Literature review of mathematics teaching design for problem solving and reasoning

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To characterize teaching designs intended to enhance students' problem solving and reasoning skills or to develop other mathematical competencies via problem solving and reasoning, a literature review was conducted of 26 articles published in seven top-ranked journals on mathematics education from 2000 to 2016. Teaching designs were characterized by a) the educational goals of the designs, b) the claims about how to reach these goals, and c) the empirical and theoretical arguments underlying these claims. Thematic analysis was used to analyze the retrieved articles. All but two studies had goals concerned with developing students' mathematical competencies. The overarching ideas of the identified emergent claims regarding the achievement of stipulated goals, concerned scaffolding students' learning and letting students construct their own mathematics. Four recurring theoretical arguments were found to support emergent claims: hypothetical learning trajectories, realistic mathematics education, theory of didactical situations and zone of proximal development.

Two central goals of mathematics education are to support students' development of the interconnected key competencies *problem solving* and *reasoning* (Ball & Bass, 2003; Kilpatrick, Swafford & Findell, 2001; NCTM, 2000; Niss & Jensen, 2002). Students should be able to engage meaningfully in problem solving, which is solving a novel task for which the solution method is not known in advance (Schoenfeld, 1985), and in reasoning, which is "the explicit act of justifying choices and conclusions by mathematical arguments" (Boesen et al., 2014, p. 75). The last three decades, there has been an increase of mathematics education research. This research has contributed with insights into how to improve teaching to help students develop their problem solving and reasoning competencies (Carpenter et al., 2004; Hiebert & Grouws, 2007; Niss, 2007). Furthermore, research has also shown that activities in which students

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are engaged in problem solving and reasoning can be an effective way to promote the development of other competencies, like conceptual and procedural understanding (Collins, 2012; Granberg, 2016; Warshauer, 2015). Despite the insights gained, there is still much we need to know to enhance teaching practice (Niss, 2007). Added to this, educational research in general has been criticized for its lack of practical applicability (Stylianides & Stylianides, 2013; van den Akker, Gravemeijer, McKenney & Nieveen, 2006). An aim by conducting intervention studies, is to bridge the gap between research insights gained, sometimes far from the classroom, and practical applicability. This review accordingly examines and characterizes intervention studies in mathematics education. Here, these intervention studies are seen as teaching designs (defined in section Design research) in which the central competencies of problem solving and reasoning are either the educational goal as such or the means for reaching another educational goal. The present results can strengthen our knowledge of the theoretical arguments that support certain claims about how to achieve specified teaching goals in mathematics education.

The teaching designs in the reviewed studies were analyzed based on their intervention goals, claims about how these goals were to be reached and arguments supporting these claims. This structure of goal, claim and supporting argument is borrowed from the area of design research (van den Akker, 2010). The research question guiding this study is presented in section Aim and research question. This review critically analyzes previous intervention study literature connected to mathematical problem solving and reasoning, and presents the goals, claims and arguments that predominate in related teaching designs.

Background and framework

This section first expands on the distinction between learning *of* or learning *via* a competency. This is important because the review aims at examining the extent to which problem solving and reasoning were learning goals in themselves (i.e., learning *of* a competency) and the extent to which those competencies were used as means for learning (i.e., learning *via* a competency). Thereafter, design research and its relationship to teaching design are characterized in order to describe the relevant terminology and justify the choice of analytical framework. Finally, the analytical categories used to analyze teaching designs are presented.

Learning of and via a mathematical competency

Mathematical *competence* is defined as "the ability to understand, judge, do and use mathematics in a variety of intra- and extra-mathematical contexts and situations in which mathematics plays or could play a role" (Niss, 2003, p. 7). A mathematical *competency* is defined as "a clearly recognisable and distinct, major constituent of mathematical competence" (Niss, 2003, p. 7). Thus, a mathematical competency is a component of general mathematical competence, a specific skill such as problem solving or reasoning. The present literature review examines how teaching designs are used in mathematics education research as a way to improve mathematics teaching and learning *of* and *via* problem solving or reasoning. The review thus distinguishes between learning *of* and learning *via* a competency. For example, a learning sequence may comprise the learning *of* a problem solving competency to become a more proficient problem solver. It could also be the case that competency in representations and connections may be developed *via* problem solving activities (Lithner, 2008).

