Investigating the fit of a model for students' understanding of fractions in a Norwegian context

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To capture the complexity of students' understanding of fractions, a model linking part-whole to the subconstructs ratio, operator, quotient and measure has been proposed. We ask if this model is compatible with students' achievements in a Norwegian context. Responses from 638 students were analysed using structural equation modelling (SEM), and a good fit of the model was obtained after removing the ratio subconstruct. In particular, part-whole is seen to be important for operator, quotient and measure. Using qualitative analysis of interviews, we found reasoning associated with ratio, with a weak link to the part-whole subconstruct.

Fractions are one of the most advanced and challenging concepts children encounter during primary school (Charalambous & Pitta-Pantazi, 2007; Misquitta, 2011). Still, the understanding of fractions is one of the most important aims of school mathematics and has been shown to be a strong predictor of further achievements in mathematics (Siegler et al., 2012). It is well recognized that the concept of fractions has multiple facets (Kieren, 1976, 1993; Lamon, 2007), and many students face difficulties in moving beyond the part-whole concept of fractions. In fact, studies show that the part-whole interpretation of fractions dominates in both students' and teachers' thinking (Park, Güçler & McCrory, 2013). Through several decades of research, a model of the rational number construct has emerged (Kieren, 1976, 1980; Behr, Lesh, Post & Silver, 1983; Marshall, 1993; Charalambous & Pitta-Pantazi, 2007), linking part-whole to the other subconstructs measure, quotient, ratio and operator.

Several studies take the five-subconstruct model as a reference point (Doyle et al., 2016; Gray & Ånestad, 2016; Pantziara & Philippou, 2012). However, the model can be understood as representing the adult view of

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fractions. It is not clear if it describes children's construction (Charalambous & Pitta-Pantazi, 2007; Olive & Lobato, 2008). Motivated by this observation, Charalambous and Pitta-Pantazi (2007) statistically tested the model in Cyprus using structural equation modelling (SEM) and obtained a good fit. They warned however, that their findings should not be overgeneralized, since students' constructions depend on the context and culture in which they are developed. They ask therefore for similar studies in other countries. As a partial answer to this challenge, we take the study of Charalambous and Pitta-Pantazi (2007) as the starting point and consider the case of lower secondary students in Norway. Several factors, such as curriculum, textbooks and teacher practices, influence students' construction of knowledge of fractions. Furthermore, students' understanding changes over time, and the time of testing will affect their achievements. Thus, it is not clear if the findings of Charalambous and Pitta-Pantazi (2007) are fully applicable to students at the lower secondary level in Norway.

The work of Charalambous and Pitta-Pantazi (2007) is cited by several authors and is also used in Norwegian studies (Bjerke, Eriksen, Rodal & Ånestad, 2013; Gray & Ånestad, 2016). We were however unable to locate studies that statistically test the model using SEM analysis in countries other than Cyprus, and we saw a need for further statistical investigation. We chose a context with older students since, in view of the Norwegian curriculum, younger students in Norway cannot be expected to answer all tasks used by Charalambous and Pitta-Pantazi (2007).

The research questions are as follows:

To what extent do Norwegian 7th to 10th grade students' achievements on different tasks involving fractions correspond with the theoretical model reported by Charalambous and Pitta-Pantazi (2007)?

How do students' achievements on each subconstruct correlate with their achievements on the other subconstructs and on fraction operations and equivalence of fractions?

Literature review

The complexity of the fraction construct is well recognized; the exact nature of the construct is, however, still debated (Lamon, 2007). The model considered in this paper, can be traced back to Kieren (1976). He connected part-whole to seven interpretations of rational numbers and concluded that students need experience with all seven interpretations. For each of them, instructional steps were suggested.

Later, Kieren (1980) pointed to the five subconstructs part-whole, quotient, measure, ratio, and operator. Building on this work, Behr et al. (1983, p. 93) investigated "whether or not subjects performing at a given stage on tasks involving one subconstruct perform at a comparable level on tasks involving other subconstructs". They also investigated which subconstructs that best could serve to develop the basic fraction concept with children and identified a "preliminary conceptualization of the interrelationships among the various subconstructs" as depicted in figure 1. "The solid and dashed arrows suggest established and hypothesized relationships, respectively, among rational-number constructs, relations, and operations" (Behr et al., 1983, p. 100).

