions.

The first week (the first day)
The groups were asked to present their working plan for the project and to report how they had got started. The following was observed:

All groups made a working plan and an appointment for a new meeting the next week. Three groups had difficulties understanding Problem 1, especially the term "excircle" (which was read as "circumscribing circle"), and "the interior of $\angle BAC$". All groups formulated a hypothesis in Problem 1. No groups proved their hypothesis in Problem 1, but instead put it aside and looked at Problem 2 or Problem 3. No groups expressed anything about emotions or feelings. One student had already solved the problems (which were given the day before). The group members wanted some time before "they were told the solutions" and decided to work individually for one more week. We talked about different ways to proceed. The groups made their own decisions.

The second week

The groups were asked to discuss their last meeting, the co-operation and progress on the problem solving. The following was observed:

Several groups had difficulties with Problem 2 and put it aside for the next meeting. Some expressed that they were stuck and needed to find another approach. Co-operating in small groups is useful and nice: "Two or more people think better than one." "It is easy to see mistakes and to find alternative ideas." "It's also a help when reaching difficulties in problem solving. It is not so easy to give up!" "Process writing is difficult when working individually between the group meetings." "Do you expect us to write about all mistakes? What do we write down and what do we drop?"

The group which delayed their first meeting decided to continue as one group. The students focused on their difficulties or mistakes and almost nothing on what they had managed or on personal feelings connected to co-operative problem solving in small groups. I stressed that it is important doing mathematics (as everything else) to appreciate the moments of (small) successes.

Responding to my questions: The students felt comfortable talking about their project in plenum. The students seemed not to be aware of the learning effect by listening to the experiences of the other groups.

The third week

The following was observed:

All groups had their meetings. All but one group made good progress. Due to a misunderstanding about their schedule this group had little time for problem solving. Sickness and heavy loads in other mathematics courses reduced the number of students in some groups. Three groups proved their hypothesis in Problem 1. One group reported that co-operative group work was interesting and promoted learning. Two groups finished the given problems. They discussed what is meant by generalizations and generating new problems.

The fourth week
We had no session in plenum. The groups had an extra group hour to finish the project. The seven ordinary groups delivered their reports on time. The three individuals received one extra week. The processing was finished by a meeting in plenum after the groups received their reports back.

Evaluation

In this section I have translated parts of the evaluation of each group into English.

Group 1: It's positive that all members have to be active and think constructively. You learn to ask questions and to argue. Working in a group has been a stimulating way of learning mathematics, especially having the opportunity (time) to make your own investigations and discoveries.

Group 2: Working in a group has been fun but also hard. We have learned that it is worthwhile 'not to give in'.

Group 3. Negative experiences:
It has been time-consuming. We have had lots of work (and problems) in other mathematics courses as well as in MAT 6 (Geometry) in addition to the project. It has been hard to stay motivated when reaching difficulties.
Positive experiences:
To see several possible solutions. Good to have the project early in the course. It's worthwhile to put a problem aside and return to it later on. Intuitive thinking is important.

We are mostly satisfied working in groups. It's a nice way to help each other. We have had problems analyzing the problems, and we missed some equipment (ruler and compass for the blackboard) in the group rooms. The project has been instructive and inspiring.

Group 4. It has been a nice way of working together. It's easier to see mistakes done by others, and it's positive all being on the same level. It's easier to discuss problems in small groups. We are not quite sure what we have learned by the project. Maybe it's too much work compared to the outcome and the total work in MAT 6. Our basic knowledge for solving the problems was too poor, especially for proving.

Group 5. It has been very instructive to cooperate in a group. Five heads gave lots of ideas, lively discussions, and very often a solution.

Group 6. It was important that everyone was well prepared and that we used the blackboard in the group sessions to keep all group members active.

Group 7. We were too few (only 2). Our recommendation is to have 4-6 students in each group and to do all problem solving without working individually outside this group.

Group 8. Working individually has not been beneficial, but I have managed OK, and it has been instructive.