Design research

Design research seeks to understand and inform both theory and practice concerning how successful a teaching design is under specified conditions and contexts (Brown, 1992; Cobb et al., 2003; Prediger, Gravemeijer & Confrey, 2015). The fundamental notion of a design research project is that it starts with design ideas. These design ideas are theoretically and empirically grounded ideas about how, for example, specified student learning is enhanced. The ideas are then tested in an intervention within a certain context (Cobb et al., 2003; McKenney & Reeves, 2012). The results of such testing, instead of simply providing an instructional sequence that "works", are intended to frame theoretical and empirical insights, and express them as *design ideas* (Cobb et al., 2003). These ideas can be of a detailed, prescriptive nature that can inform both researchers and teachers on how, for example, to design tasks and teaching that enhance particular student learning (Cobb et al., 2003; Kali, 2008; McKenney & Reeves, 2012).

Over the past two decades, design research has gained prominence (Anderson & Shattuck, 2012; Cobb et al., 2003). McKenney and Reeves (2013) stressed the need for thoughtful and in-depth analysis of full-text reports of design research. Design research should build and be built on theory (Cobb et al., 2003; McKenney & Reeves, 2012; van den Akker, 2010). The term *theory*, in this context, refers to the grounding hypothesis or idea about something (Thomas, 1997). The design process may draw on various kinds of *grand theories*. Typically, a grand theory would uphold certain ideas about human development and learning, for example, Piagetian perspectives on knowledge construction or sociocultural theory. However, grand theories are often too general to inform the construction of an educational design. Hence, a specific intermediary framework linking a grand theory or theories and the design process is more suitable (Cobb et al., 2003; Ruthven, Laborde, Leach & Tiberghien, 2009). Such *intermediary frameworks* "extract, coordinate, and contextualize relevant aspects of several grand theories that are pertinent to developing, analyzing, and evaluating teaching designs" (Ruthven et al., 2009, p. 330). An example of an intermediate framework is Brousseau's theory of didactical situations (TDS) (Ruthven et al., 2009). Central to TDS is the teacher's responsibility for creating a learning situation in which the teacher can hand over the responsibility for learning to the student (Brousseau, 1997). This intermediate framework has been strongly influenced by Piagetian theory, which, as stated earlier, can be seen as a grand theory (Brousseau 1997; Ruthven et al., 2009).

This literature review is not restricted to reviewing design research in mathematics education, but uses the structure of a design research study to analyze the reviewed articles' reporting of intervention studies in mathematics education. The structure of design research studies is presented in section Analytical categories detecting teaching designs which also presents how the teaching designs in the reviewed articles were analyzed and characterized. The focus of this literature review is on the design ideas connected to teaching, here called *teaching design* – for example, a plan or program functioning as a guide for the construction and implementation of an activity. A teaching design conveys information about essential components of a teaching product, which is the outcome of a teaching design. A teaching sequence is one example of a teaching product.

Analytical categories detecting teaching designs

Theory cannot in itself provide a straightforward recipe for designing effective learning environments (National Research Council, 2000). Analogously, "physics constrains but does not dictate how to build a bridge" (National Research Council, 2000, p. 131). A teaching design must include a goal for learning that can direct the design (National Research Council, 2000), but to reach that goal, there must be an idea, a claim, as to how the goal is to be reached (van den Akker, 2010). For example:

> The claim that invented representations are good for mathematics and science learning probably has some merit, but it specifies neither the circumstances in which these representations might be of value nor the learning processes involved and the manner in which they are supported. (Cobb et al., 2003, p. 11)

A claim should therefore specify what objects are required to enable the goal to be reached and a specification of the relationships between the objects (Schoenfeld, 2002). The claim, in turn, is to be supported by theoretical and empirical arguments (Lithner, 2017; McKenney & Reeves, 2012; van den Akker, 2010). The empirical arguments ought to be anchored in both previous research and the empirical findings of the study in question (Lithner, 2017; van den Akker, 2010). An intervention study should therefore provide information about goal(s), claim(s) and supporting theoretical and empirical argument(s) (Lithner, 2017; van den Akker, 2010).