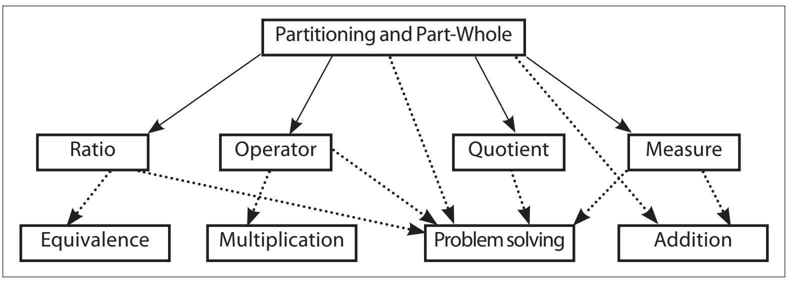


Figure 1. *Conceptual scheme linking the five subconstructs of fractions to the different operations of fractions and to problem solving (Behr et al., 1983, p. 100)*

Marshall (1993, p. 268) developed detailed descriptions of the five subconstructs to provide the "scaffolding for assessment questions". Building on these and the previous works, Charalambous and Pitta-Pantazi (2007, p. 294) detailed an operationalization of each subconstruct. They set out to "examine students' constructions of the different subconstructs of fractions, the associations within these subconstructs, and any potential links of these subconstructs to fraction operations and fraction equivalence."

We will here give a short description of some of the main features of each subconstruct and refer the reader to Charalambous and Pitta-Pantazi (2007) for further details. In brief, the five subconstructs are characterized as follows: Part-whole emphasizes that a rational number may be interpreted as a partition of a (continuous or discrete) set. Typical questions are to ask the student to identify a fraction of a given whole, or to find the whole from a given fraction. In the ratio subconstruct, fractions are comparisons of two quantities. It is therefore considered a comparative index rather than a number (Charalambous & Pitta-Pantazi, 2007; Carraher, 1996). Two main problem types are comparison problems and missing value problems (Lamon, 2007). In a comparison problem the goal is to determine an order relation between the ratios *a*/*b* and *c*/*d*, while the missing value problem provides three of the four values in the proportion $a/b = c/d$, asking for the fourth. Fractions are interpreted as operators when regarded as functions applied to numbers, objects or sets (Charalambous & Pitta-Pantazi, 2007; Behr et al., 1983; Marshall, 1993). Typical questions are to ask a student to scale a figure by a fraction, reduce a recipe by a factor, or if multiplying by *a* and dividing by *b* is the same as multiplying by *a*/*b* (Charalambous & Pitta-Pantazi, 2007; Marshall, 1993). In the quotient subconstruct a fraction is seen as a result of a division (Charalambous & Pitta-Pantazi, 2007; Kieren, 1993). While *a*/*b* in the part-whole subconstruct represents parts of a whole, in the quotient subconstruct ”*a* represents something that is itself to be partitioned” (Marshall, 1993, p. 273). Finally, the measure subconstruct emphasizes that fractions are numbers and that a unit fraction is used repeatedly to determine a distance from a preset starting point (Charalambous & Pitta-Pantazi, 2007; Lamon, 2001; Marshall, 1993). A typical question is to ask the student to represent a fraction by a point on the number line.

In their operationalization, Charalambous and Pitta-Pantazi (2007, p. 295) interpret the part-whole subconstruct as the ”fundament for developing understanding of the four subordinate constructs of fractions”. The solid arrows in figure 1 depict this relationship. Similarly, for instance, the dashed arrow from operator to multiplication indicates that the operator subconstruct of fractions is hypothesized to be helpful for developing understanding of the multiplicative operations on fractions. They tested these dependencies against the Cypriot students’ achievements, using structural equation modelling (SEM), and were able to fit a model interrelating the five subconstructs, verifying the solid arrows in figure 1. Furthermore, links were found to fraction operations and fraction equivalence verifying most of the dashed arrows in figure 1. Charalambous and Pitta-Pantazi (2007, p. 309) concluded by asking if the good fit with the model justifies ”the traditional instructional approach using the part-whole notion as the inroad to teaching fractions”, but warns that the structure of the curriculum may be responsible for the ”hierarchy” in the model. Still, they argue that the good fit of the model with their data provides support for the claim that ”the part-whole interpretation of fractions should be considered a necessary, but not sufficient condition for developing an understanding of the remaining notions of fractions” (Charalambous & Pitta-Pantazi, 2007, p. 310). Thus, they conclude that there is a need for instruction to emphasize the other subconstructs of fractions, and especially those that their model fit suggests are less related to part-whole.

In the Norwegian context Bjerke et al. (2013) investigated fraction misconceptions among primary school students and found that students in grade 6 and 7 have insufficient knowledge of fractions. Grønmo and Bergem (2009, p. 59) analysed TIMSS 2007 for 8th graders and found that Norwegian students had a particularly weak achievement on ratio tasks. It should be noted that the TIMSS data from 2007 are relevant to the situation before the current Norwegian national curriculum, LK06 (Kunnskapsdepartementet, 2013), came into effect in 2006. Gray and Ånestad (2016) found, when investigating the fraction tasks given at the National tests for grade 8 in 2011, that all of the five subconstructs, to some extent, are emphasized.