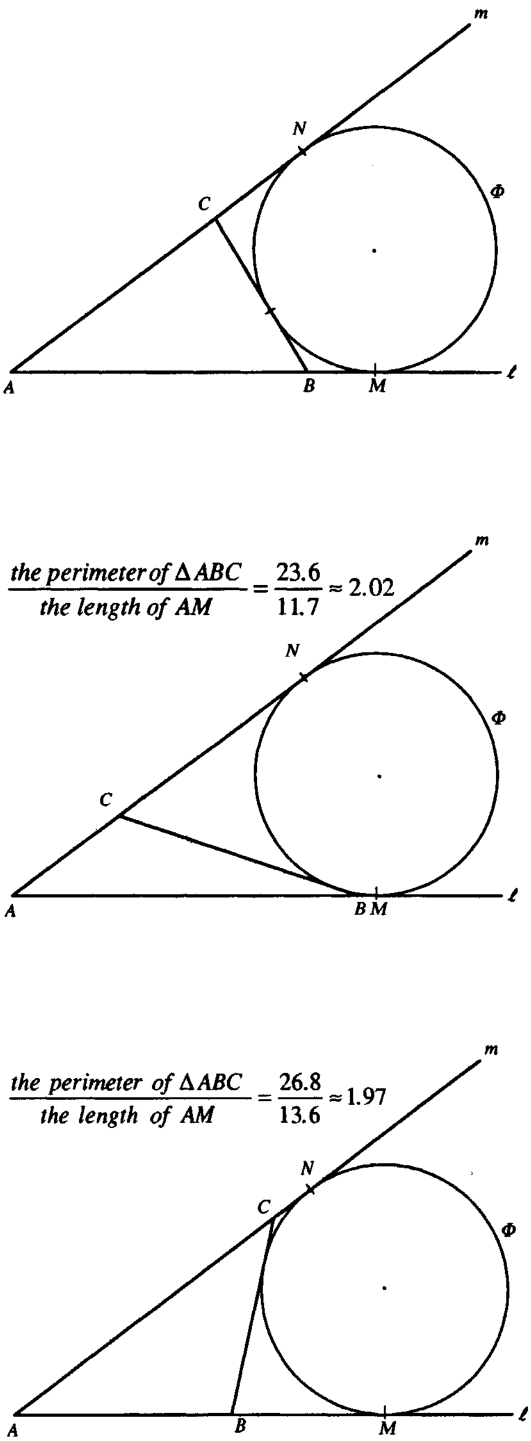
Group 9. It has been an enjoyable project. I have worked with the problems over a long period of time and continually discovered new ways of looking at the problems (to attack and solve the problems). At last I had to decide "it's good enough" and hand in the report. Using the computer for word processing and making drawings has been good. I have also learned to use the program GeomeTriks in a meaningful context. Working on Problem 2 d) I missed being in a group, as well as when I tried to generalize and generate new problems.

Group 10. I missed to participate in a group; to discuss the problems, to be encouraged and inspired and to learn from other group members. Working individually, it's also very difficult to find different ways of looking at a problem. My basic knowledge was not good enough. I have experienced geometry as many-sided, but interesting.

Comments

The students used words such as "stimulating", "fun", "instructive", and "inspiring" to describe their experiences when working on the project in small groups. Negative experiences were expressed by the words "hard" and "time-consuming". The students experienced that it is hard to stay motivated when reaching difficulties and they experienced that working in groups is a nice way to support each other. All being on the same level, it was easier to discuss problems and find solutions. The students experienced that in problem solving it is worthwhile "not to give in" and "to put a problem aside and to return to it later". To conclude, the students experienced the important roles of motivation and their own attitudes in the process of problem solving. They also reflected on what is good for their own learning.

Figur 7. Three different triangles from the report of Group 7 on problem 1.