To characterize the teaching designs found in the retrieved articles, the review used a version of van den Akker's (2010) characterization of teaching designs as modified by Lithner (2017). Lithner's modified version used the categories *goals*, *claims* and *arguments*, which convey information about:

- the goal to be attained through the suggested teaching;
- the claim made as to how the stipulated goal is to be reached, i.e., the means for reaching a learning goal, including information about the overarching ideas of the intervention and how the intervention methodology is to be executed; and
- theoretical and empirical arguments supporting the claims about reaching the stipulated goal; empirical arguments consist of empirical evidence from both previous research and the research conducted for the study in question.

Aim and research question

Teaching designs can have different educational goals, make different claims as to how to reach these goals and cite different arguments supporting the claims made. The aim of literature review is to characterize teaching design in mathematics educational research, connected to problem solving and/or reasoning. The research question is accordingly: What characterizes the teaching design research that aims to support students' learning *of* problem solving or reasoning and *via* problem solving or reasoning?

Method

The literature review method essentially followed the method introduced by Gough, Oliver and Thomas (2013).

Search strategy

The search included results from journal articles published between 2000 and 2016. Only journal articles were included (excluding doctoral theses, books and conference proceedings) because the most important research results are published as journal articles (Ryve et al., 2015). The search was conducted in the following journals on mathematics education: *Educational Studies in Mathematics*, *For the Learning of Mathematics*, *The Journal of Mathematical Behavior*, *Journal of Mathematics Teacher Education*, *Journal for Research in Mathematics Education*, *Mathematical Thinking and Learning* and *ZDM Mathematics Education*. The rationale underlying this selection was that these journals were internationally top ranked by Toerner and Arzarello (2012), meaning that only top-quality articles would be reviewed. The search was conducted via the Mathematics Education Database (MathEduc Database, 2017). Classification codes provided by the database were used to delimit the search. The classification codes included "teaching methods and classroom techniques", "lesson preparation", "educational principles", "investigating and problem solving" and those referring to the 1st–13th school years.

For an article to be included in the review, its title, keywords, or abstract had to contain a two-term combination (term 1 plus term 2) of terms shown in table 1 and report on an intervention study. During pilot testing of the search method, a single-term search was tried, resulting in too many irrelevant hits, as well as a three-term search, which missed potentially relevant hits. Using a two-term combination search method seemed to result in balanced search results. The two-term combination search method was previously used in a literature review by Ryve et al. (2015). During pilot testing of the two-term combination search method, more search terms were included. The results of this testing indicated that some search terms overlapped and did not render more unique hits. For obvious reasons, the combination "design*, design*" was omitted. The term "mathematic*" was not used, because only journals on mathematical educational research were included in the search.

The first search identified 187 unique articles. First, to ascertain whether the inclusion criteria were potentially met, all the articles were checked by reading the title and keywords. If there was any doubt as to whether the article should be included in the review, it was included in a secondary check. This first check resulted in 84 selected articles. Second, the abstracts of all 84 articles were read to check whether the inclusion criteria were still met. If there was any doubt, the whole article was read. This resulted in a final selection of 26 articles to be reviewed (see appendix for list of reviewed articles).

Table 1. *The search terms used in the literature review*

Term 1	Term 2
Teach*	Design*
Instruction*	Method*
"Problem solv*"	Principle*
Reason*	
Design*	

Method for extracting and thematizing goals, claims and arguments

For each article, the *goals*, *claims*, and theoretical and empirical *arguments* for the teaching design were identified by closely reading the full text of each article. To extract the goal of a design, the following analytical question was posed: "What goals are to be attained through the suggested teaching/intervention?" The question was considered answered when the design intervention's purpose was identified (van den Akker, 2010). To extract the claims of a design, the following analytical question was posed: "What claims are made concerning how the goals are to be reached?" The question was considered answered when (i) a description of the characteristics of the proposed intervention (i.e., a description of the overarching ideas of the intervention) and (ii) the methodology for how this was implemented had been identified (van den Akker, 2010). The arguments supporting a teaching design were of three types: theoretical arguments, empirical arguments from previous studies and empirical arguments in the study itself. To extract the different types of supporting arguments, the following analytical question was posed: "What supporting arguments are given to indicate that the goals were achieved through the claimed teaching/intervention?" The question was considered answered when (i) theoretical arguments and (ii) empirical arguments from the study in question (presented as results) and from previous studies were identified.