Norwegian studies mentioned above (Bjerke et al., 2013; Gray & Ånestad, 2016) utilize the conceptualization of the fraction construct in the five subconstructs part-whole, ratio, operator, quotient, and measure. Using the five-subconstruct model as a point of reference they do not question the applicability in a Norwegian context. However, several factors, such as curriculum, textbooks and teacher practices, influence students' construction of fractional knowledge. Hence it is not clear if the findings of Charalambous and Pitta-Pantazi (2007) are fully applicable in a Norwegian context. In line with the challenge of Charalambous and Pitta-Pantazi (2007) the present study seeks further understanding of the applicability of the model by investigating it in a Norwegian context.

Methods

The authors are teacher educators in Norway. We worked with our pre-service teachers to collect the data for this study. We developed a test, and the pre-service teachers administered it during their training period. This was done by agreement with the respective schools and it was possible to use 90 minutes for the test. Furthermore, the tests were coded by the pre-service teachers under the supervision of the authors.

The setting of the study

To situate our study, it is relevant to give some background on the Norwegian national curriculum that all schools in Norway follow, LK06 (Kunnskapsdepartementet, 2013). It was effective from 2006 but will be replaced in 2020. There are learning objectives for 2nd, 4th, 7th and 10th grade, and the students' work with fractions from 3rd to 10th grade. In Cyprus, the part-whole subconstruct is introduced from 1st grade and division of fractions is introduced in 5th grade (Charalambous & Pitta-Pantazi, 2007). In Norway, division of fractions is not in the learning objective after 7th grade but mentioned after 10th grade.

Several Norwegian textbook series, for example "Faktor" by Hjardar and Pedersen (2006), and "Grunntall" by Bakke and Bakke (2006), devote a large chapter to fractions at the beginning of 8th grade. However, the textbook series ”Sirkel” by Torkildsen and Maugesten (2006), concentrate on fractions in 9th instead of 8th grade, and are focusing on aspects of fractions in connection with other topics in the 8th grade book. Fractions also have a prominent place in textbook series for grade 3 to 7. The textbook series ”Multi” by Alseth, Nordberg and Røsseland (2006), are used in 60 % of the schools in grade 1 to 7 (Hagesæther, 2013), and for every grade from 3rd to 7th there is a separate chapter for fractions. Notably, for the 6th grade there are two chapters, ”Fractions”, and ”Positioning and ratios”. The textbook series, ”Grunntall” (Bakke & Bakke, 2006), devote a separate chapter to fractions for each year from 5th to 7th grade.

We decided to focus mainly on 8th and 9th graders. Choosing this older group means that all of the sample tasks Charalambous and Pitta-Pantazi (2007) provided, including division of fractions, are included in the curriculum. There is also the important question of whether we want to test students when they are in the process of learning fractions, and thus have mastered only parts of the rather large subject; or when, according to the learning objectives in the curriculum, they should be able to master the subject more thoroughly, and also at a stage where teachers have tried to ensure that all the students have the same basic knowledge of fractions before they start on the more advanced topics in the secondary school. If we had tested younger Norwegian students, we would have expected a disproportionate dependence on knowledge of the part-whole construct, since some of the early teaching of fractions focus heavily on this construct. A downside to testing older students is however that some subconstructs will be mastered exceptionally well, and thus the test will not have the same predictive power.

The development of the test

When developing the test, we took into account the age of the students, and that a maximum of 90 minutes was allowed for the test. Our test consisted of 35 tasks. Of these, 19 were, except for the changing of some numbers, the representative tasks listed in the paper by Charalambous and Pitta-Pantazi (2007), two or three for each subconstruct and fraction operation. Most of these were also known or inspired from previous works, e.g. Lamon (1993), Marshall (1993) and Noelting (1978). The task in figure 2 is an example of one of the adapted tasks.