Presentations in plenum

The 1993–project was finished by two sessions in plenum where:

i) I commented on the group reports and the group evaluations in general terms.

ii) Some problems were presented by the students. (The groups appointed a member to do this after having been asked in advance to prepare such a presentation.)

Problem 1 solved by group 7

Group 7 agreed to have their report on Problem 1 presented on an overhead. They asked to be excused from presenting it themselves.

Problem 1 (Translated from the report of Group 7)
Φ is in the interior of $\angle BAC$ tangent to both $\ell\,(A,B)$ and $\ell\,(A,C)$. Since Φ is an excircle to ΔABC it's also a tangent to $\ell\,(B,C)$ and exterior to ΔABC.

Let M be the point of tangency between Φ and $\ell\,(A,B)$. The problem is to find the ratio of the length of AM to the perimeter of ΔABC.

We started to experiment with different triangles, establishing data for making qualified guesses. We also went to the extremes of a triangle to control our hypothesis.

Let the circle Φ and the tangents AM and AN be fixed. Moving B towards M, C will move towards A, that is $|AB| \to |AM|$ and $|AC| \to 0$.

Then $|BC| \to |AM|$ and the perimeter converge to $2\,|AM|$. Therefore

$$\frac{\text{the perimeter of } \Delta ABC}{\text{the length of } AM} \to \frac{2|AM|}{|AM|} = 2$$

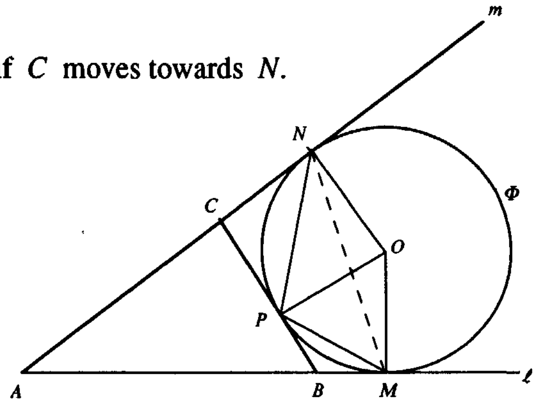
By symmetry the same will happen if C moves towards N.
We conclude with our first

Hypothesis:

$$\frac{\textit{the perimeter of } \Delta ABC}{\textit{the length of } AM} = 2$$

i.e.

$$\textit{the perimeter of } \Delta ABC = 2 \cdot |AM|$$



Figur 8. A new drawing in the seeking for similar triangles.

In our attempt to prove the hypothesis we made figure 8 looking for similar triangles. We didn't succeed but discovered certain line segments to be congruent. It looks like $|BM| = |BP|$ and $|PC| = |CN|$.

We continued to look for equal angles (in isosceles triangles). From here, our thinking process is shown in our proof.

Proof

ΔMNO, ΔPNO and ΔPMO are isosceles triangles because two sides in each triangle equals the radius of the excircle. A tangent to a circle is perpendicular to the radius in the point of tangency. Therefore ΔAMN is isosceles because $\angle AMN$ and $\angle NMO$ are complementary and $\angle ANM$ and $\angle MNO$ are complementary. In a similar way ΔPBM and ΔPCN are isosceles.

Therefore, we have $|AN| = |AM|$, $|BM| = |BP|$ and $|PC| = |CN|$.

$$\begin{aligned}\text{The perimeter of } \Delta ABC &= |AB| + |BP| + |PC| + |CA| \\ &= |AB| + |BM| + |CN| + |CA| \\ &= |AM| + |AN| \\ &= |AM| + |AM| \\ &= 2\,|AM|\end{aligned}$$

Comments

It's a nice piece of work, and it's representative of most of the groups. Also, by having no reflection on a unifying idea such as the congruence of the two tangents from an exterior point to a circle, although they prove it (three times) and use it in the proof. The report has no generalization or attempt to formulate new problems or result.

Five solutions of Problem 2b

To have the students experience that a simple problem may be looked at and proved in different ways, *five* groups were asked to present their proof of Problem 2b.