Themes were then extracted from the identified goals, claims and arguments. Thematic analysis (Braun & Clarke, 2006) was used to do this, by means of (i) close reading of the full article, (ii) generating initial codes, and (iii) combining codes into themes. Examples of initial codes (with examples of relevant articles in parentheses) resulting in the goal theme "problem solving" were: "ways to work" (Abdu, Schwarz & Mavrikis, 2015, article 1); "strategies" (Koichu, Berman & Moore, 2004, article 10; Lynch & Star, 2014, article 13); and "problem solving" (Csíkos, Szitányi & Kelemen, 2012, article 4; Lee, Yeo & Hong, 2014, article 12). After close reading and rereading of the selected articles, the initial codes were all connected to the goal of developing students' problem solving competency. The results section presents examples illustrating why certain articles were categorized under certain themes.

Results

This section first presents a table (table 2) listing the reviewed articles, showing the various goals, claims and theoretical arguments identified when analyzing the articles, as well as whether the claims of the articles used problem solving or reasoning to reach the stipulated goals. Thereafter, three sections present descriptions and examples of the reviewed articles' various goals (section Goals as part of a teaching design), claims (section Claims as part of teaching design) and theoretical and empirical arguments (section Theoretical arguments and examples of empirical arguments as part of a teaching design). A number in parentheses after each article connects the article to the results in table 2; the full article reference is found in the appendix. To illustrate the results, examples of reviewed articles are cited, rather than presenting an exhaustive treatment of all reviewed articles.

Goals as part of a teaching design

All but two studies had goals connected to mathematical competencies (these two studies are indicated as "Other" in table 2 in the "Goal" column). Nine studies had goals connected to problem solving, one to reasoning, and 14 to other competencies (e.g., communication, representations and connections). Therefore, ten studies (nine problem solving plus one reasoning studies) had the learning goal "learning *of* the competencies problem solving or reasoning".

An example of a study with a learning goal connected to problem solving was one by Visnovska and Cobb (2015) (article 25), in which the authors presented the results of an in-service training program. The aim was to study a means of supporting teachers' development of instructional practices that involved diagnosing students' reasoning and adjusting the instruction according to the diagnosis. The in-service training was to promote the development of students' problem solving competency. A study with a competency goal other than problem solving and reasoning was conducted by Ridlon (2009) (article 18), who reported on a two-year classroom-intervention study with the goal of improving students' communication and thinking skills by letting students work on problem solving tasks in collaborative groups. Here, problem solving was used as a means to develop other competencies – i.e., learning *via* a competency.

Table 2. *Articles included in the full-text literature review*

Article	Goal	Claim		Theoretical argument
		Overarching idea	Via PS or reasoning	
1	PS	TS	PS	ZPD
2	Other competency	Other	PS	RME
3	Other	TU	PS	Other
4	PS	Other	PS	Arguments not explicit
5	Other	EM	Reasoning	RME
6	Other competency	EM	Reasoning	RME
7	Other competency	SS	Reasoning	TDS
8	PS	TU	PS	Arguments not explicit
9	Other competency	TS, SS	Reasoning	ZPD
10	PS	TU	PS	Arguments not explicit
11	Other competency	TS	PS	ZPD
12	PS	SS	PS	Arguments not explicit
13	PS	TU	Reasoning	Arguments not explicit
14	Reasoning	SS	Reasoning	ZPD
15	Other competency	TU	PS	Arguments not explicit
16	Other competency	TU	PS	Arguments not explicit
17	Other competency	TS	Reasoning	ZPD & HLT
18	Other competency	TU	PS	Arguments not explicit
19	Other competency	Other	Reasoning	Other
20	PS	TU	PS	Arguments not explicit
21	Other competency	EM	Reasoning	RME & HLT
22	Other competency	TS	Reasoning	RME & HLT
23	Other competency	TU	PS, Reasoning	Arguments not explicit
24	Other competency	TU	PS	TDS
25	PS	TS	Other	ZPD & RME
26	PS	Other	PS	Other
Total: 26	PS: 9 Reasoning: 1 Other: 2 Other competency: 14	SS: 3 TS: 5 SS & TS: 1 EM: 3 TU: 10 Other: 4	PS: 14 Reasoning: 10 PS & reasoning: 1 Other: 1	ZPD: 4 ZPD & RME: 1 ZPD & HLT: 1 RME: 3 RME & HLT: 2 TDS: 2 Arg. not explicit: 10 Other: 3