The remaining 16 tasks were developed by the authors. When designing these tasks, the definitions of the subconstructs in Charalambous and Pitta-Pantazi (2007) were used, and emphasis was given to complement the 19 adopted tasks, taking into consideration the Norwegian

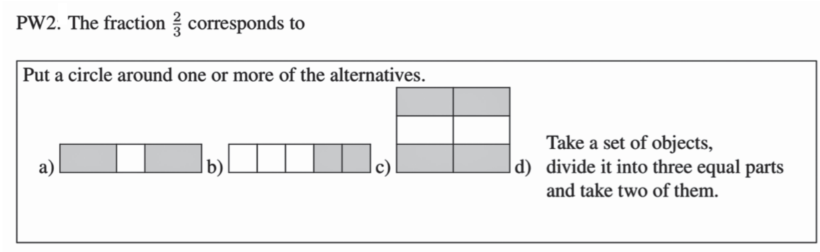


Figure 2. *Sample task (part-whole subconstruct) Charalambous and Pitta-Pantazi (2007, p. 312)*

curriculum and the age of the participants. Three tasks regarding percentages were included, measuring different constructs, since we view percentages as an important part of the students' work with fractions. Furthermore, one task with units of measurement was included. The test also contained two tasks with algebra for students in secondary school (age 13 to 15), but these two were omitted in the final analysis due to few correct answers. Some of the tasks could be classified as word problems, thus language and problem-solving skills would be expected to influence students' performance. Examples of new tasks aimed at measuring understanding of subconstructs include:

> PW1
> A football team has won $\frac{1}{3}$ of their matches this year. They won four matches. How many matches have they played? [Part-whole]
>
> M1
> Write the fractions $\frac{1}{5}$, $\frac{5}{4}$, $\frac{1}{8}$ in order from least to largest. [Measure]
>
> O1
> When importing board games, an import tax of 3 % is added, and then 25 % sales tax. Would the total price be greater, lesser or equal if the sales tax had been added first? [Operator]

Our aim was to create tasks that corresponded to one and only one of the subconstructs. However, because the subconstructs are interconnected and since common tasks often can be solved by several approaches, this is difficult in practice. This challenge was also illustrated in the study of Charalambous and Pitta-Pantazi (2007). During the course of their analysis, they decided to associate, some tasks originally designed to measure a single subconstruct, to more than one subconstruct. We opted, however, to associate each task strictly to one subconstruct. The classification assigned six tasks to part-whole, two to ratio, three to operator, five to quotient, five to measure, and six to equivalence. The rest were assigned to operations. The tasks were validated by the two authors and a third researcher. Since we have access only to parts of the test used by Charalambous and Pitta-Pantazi (2007), we cannot compare tasks in full detail. Their number of tasks, 50 (for a test of 160 minutes), cannot be directly compared with our number of tasks, 35 (for a test of 90 minutes). For example, we counted the task PW2 in figure 2 above, as one task and awarded full score for answering c and d, while half score if only one of them were circled. Charalambous and Pitta-Pantazi (2007) counted this as one task for each of the answer alternatives, thus a total of four tasks, in our view overemphasizing the task. Since we had fewer tasks, our test has less redundancy for simple mistakes. Also, five of our tasks concerned algebra and percentages. However, we associated the tasks to the same subconstructs as in the study of Charalambous and Pitta-Pantazi (2007), allowing us to compare the fits of the models.

Participants and data collection

The test was aimed at 8th and 9th grade students; however, some 7th and 10th grade students were also included. This meant that we obtained a larger pool of data, and that we could do comparisons between different grades.

In addition to the scoring of the individual tasks, grade level, textbook used, and gender were registered. The tests were answered anonymously and no personal information was collected. Over a period of fourteen months we obtained responses. One of the pre-service teachers misplaced answer sheets before they were coded adequately and two responses did not contain any correct answers. The results for these students are not part of the analysis. Among the remaining 638 students, there were 38 seventh graders, 218 eighth graders, 207 ninth graders and 150 tenth graders. For 25 students, data of grade was missing, although they came from a pool of eight and ninth grade students.

In addition to the written test, 33 students among the 8th and 9th graders were selected for interview. The pre-service teachers administering the test at four of the schools were asked to interview students, and together with their mentors they decided which of the students to interview. The interviewers were asked to focus on one or more constructs and ask questions to shed light on the students' conceptualization of fractions; the students' computational skills; how the students reasoned to find a solution strategy; which solution strategies they used; and to what extent the students reflected on their choices of solution strategy and their reasoning. The pre-service teachers gave written accounts of the interviews.

Data analysis

For the quantitative analysis, a score of 0 was assigned to a blank answer or wrong answer, and 1 to a correct answer, as it was not feasible to grade the tasks continuously. For some answers a score of 0.5 was assigned. Even though each task was scored discretely, the latent variables in the structural equation model are viewed as continuous.

We have used descriptive statistics and SEM-modelling. The data analysis was carried out using R: A language and environment for statistical computing (R Core Team, 2018) and lavaan (Rosseel, 2012, 2018). The structural equation models were described using the lavaan model syntax, and maximal likelihood was used as the estimator.