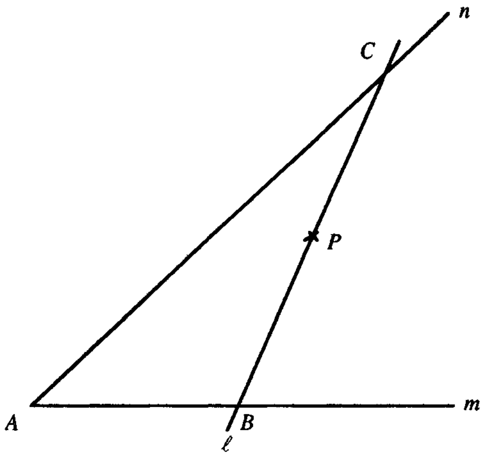


Figure 9. Problem 2 b

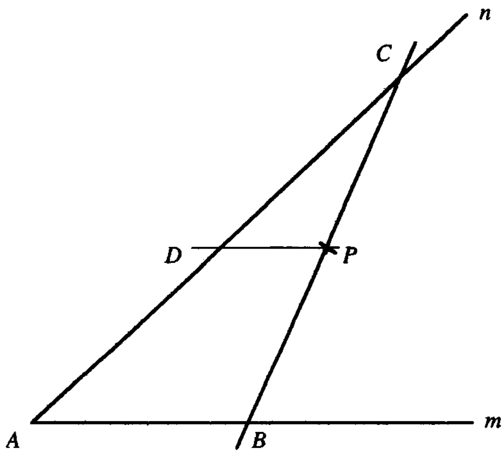


Figure 10. The drawing of Group 2 on problem 2 b.

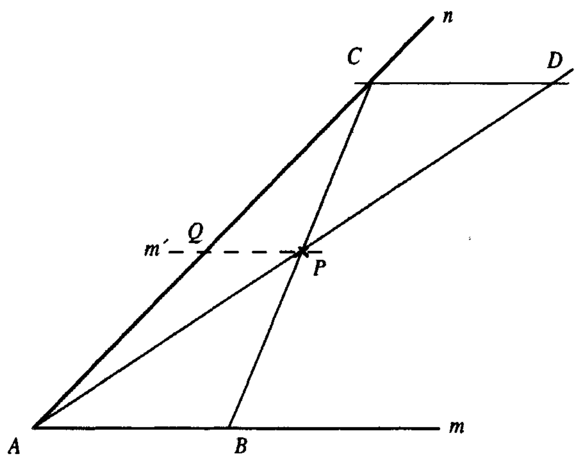


Figure 11. The drawing of Group 3 on problem 2 b.

In Problem 2b an angle with vertex A and legs m and n is given. P is an arbitrary point interior to the angle. A line ℓ through P meets m in B and n in C. The problem is to find (construct) the position for ℓ such that P is the midpoint of BC.

Proof by Group 2
Construct a line through P parallel to m, which meets n in D. Choose C on n such that $|AC| = 2\,|AD|$. Let B be the point of intersection between m and $\ell\,(P,C)$. Then $\Delta\,ABC$ and $\Delta\,DPC$ are similar and the ratio between corresponding sides is 2:1.

Therefore, $|BC| = 2|PC|$, and P is the midpoint of BC.

Hypothesis by Group 3
Construct a point D on $\ell\,(A,P)$ such that $|AD| = 2 \cdot |AP|$. The parallel to m through D meets n in C, and the line $\ell\,(P,C)$ meets m in B. Then P is the midpoint of BC.

The group had no proof, but it became obvious that the arguments given by Group 2 above would work here as well:

Let m' be parallel to m through P. Then m' intersects n in a point Q and $\Delta\,APQ$ and $\Delta\,ADC$ are similar. Therefore Q is the midpoint of AC, and as in the proof of Group 2, P is the midpoint of BC.