Notes. The articles' goals, claims and theoretical arguments are stated, and whether the goal was to be reached via problem solving or reasoning. The various goals, claims and theoretical arguments are defined in sections Goals as part of a teaching design, Claims as part of teaching design, and Theoretical arguments and examples of empirical arguments as part of a teaching design, respectively.

Abbreviations: PS – Problem solving, TS – Teacher-led scaffolding, EM – Emergent models, SS – Students' self-assisted scaffolding, TU – Teaching unit, ZPD – Zone of proximal development, RME – Realistic mathematics education, TDS – Theory of didactical situations, HLT – Hypothetical learning trajectories.

Claims as part of teaching design

In analyzing the studies, four overarching ideas of emergent claims as to how the stipulated goals were reached: teacher-led scaffolding, student self-assisted scaffolding, emergent models and teaching units. These overarching ideas are characterized in the subsections that follow. Four of the 26 studies did not make claims fitting into these categories. In 25 cases, the teaching design goal was reached *via* problem solving or reasoning, for example, in the study by Ridlon (2009) (article 18) mentioned in section Goals as part of a teaching design. It was more common to use problem solving and reasoning as means (i.e., learning *via* problem solving or reasoning, 25 studies) than to have problem solving or reasoning as a teaching goal (i.e., learning *of* problem solving or reasoning, ten studies).

The analysis showed that a similar claim could be connected to different goals. For example, goals connected to problem solving and goals connected to other competencies could both be claimed to be reached via some type of scaffolding. Claims made about teacher-led scaffolding and student self-assisted scaffolding were most frequently connected to goals about problem solving or reasoning, while claims about emergent models were most frequently connected to goals about other competencies. The overarching idea of teaching units was connected to goals about both problem solving and other competencies. In the following subsections, the four overarching ideas of claims are presented.

Scaffolding via the teacher

The first overarching idea of claims concerned reaching design intervention goals by means of teacher-led scaffolding. Wood, Bruner and Ross (1976) characterized *scaffolding* as a "process that enables a child or novice to solve a problem, carry out a task or achieve a goal which would be beyond his unassisted efforts" (p. 90). This was exemplified in Prediger and Pöhler's (2015) study (article 17), whose goal was to develop students' conceptual understanding in mathematics and language learning in a multilingual context. Their claim was for a scaffolding strategy characterized by the teachers' (i) ongoing diagnoses of the students' thinking and learning, (ii) adapting their instruction to the students' thinking and learning, and (iii) systematically fading out their interactional support. The most frequently used move was to ask students for clarification in spoken language. A second example of a claim connected to teacher-led scaffolding, was made by Visnovska and Cobb (2015) (article 25) in a study reporting on a professional development program in which the teachers were to become more proficient in whole-class scaffolding. The goal was to improve teachers' instruction (see section Goals as part of a teaching design). Methodologically, this was implemented by means of high-quality instructions, open-ended tasks and supporting the students based on their specific difficulties – so-called adaptive support. Within this approach, teachers were expected to diagnose students' mathematical reasoning by focusing on what students did and said as they participated in classroom activities. Visnovska and Cobb (2015) reported that the professional development program led teachers to more proactively plan how to support the emergence of students' reasoning.

Abdu et al. (2015) (article 1) reported on a study with two goals of teaching students: to solve mathematical problems and to learn how to learn together. During two iterations, a group of teachers enacted scaffolding strategies in a whole-class context that fostered metacognitive skills and the social dimension that encourages group learning. The study showed that the strategic organization of learning together positively influenced the students' ability to solve mathematical problems together.