A qualitative analysis with focus on the ratio subconstruct was conducted, since this was the construct which eventually had to be excluded in the quantitative analysis, and all 19 interviews which included solution strategies on tasks under the ratio construct were analysed. Ten of these students only gave an answer to one of the tasks in the interview. The strategies used by the 19 interviewed students were assigned to one of four categories.

Results, findings and discussion

Means, standard deviations and correlations

For each student, the arithmetic mean score for the tasks corresponding to each subconstruct was computed, thus obtaining a score on each construct for each student. Then, for each construct, the average of these scores were calculated, along with the standard deviation, and is tabulated in table 1.

Table 1. *Average success rate of constructs. Algebraic tasks are not included*

Subconstructs	P-W	Ratio	Opera	Quoti	Meas	Equiv	Multi	Add	Total
M	0.64	0.81	0.36	0.54	0.39	0.66	0.22	0.40	0.49
SD	0.24	0.28	0.31	0.31	0.25	0.30	0.22	0.49	0.19

We have also tabulated the means and standard deviations using only the tasks adopted from Charalambous and Pitta-Pantazi (2007), see table 2, and note that the means are not very different.

Since the tests are not identical and the setting is different, we cannot directly compare with the means reported in the paper by Charalambous and Pitta-Pantazi (2007) and by Leung (2009). Nevertheless, we point out

Table 2. *Average success rate of constructs when using only the 19 tasks taken from Charalambous and Pitta-Pantazi (2007)*

Subconstructs	P-W	Ratio	Opera	Quoti	Meas	Equiv	Multi	Add	Total
M	0.55	0.81	0.39	0.50	0.32	0.64	0.21	0.40	0.47
SD	0.34	0.28	0.37	0.35	0.26	0.32	0.29	0.49	0.20

that the lower average score on the measure subconstruct relative to the part-whole is in line with both of these studies.

In table 3 the correlations between students' mean scores of the tasks related to each subconstruct are given. As expected, the mean scores for each subconstruct are positively correlated.

Table 3. *Correlation r(636) (p < .01) between the students' average score on tasks associated to each subconstruct*

Subconstructs	P-W	Ratio	Opera	Quoti	Meas	Equiv	Mult	Add	Total
Part-Whole		.25	.30	.30	.26	.44	.18	.27	.61
Ratio	**.28**		.16	.21	.23	.26	.12	.11	.45
Operator	**.41**	**.18**		.32	.28	.35	.29	.21	.60
Quotient	**.47**	**.25**	**.37**		.37	.42	.22	.29	.71
Measure	**.49**	**.26**	**.38**	**.46**		.32	.23	.22	.66
Equivalence	**.59**	**.28**	**.37**	**.47**	**.45**		.23	.43	.74
Multiplication	**.48**	**.22**	**.47**	**.43**	**.46**	**.49**		.21	.47
Addition	**.33**	**.11**	**.20**	**.32**	**.30**	**.40**	**.36**		.52
Total	**.79**	**.41**	**.62**	**.74**	**.72**	**.78**	**.74**	**.50**	

Note. The lower left triangle, in boldface, is the correlation with all tasks included except for the tasks involving algebra. The upper right triangle is the correlation with only the tasks from the appendix of Charalambous and Pitta-Pantazi (2007) included.

In table 3 we see that multiplication has a higher correlation with other subconstructs as compared to addition when all tasks are included. The reason for this may be that several of the tasks on multiplication were word problems, and that more complicated tasks tend to draw on knowledge from several different areas.

The larger correlation ($r(636) = .59$, $p < .001$) between part-whole and equivalence may indicate that the part-whole subconstruct is more central to the equivalence tasks than the measure and quotient subconstructs. Moreover, equivalence is a strong predictor of the students' achievement on the test. However, the task concerning addition of fractions had only a moderate correlation ($r(636) = .40$, $p < .001$) with the tasks concerning equivalence. On the other hand, we observed that the highest correlations with knowledge of adding fractions were with specific tasks related to equivalence adopted from Charalambous and Pitta-Pantazi (2007, p. 314) and with one of the tasks the authors classified as related to measure where the students were asked to order three fractions according to their size. We note that this is in agreement with the model by Behr et al. (1983) and Charalambous and Pitta-Pantazi (2007). The solving of this task requires finding a common denominator and it is therefore not surprising that this shows some correlation with adding fractions.

Among the equivalence tasks, we noted a difference between tasks using small numbers, where it is easy to use trial and error, and nearly equivalent tasks with larger numbers, whose predictive power seems better. It should be noted that Charalambous and Pitta-Pantazi (2007, p. 314) used numbers with greater value in one of the tasks on equivalent fractions we adapted.