Another idea is to rotate 180° about P. Then the image of C will be on $\ell\,(C,P)$ and m. Therefore, the image of C is B, so $|PC| = |PB|$. An argument by congruence like the one given by Group 10 also works.

Hypothesis by Group 5
The perpendiculars from P on to m and n intersect in D and E respectively. Choose a point F on m and G on n such that $|AF| = |PE|$ and $|AG| = |PD|$. Then the line ℓ parallel to $\ell\,(F,G)$ through P gives B and C.

Through lots of examples the group found this hypothesis which they believed in, but had not been able to prove. It became a challenge for all of us. We will return to this later.

Proof by Group 6
If P' and C' are the feet of the perpendiculars from P and C onto m respectively, then $\Delta CC'B$ and $\Delta PP'B$ are similar and $\frac{|CC'|}{|PP'|} = \frac{|CB|}{|PB|} = 2.$

Therefore $|CC'| = 2|PP'|$. This proves the following construction of B and C.

Make a parallel m' to m in the distance twice the distance from P to m. m' intersects n in C, and $\ell\,(P,C)$ intersects m in B.

Proof by Group 10
Let Q be the point on $\ell\,(A,P)$ such that $|AQ| = 2\,|AP|$. Then P is the midpoint of AQ.

The parallels to m and n through Q intersect n in C and m in B respectively. The quadrangle $ABQC$ is a parallelogram. If the diagonals intersect in a point P', then the triangles ABP' and QCP' are congruent because $|AB| = |QC|$, $\angle BAP' = \angle CQP'$ and $\angle ABP' = \angle QCP'$. Therefore $|AP'| = |P'Q|$, so $P' = P$ and $|BP| = |PC|$. P is the midpoint of BC.
Notice: This is a short version of the proof given by Group 10.

Comments

The students were surprised to see that even a simple problem could produce such a diversity of arguments. The proofs of Group 2, 3 and 6 are based on similarity, and the proof of Group 10 is based on congruence. The hypothesis of Group 5 seems to be based on a different idea. Anyway, it was a nice challenge to all of us.

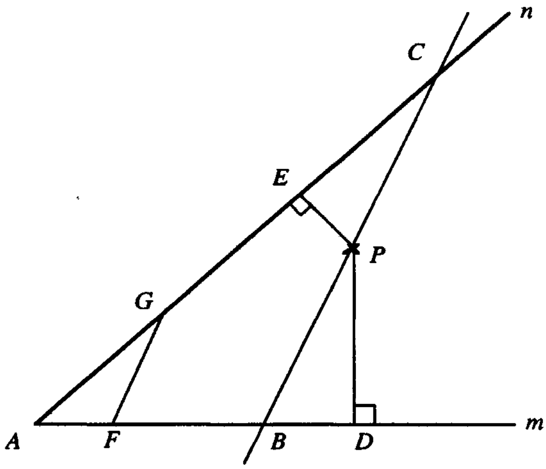


Figure 12. The drawing of Group 5 on Problem 2 b.

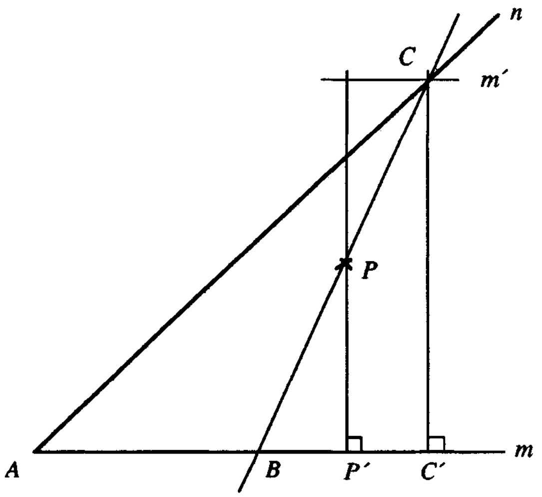


Figure 13. The drawing of Group 6 on Problem 2 b.