Student self-assisted scaffolding

The second overarching idea of claims concerned reaching design intervention goals by teaching students how to learn by using scaffolding strategies or solution method plans aimed at scaffolding their problem solving. For example, Lee et al. (2014) (article 12) reported on a study claiming that the goal had been reached by scaffolding students via a prompting scheme focused on understanding and planning. The claim was supported by the empirical results, which indicated that participating students from the experimental group developed problem solving strategies better than did a reference group. However, the authors stated that the intervention might have been slightly too short for the students to internalize these strategies.

Emergent models

The third overarching idea of claims concerned emergent modeling. According to Gravemeijer (1999), the essence of emergent modeling is letting students take a situation-specific problem and model it informally as a point of departure, and gradually, with teacher support, letting the model develop into more formal mathematics. In this way, the students gain new mathematical knowledge and understanding. According to Gravemeijer (1999), that emergent modeling is useful is a claim connected to realistic mathematics education (see section Theoretical arguments and examples of empirical arguments as part of a teaching design). For example, Doorman and Gravemeijer (2008) (article 6) investigated an intervention in which teachers used an instructional sequence designed to help students move from informal mathematics to more formal mathematics, in this case, reasoning about change. In another example, Stephan

(2015) (article 22) reported on a teaching design intervention whose goal was to let students meaningfully learn addition and subtraction. Both Doorman and Gravemeijer (2008) and Stephan (2015) referred to emergent modeling as the means for reaching the teaching goal.

Teaching unit

The final overarching idea of claims concerned various teaching units in which teachers executed some kind of detailed instructions. This overarching idea was found in various studies but did not comprise claims connected to teacher-led scaffolding, student self-assisted scaffolding, or emergent models. This overarching idea of claims represents a rather varied group, since the way the teaching units were constituted varied considerably from study to study, but was united by the common theme of using detailed teaching instructions. For example, Tempier (2016) (article 24) presented a design study whose goal was for students to overcome difficulties with the decimal number system. The claim made was that this could be done by providing the teachers with instructional resources intended to produce autonomy and support student decision making. Tempier (2016) was quite detailed as to how the teaching sequence should be conducted.

Theoretical arguments and examples of empirical arguments

This section presents the theoretical arguments that emerged, together with examples of empirical arguments supporting the claims made. Connections are also made between the theoretical arguments and the various claims. The results concerning the arguments used to support the claims are summarized in table 3.

Sixteen out of 26 studies based their claims on theoretical arguments. The studies that did not heavily rely on theoretical arguments justified their designs by referring to previous empirical results. Four theoretical arguments were identified: Hypothetical learning trajectories (HLTs), Realistic mathematics education (RME), Theory of didactical situations (TDS) and Zone of proximal development (ZPD). The basis of HLTs (Clements & Sarama, 2004; Simon, 1995) is the understanding that, for example, teachers anticipate the learning paths of their students in order to plan teaching and facilitate learning. Central to RME (Freudenthal, 1973, 1991; Gravemeijer, 1994) is using mathematics to understand and solve real-world problems, in order to gradually develop formal mathematics (Freudenthal, 1973). For example, in the study reported by Visnovska and Cobb (2015) (see section Scaffolding via the teacher, article 25), the theoretical argument supporting the claim made rested on

RME. Visnovska and Cobb (2015) stated: ”The supports for the teachers’ learning included an instructional sequence on statistical data analysis that reflected RME design principles and that had been developed and refined during two prior classroom design experiments” (p. 1135). RME emphasizes the importance of understanding teaching from the students’ point of view and strives to support the progressive development of their mathematical activity. Arguments connected to HLTs and RME seemed to complement each other, since they are both used to support claims made in two of the retrieved studies (Stephan & Akyuz, 2012, article 21; Stephan, 2015, article 22). Stephan and Akyuz (2012) wrote: ”The approach that undergirds the design of the integers instructional sequence is Realistic Mathematics Education” (p. 432); the authors continued, saying that HLTs ”served as the backbone for this study” (p. 433). A third theoretical argument is TDS (Brousseau, 1997), which was defined in section Design research. Tempier (2016, p. 264f) (article 24) used TDS to support his teaching design:

> Based on the example of didactical engineering developed [...] by Brousseau, I can say that didactical engineering is characterized by collaboration between teachers and researchers for studying didactical situations; collaboration in which teachers and researchers have distinct roles. [...] Through the individual teachers involved in [the] experiments [...] I aim to develop a resource for potential larger scale use.