SEM-models

We tested several SEM-models. In lavaan we defined latent variables for each of the subconstructs, and it was specified that the latent variables were measured by the scores on the tasks associated with the given subconstruct. Regressions were specified according to the graphical models as in figure 3 and figure 4.

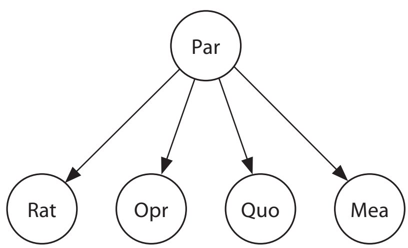


Figure 3. *SEM-model showing the Part-whole (Par), Ratio (Rat), Operator (Opr), Quotient (Quo), and Measure (Mea) subconstructs*

We were unable to get a good fit with the full model of Behr et al. (1983) and Charalambous and Pitta-Pantazi (2007). For this reason, we decided to investigate a restricted model given in figure 4. A closer analysis revealed that the two tasks concerning ratio were problematic. Both were adapted from Charalambous and Pitta-Pantazi (2007, p. 312). In order to obtain a good fit, a substantial amount of the variance of students' scores on tasks concerning the ratio concept have to be explained by the variance of their scores on tasks concerning part-whole. However; the correlation between students score on part-whole and ratio was only $r(636) = .25$, $p < .001$,

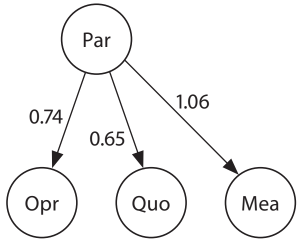


Figure 4. *Fitted SEM-model showing the Part-whole (Par), Operator (Opr), Quotient (Quo), and Measure (Mea) subconstructs ($\chi^2/df = 1.619$, RMSEA = 0.031, CFI = 0.938)*

Note. The weights in the figure are an estimate of the regression coefficients (SD of these estimates is between 0.11 and 0.15).

less than the correlation between part-whole and any other construct. Also; there must be sufficient covariance between the scores on tasks under the same subconstruct, and this was not the case. In fact, we could not reject the hypothesis that the responses on the two tasks measuring the ratio construct, R1 and R2, were independent. Using a contingency table and Pearson's test without Yates' continuity condition we obtained $\chi^2(1, N = 638) = 0.21, p = .646$.

High success rate on the ratio tasks was another obstruction to a good fit. If most students answer a given task correctly, regardless of their achievements on other parts of the test, the task have weaker predictive power. For the ratio tasks the mean score was 0.81 ($SD = 0.28$), compared to an average of 0.49 ($SD = 0.19$). Also, task R1 and task R2 had low correlation with the total score, $r(636) = .30$ and $r(636) = .28$ respectively. There were only five tasks with lower correlation with the total score. Expecting that the success rate on the tasks would be higher for 10th graders than for the lower grades, we broke down the results for each grade and calculated the correlation, see table 4, and found that the correlation between the two tasks was low for all grade groups.

Table 4. *Correlation between Task R1 and Task R2 broken down by grade*

Grade	7	8	9	10	All
# students	38	218	207	150	638
Mean score Task R1	0.737	0.835	0.821	0.867	0.828
Mean score Task R2	0.711	0.739	0.826	0.873	0.798
Correlation	-.250	.101	-.114	.086	.018

Note. There were 25 students where data for grade were missing.

In view of the possibility that a high success rate for the answers of the tasks led to the independence hypothesis not being rejected, we also tested the independence of task R1 and task R2 for the 8th graders and obtained χ^2 (1, N = 218) = 2.22, p = .137 using a contingency table (there were too few 7th graders). The lack of significant correlation between the ratio tasks may indicate that random mistakes are more common than mistakes rooted in misconceptions. Similarly, since the ratio tasks only had two possible answers, guessing could be contributing to a lower correlation.

In summary, there are challenges to the reliability of the ratio tasks. However; it is still possible that Norwegian students at 7th to 10th grade do not rely on part-whole reasoning as much as the Cypriot students. Our qualitative analysis of 19 interviews in the next section seems to support this.

The model used by Behr et al. (1983) and Charalambous and Pitta-Pantazi (2007) asserts that part-whole plays a prominent part in the understanding of the other subconstructs. As part-whole is the starting point for fractions in both the Norwegian and the Cypriot curriculum, in the latter one introduced from the first grade (Charalambous & Pitta-Pantazi, 2007), one could think that younger students make more use of strategies depending on part-whole reasoning than older students, making it is easier to obtain a good fit for the model with younger students compared to older students. Likewise, independent focus on the ratio subconstruct, as indicated in a separate chapter in the most popular Norwegian textbook for 6th grade written by Alseth et al. (2006), may also contribute to the development of an understanding that does not rely as heavily on part-whole as in Cyprus. Thus, both the maturity of the students and curricular factors may have contributed to the lack of fit with the ratio subconstruct included. That the rest of the subconstructs still have a significant dependence on part-whole may be seen as a stronger support for the rest of the model than if we had tested younger students.