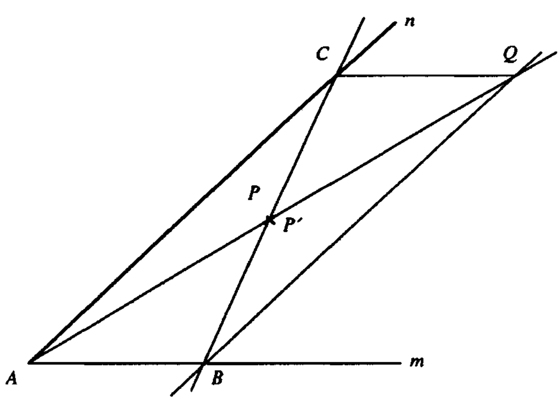


Figure 14. The drawing of Group 10 on Problem 2 b.

Figure 15. The first proof of an individual on Problem 2 b.

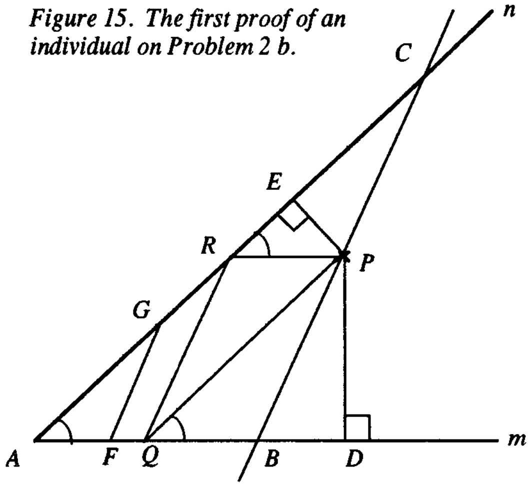
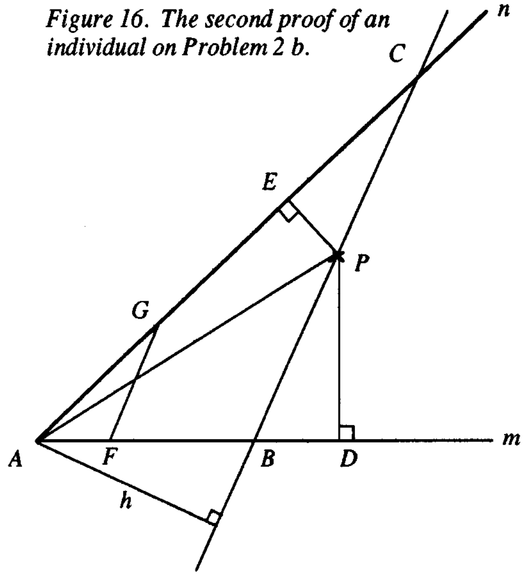


Figure 16. The second proof of an individual on Problem 2 b.



During the same lecture an individual from another group came up with a very unique proof! It is as follows:

Let BC be parallel to FG through P where $|AF| = |PE|$ and $|AG| = |PD|$. Choose Q on m and R on n such that $AQPR$ is a parallelogram. Then

$$\frac{|AB|}{|AC|} = \frac{|AF|}{|AG|} = \frac{|PE|}{|PD|} = \frac{|PR| \sin\alpha}{|PQ| \sin\alpha} = \frac{|PR|}{|PQ|} = \frac{|AQ|}{|AR|}$$

where $\angle BAC = \angle DQP = \angle PRE = \alpha$.
Therefore, QR is parallel to BC and $|PB| = |QR| = |PC|$.