A final theoretical argument was found in articles that referred to ZPD (Vygotsky, 1978) in connection with scaffolding (Puntambekar & Hubscher, 2005; Smit, van Eerde & Bakker, 2013; van de Pol, Volman & Beishuizen, 2010; Wood et al., 1976). ZPD is defined as the ”distance between the child’s actual developmental level as determined by independent problem solving and the higher level of potential development as determined through problem solving under adult guidance and in collaboration with more capable peers” (Vygotsky, 1978, p. 86). ZPD was used as a theoretical argument supporting the claim by Kazak, Wegerif and Fujita (2015) (article 9), who reported on a study in which students were to scaffold each other via communication: ”the group is able to spontaneously reproduce the role of the teacher who, from a Vygotskian perspective, offers scaffolds for problems that are within the ZPD of the learner” (p. 1270). As seen in table 3, the most common theoretical argument was ZPD, and HLTs were only used in combination with other theoretical arguments. The studies most likely to lack a supporting theoretical argument were those making teaching unit type claims (see section Teaching unit). Overall, the reviewed articles made many references to previous

Table 3. *The overarching ideas of the claims supported by theoretical and empirical arguments (number of articles in parentheses)*

Claims: overarching idea	Theoretical argument	Examples of empirical arguments
Teacher-led scaffolding (6)	ZPD (5*)	Positive results were obtained using scaffolding in combination with small group strategies (Abdu et al., 2015). Scaffolding and tinker plots enhanced conceptual understanding (Kazak et al., 2015). Student efforts to present and defend the task and solutions were partly successful (Kotsopoulos & Lee, 2012).
	RME (2*)	The teachers proactively planned how to support the emergence of student reasoning (Visnovska & Cobb, 2015).
	HLT (2*)	Results indicate that students could successfully use their own experience to grasp the meaning of addition and subtraction (Stephan & Åkyuz, 2012). Results indicate "the relevance of a key characteristic of effective micro-scaffolding, namely reference to a hypothetical learning trajectory as macro-orientation" (Prediger & Pöhler, 2015, p. 1179).
Student self-assisted scaffolding (4)	ZPD (2)	The scaffolding support tool enabled a shift in the understanding of the worked problem (Kazak et al., 2015).
	TDS (1)	The interactive visualizations activated the formation of intuitive access to concepts of calculus (Hoffkamp, 2011).
Teaching units (10)	TDS (1)	The "situations can potentially help students to learn place value concept" (Tempier, 2016, p. 261).
Emergent models (3)	RME (3*)	Understanding was enhanced when building on students' informal notion of speed (de Beer, Gravemeijer & van Eijck, 2015). Emergent models supported the development of student reasoning (Doorman & Gravemeijer, 2008).
	HLT (1*)	Results indicate that students could successfully use their own experience to grasp the meaning of addition and subtraction (Stephan & Åkyuz, 2012).

Note. * One or more of the articles used more than one theoretical argument; see table 2 for details.

empirical studies to support their teaching design claims. The articles also cited empirical arguments derived from their own empirical investigations in proving the claims made. Table 3 includes examples of empirical arguments supporting the claims made.

Conclusion and discussion

Most of the reviewed studies' goals concern an emphasis on, and an urge to improve, the teaching and learning of problem solving, reasoning, and other mathematical competencies, an emphasis in line with the conclusions of Carpenter et al. (2004), Hiebert and Grouws (2007) and

Niss (2007). Though this might be seen as an expected result, considering the selection criteria, this indicates that the reviewed articles mirror an international trend in mathematics education research. The literature review shows that problem solving and reasoning are successful ways of achieving teaching design goals, even if the goals are other than problem solving or reasoning. This may be because the mathematical competencies are intertwined (Niss, 2003), so it is possible to develop one competency by means of another, as previously shown by, for example, Collins (2012), Granberg (2016) and Warshauer (2015). For example, in the reviewed studies, reasoning was seldom an explicit teaching goal, but was frequently used as a means to reach a learning goal. The review showed that claims concerning the use of scaffolding strategies are connected to different kinds of goals, possibly because scaffolding is an effective teaching strategy for student learning in general or at least in a wide range of situations.