We concluded that quantitatively, we were unable to identify either the ratio factor or the association between ratio and part-whole in the current dataset, and hence we omitted the ratio subconstruct from the SEM-model.

With the ratio factor removed we got a very good fit (χ^2/df = 1.619, $RMSEA$ = 0.031, CFI = 0.938) of the model in figure 4. Using only the relevant tasks from Charalambous and Pitta-Pantazi (2007) we also obtained a very good fit of the model (χ^2/df = 1.429, $RMSEA$ = 0.026, CFI = 0.973). We view this as a confirmation of a part of the model of Behr et al. (1983) adapted by Charalambous and Pitta-Pantazi (2007) as we are confirming the factors operator, quotient and measure and their dependence on the part-whole factor. In particular, it confirms that the part-whole interpretation of fractions plays a dominant role.

Even though the measure aspect does not derive directly from part-whole (Stafylidou & Vosniadou, 2004), we see that measure is the subconstruct that has an edge with the highest weight from part-whole, indicating the central position of both the part-whole and the measure subconstructs. This is different from the fit of Charalambous and Pitta-Pantazi (2007) where measure is the subconstruct with the weakest association to part-whole. There has been some discussion among researchers as to whether part-whole and measure should be viewed as separate subconstructs (Lamon, 2007).

We note that the model presented in figure 1 include ratio and in fact assume that understanding of equivalence depends mainly on knowledge of the ratio construct. Since we had to remove the ratio construct from the SEM-model, we restricted ourselves to statistical confirmation of the sub-model given by figure 4.

A qualitative analysis of the ratio subconstruct

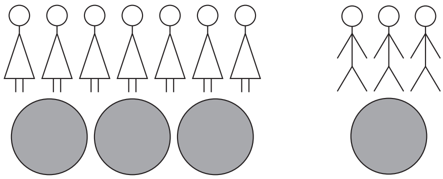
As we were unable to include the ratio construct in the SEM-model, but wanted to investigate the relation, we decided to use a qualitative analysis of this subconstruct. The two tasks included have been extensively used in the literature. Task R1 was adapted from Charalambous and Pitta-Pantazi (2007, p. 312) who used the following task inspired by an experiment of Noelting (1978).

> R1
>
> John and Mary are preparing orange juice for their party. Presented below are the recipes they used. What recipe will make the juice the most "orangey"?
>
> – John's recipe: Two cups of concentrate juice – five cups of water.
> – Mary's recipe: Four cups of concentrate juice – eight (eleven) cups of water.

We altered the task by replacing eight with eleven. Thus, we avoid that the strategy of comparing four cups to two cups of concentrate juice without considering the water yields a correct answer. Task R2 is identical to a task used by Charalambous and Pitta-Pantazi (2007, p. 312) which originates from Lamon (1993).

In both task R1 and task R2 it is possible to obtain the correct answer even though the student has a misconception. The success rate of 82.8 % and 79.8 % respectively, was contrary to what we expected, since task R1 was changed in order to make it harder.

R2
Who gets more pizza, the boys or the girls?



Problems concerning ratio have been classified into comparison problems and missing value problems, similar to fraction comparison and equivalence tasks respectively (Lamon, 2007, p. 637). We follow Charalambous and Pitta-Pantazi (2007) and do not include the missing value problems in the ratio subconstruct task analysis. Of the 33 interviewed students, a total of nine told how they reasoned on both ratio tasks, four how they reasoned on task R1, and six how they reasoned on task R2. The answers were classified into four categories: norming, comparative index, formal part-whole, and guessing. The most common method of reasoning is what is classified as norming (Lamon, 2007), including informal use of external ratio. A student using the norming process will "choose one of the ratios and use it to interpret the other ratio" (Lamon, 2007, p. 644). An example of this is the interviewed 9th grade student justifying his answer of task R2 by explaining:

> The three boys share a pizza. If the girls divide themselves in groups of three on the pizzas they have, we are left with a girl that gets a whole pizza, therefore the girls get more pizza. Or, we can say that there are two girls to each pizza, and then there is a third girl that joins in on one of the pizzas. The boys are three to the one pizza they have, and therefore the girls get more pizza.