At our next meeting (lecture) the same student came with another elegant proof based on a new idea:

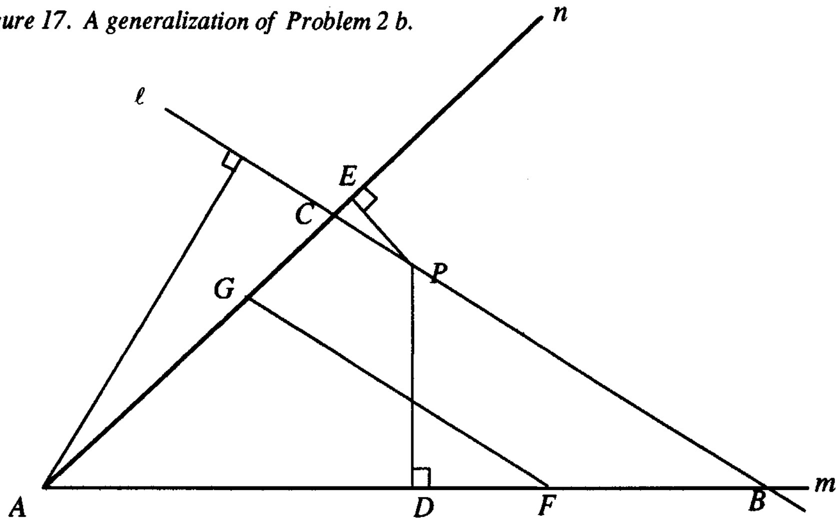
Let BC be parallel to FG through P, where $|AF| = |PE|$ and $|AG| = |PD|$.

Then $\dfrac{|AB|}{|AC|} = \dfrac{|AF|}{|AG|}$ and

$$\frac{|BP|}{|PC|} = \frac{\frac{1}{2}|BP|\cdot h}{\frac{1}{2}|PC|\cdot h} = \frac{Area(\Delta\, ABP)}{Area(\Delta\, APC)} = \frac{\frac{1}{2}|AB|\cdot|PD|}{\frac{1}{2}|AC|\cdot|PE|} = \frac{|AF|}{|AG|}\cdot\frac{|AG|}{|AF|} = 1$$

i.e. $|BP| = |PC|$

Figure 17. A generalization of Problem 2 b.



Problem 2b may also be generalized as follows:

Let t be a positive rational number. The problem is to construct ℓ such that $\dfrac{|BP|}{|PC|} = t$.

The proof given by Group 2 and 6 were easily extended to prove this generalized version.

Let us finish by referring to the proof given by the student mentioned above:

Let BC be parallel to FG where $|AF| = t \cdot |PE|$ and $|AG| = |PD|$. Then

$$\frac{|AB|}{|AC|} = \frac{|AF|}{|AG|} = t\frac{|PE|}{|PD|} \quad \text{and}$$

$$\frac{|PB|}{|PC|} = \frac{Area(\Delta\, ABP)}{Area(\Delta\, APC)} = \frac{\frac{1}{2}|AB|\cdot|PD|}{\frac{1}{2}|AC|\cdot|PE|} = t \cdot \frac{|PE|}{|PD|} \cdot \frac{|PD|}{|PE|} = t.$$

Conclusions

This article started by pointing out two major challenges in mathematics teaching at all levels, namely to have students experience a more true picture of doing mathematics and to create learning environments of acceptance, safety and trust.

We have chosen co-operative work on open ended problem solving in geometry as one way of meeting these challenges. It's a method of experiencing the whole process of doing mathematics. Using this method, we must respect the students way of thinking and need for time. Being a time consuming method, it probably needs to be used together with other teaching methods.

Choosing geometry as the subject for open ended problem solving is based on our own experiences. Elementary geometry is excellent for problem solving from scratch, and it is possible to reach interesting and deep results without a battery of tools.

For many years we have been organizing our teaching in mathematics as a blend of lecturing and problem solving in small groups. It is our experience that co-operation in small groups is a specially good instrument for developing positive learning environments.

In this article we have described how these ideas are brought together as a structured project within a geometry course at the college level. We have reported about our experiences from 1986, 1990 and 1993, how the project has developed, the problems given and how the students have responded.