The claims regarding goal achievement, cluster into four overarching ideas: teacher-led scaffolding, student self-assisted scaffolding, emergent models and various teaching units. These claims could, in turn, be connected to the theoretical arguments' hypothetical learning trajectories, realistic mathematics education, theory of didactical situations and zone of proximal development. The empirical arguments given in the individual studies can be seen as pieces of evidence that the tested teaching designs were individually successful.

The review shows that most of the studies base their designs on theoretical arguments and the studies that cite theoretical arguments to support their claims often refer to an intermediate theory. The most frequent theoretical argument used to support claims relates to ZPD (Vygotsky, 1978). Some of the reviewed articles expand on the notion of adaptive scaffolding, which can be understood as more detailed and individualized support. In this context, adaptive scaffolding seems to tie scaffolding and ZPD together with HLT (Clements & Sarama, 2004; Simon, 1995), in that HLT can support the construction of scaffolding. Several of the articles cite examples of how students can be scaffolded in their mathematical learning, without taking away the mathematical and cognitive challenge. One element of scaffolding is the gradual fade-out of teacher support, which can be seen as part of letting students engage in productive struggle. Students are engaged in *productive struggle* if they "figure something out that is not immediately apparent [...] [and are] solving problems that are within reach and grappling with key mathematical ideas that are comprehendible but not yet well formed" (Hiebert & Grouws, 2007, p.387). Letting students engage in meaningful mathematics, and doing this through productive struggle, has been shown to be crucial for their learning (Hiebert & Grouws, 2007; Schoenfeld, 1985). Potential productive struggle can be a way to develop problem solving and reasoning (Jonsson, Norqvist, Lithner & Liljekvist, 2014; Schoenfeld, 1985). Productive struggle could also be claimed to be a way to develop other mathematical competencies. None of the studies connects the TDS notion of *situation of devolution* (Brousseau, 1997) to the scaffolding aspect of systematically fading out the teacher's interactional support, even though these concepts seem related. In a situation of devolution, the student systematically receives conditions, rules, goals and criteria for success from the teacher, and thereby the responsibility for learning is handed over to the student (Brousseau, 1997).

The groups of identified claims and references to theoretical arguments seem to connect to some of the key strategies of formative assessment (Black & Wiliam, 2009), although only one article explicitly refers to it. A key strategy to help a student reach a stipulated goal is for the teacher to provide feedback, to help the student progress as a learner. This strategy can be linked to the elements of scaffolding diagnosis and adaptive support (Puntambekar & Hubscher, 2005; van de Pol et al., 2010). Another key strategy of formative assessment is activating students as owners of their learning process. This strategy can be linked both to claims about emergent models, in which a central element is that students construct their own mathematics, and to scaffolding and the concept of systematically fading out the teacher's interactional support (Puntambekar & Hubscher, 2005). This key strategy can be linked to the TDS notion of devolution, because activating students as owners of their learning process includes handing over responsibility to the student.

Ten of the reviewed articles did not present theoretically clear arguments supporting their teaching designs. According to McKenney and Reeves (2012), teaching designs should be based on theory to be stable. Stylianides and Stylianides (2013) stated that one key to letting research influence classroom practice is to test theory-based solutions, to show what things work and to explain why they work. Cobb et al. (2003) stated that one characteristic goal of teaching design research is the advancement of theory. It could therefore be questioned whether theory can be advanced if one's teaching design is not clearly grounded in theory from the start. Most of the studies that did not explicitly use theoretical arguments to support their claims belonged to the teaching unit claim category, in which more detailed teaching instructions were used during the intervention. This connection could be interpreted as indicating that if the aim of a teaching intervention is to advance theory and teaching based on theory, detailed teaching sequence instructions might not be the best way to achieve this.

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Appendix

The following articles were included in the review. The numbers refer to the article numbers in table 2.

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