This thinking is related to the rate concept and is also called external ratio (Charalambous & Pitta-Pantazi, 2007; Lamon, 2007). The student also used the norming process when reasoning on task R1, and understands that there is a ratio and scales in order to compare:

> If John's juice is to approach Mary's, he doubles the recipe and then gets 4 parts oranges and 10 parts water. He then still has less water than Mary and therefore stronger juice.

This student is representative for those seven and ten students classified as using the norming method or the external ratio informally to answer R1 and R2 respectively.

In contrast another student (8th grader) divided the number of pizzas by the number of persons and compared the results, thus formally using the comparative index (Charalambous & Pitta-Pantazi, 2007). Three students answered R2 in similar ways. None used this approach for R1. A slightly different approach is seen in the following reasoning on task R1:

> John has $\frac{2}{7}$ oranges, Mary has $\frac{4}{15}$.

The same student (9th grader) remarked on task R2:

> The boys get $\frac{1}{3}$ pizza each, the girls get $\frac{3}{7}$. The girls therefore get the most.

This student uses a formal approach, and instead of using for example the ratio 4 to 11, considers $\frac{4}{15}$, i.e. a part-whole interpretation. One other student reasoned on R1 using these formal rules. There is a difference between ratio and the other constructs in the way they combine through arithmetic operations. However, students having a good command of the ratio concept easily switch between ratio and part-whole comparison (Lamon, 2007, p. 659). Studies suggest that older students tend to replace informal reasoning strategies by rules and algorithms (Lamon, 2007, p. 645). There were four students who said they just guessed on task R1 and one on task R2.

We must be careful in drawing conclusions from the small sample of interviews. However; we see the ratio subconstruct as distinguishable from the other four subconstructs, in that the majority used norming (including use of external ratio) or the comparative index. The limited use of part-whole reasoning is consistent with the low correlation between the ratio tasks and the other tasks and may indicate that ratio is constructed separately from the part-whole interpretation.

To further understand the students' reasoning, we investigated their textbooks, ”Faktor” (Hjardar & Pedersen, 2006) and ”Grunntall” (Bakke & Bakke, 2006), and found that none of the interviewed students had a textbook focusing especially on ratio the year they were interviewed. This could partially explain the amount of informal reasoning on the ratio tasks. Also, Grønmo and Bergem (2009, pp. 71–72) found that ”the Nordic group” of countries, emphasize mathematics related to daily life, including estimation, at the expense of classical formal mathematics. Thus, it should not be surprising that the Norwegian students used non-formal reasoning on the ratio tasks and did depend less than Cypriot students on the understanding of the part-whole subconstruct when solving tasks on ratio. However, we also found some indications that the ratio subconstruct is linked to the part-whole subconstruct.

Conclusions

Using structural equation modelling, we were able to obtain a very good fit with a simple model (figure 4) showing that part-whole has an important role in Norwegian 7th to 10th graders' understanding of fractions, thus confirming the prominent position of the part-whole subconstruct reported in other studies. The strengths of the associations between part-whole and the different subconstructs are; however, different from those reported by Charalambous and Pitta-Pantazi (2007), and this is interpreted as an influence of cultural and curricular factors and the maturity of the students. We were unable to obtain a good fit when including the ratio subconstruct. Analysing 19 task interviews, we found mainly informal reasoning viewed as specific to the ratio subconstruct, but also some reasoning relating the ratio subconstruct to the part-whole. We observed that these findings were consistent with emphasis in textbooks and broad characteristics pointed out by Grønmo and Bergem (2009).

We found that the mean scores for each subconstruct are positively correlated, and there is a particularly high correlation between tasks measuring the understanding of part-whole and equivalence of fractions.

Several studies conducted in Norwegian contexts (Bjerke et al., 2013; Gray & Ånestad, 2016) utilize the five-subconstruct conceptualization of fractions. Our partial verification of the model in a Norwegian context adds further validity to these and other studies.

We argue that our findings have implications for teaching: One can expect that a good understanding of part-whole will contribute to the understanding of the other subconstructs. However, part-whole does not fully explain the variation in the other subconstructs; hence the students need learning experiences connected with each of the subconstructs and especially with those that the model-fit suggests are less related to part-whole (see figure 4). This is in alignment with other studies (Charalambous & Pitta-Pantazi, 2007, p. 310; Lamon, 2007, p. 635).

For further research, we conjecture that the model linking part-whole to the subconstructs ratio, operator, quotient and measure has wide applicability. However, we believe the amount of variance that part-whole explains in the other constructs will depend on factors such as context and the age of the students tested. Thus, caution must be exercised when the model is applied. In some contexts, the association between part-whole and the other subconstructs may be weaker than under other circumstances.

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