Our hypotheses about co-operating in small groups on open ended problem solving in geometry have been confirmed:

- It *is* possible to have students experience the whole process of doing mathematics.
- Geometry *is* a rich resource for finding problems to do so.
- It *is* possible to create learning environments where students cooperate and give each other support to go through the process together.
- It *is* possible to do this within an ordinary first year college geometry course.

Considering all the projects we have observed the following:

Most students take the project seriously and enjoy it even if the project is not graded. It also seems to motivate further work in the course and creates a positive learning environment. The quality of the final reports have been variable, but mostly high. All final reports contained analysis and proof writing. Each year some groups have no log or process writing at all. Only a few reports contained generalizations and evaluation.

Considering the 1993–project we have observed the following:

There has been observable progress on process writing and evaluation. All reports contained evaluation, and all but one contained process writing.

Our emphasis on processing and evaluation seems to have had an effect on the students' motivation and understanding of doing mathematics.

Only three reports contained generalizations or attempts to formulate new problems generated by the problem solving process. This number is however more than the total number of reports containing generalizations from 1986 to 1990. But it is still not satisfactory.

The processing and evaluation were mainly concentrated on co-operation in small groups, problem solving, process writing, and the project in general. We want to emphasize the positive experiences by co-operating in small groups, and the students' need for encouragement and support in the process of doing open ended problem solving. The students have learned by experience the importance of perseverance and hard work.

We give credit to the students who participated, and in the following we present four attitudinal characteristics that are essential for successful projects.

First, it requires courage to engage in open ended problem solving (like all creative processes). Therefore, a supportive attitude in the groups is essential, and it is important to focus on feelings as well as on the mathematics involved.

Second, it requires perseverance and hard work. Often, this is underestimated by the students. In our opinion it is, perhaps, the most important ability in doing mathematics. Therefore, the groups have to organize their work with a time schedule for the project period and make sure it is followed.

Third, it requires an open and positive attitude. This is essential for creating a learning environment of acceptance, safety and trust as well as for each individual's learning process.

Fourth, it requires responsibility. This is essential for each individual's learning process, for the dynamic in the groups and for the climate and communication in general.

Recommendations

I conclude by giving five recommendations to teachers of future projects. First, it requires lots of efforts to find and test potential project problems. Second, it is important to motivate the students for the project and ask for feedback throughout the project period. Third, it is important to emphasize processing on the group work in lecture sessions and to do it in a non-competitive way. Fourth, it is important to give a precise description of what the final reports are expected to contain and to underscore the requirements of process writing and evaluation. Fifth, it is important to emphasize improving the students' understanding of generalization and their ability to formulate new problems, as well as to emphasize the requirement of including some writing on this topic in the final reports.

Acknowledgements

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Problemløsning i geometri

Sammendrag

Artikkelen beskriver et prosjekt om problemløsning ved Agder distriktshøgskole. Prosjektet inngår som del av et geometrikurs og bygger på samarbeid i smågrupper. Artikkelen inneholder begrunnelse for prosjektet, den beskriver hvordan prosjektet er organisert og hvordan det har utviklet seg over tid. Hovedkonklusjonene er:

- Det er mulig å legge til rette for at studenter kan erfare matematikk som en helhetlig, skapende prosess.
- Problemer hentet fra geometri er velegnet for å gjøre dette.
- Det er mulig å utvikle læringsmiljø der studenter samarbeider og gir hverandre støtte i den kreative prosessen.
- Det er mulig å gjøre dette innen rammene av ordinære kurs i det første studieåret ved høgskole/universitet.
- Det har over tid vært observerbar framgang i studentenes prosessskriving og prosjektevaluering. Studentenes forsøk på å generalisere og å formulere nye problem har hatt en viss framgang, men er ikke tilfredsstillende. Vektleggingen på prosess-skriving og evaluering synes å ha positiv effekt på studentenes motivasjon og forståelse av matematikk.
- Studentene erfarer et stort behov for oppmuntring og støtte når de arbeider med problemløsning.
- Studentene anbefaler sterkt samarbeid i smågrupper som arbeidsform.

Forfatter

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