Preschoolers exercising mathematical competencies

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The mathematical ideas that emerge in children's free and guided play can be both complex and sophisticated, and if they are linked to formal mathematics, they can be a powerful basis for mathematical development. To form such links, one needs knowledge of how children use and express these ideas. This is especially true in the intersection of arithmetic and geometry, where the intermingling of numerical and spatial concepts and skills is not yet fully understood. This study aims to gain understanding of children's mathematical practices by describing the interplay of key mathematical ideas, and more specifically how young children exercise mathematical competencies in the intersection of early arithmetic and geometry. The results show that children can use spatial representations when reasoning about numbers, and that they are able to connect spatial and numerical structures. Furthermore, it is shown that children not only use and invent effective procedures, but also are able to explain, justify and evaluate such procedures.

As is well known by now, young children develop complex mathematical knowledge and abilities already before entering first grade (Ginsburg & Seo, 1999; Hachey, 2013; Kilpatrick, Swafford & Findell, 2001; Mulligan & Mitchelmore, 2015; Perry & Dockett, 2015). While children do develop mathematical ideas in free play (Sumpter & Hedefalk, 2015), highlighting mathematical aspects of their activities can stimulate and challenge their reasoning (Björklund, 2008; Lee & Ginsburg, 2009; Wager, 2015; Wager & Parks, 2016). In order to build on children's own ideas to support their mathematical learning, the ability to notice mathematical thinking is key (Sherin, Jacobs & Philipp, 2011). Since children's activities may bridge over various mathematical concepts and ideas, it is a problem that the main part of research on early childhood mathematics studies individual

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topic areas, especially arithmetic and numerical knowledge (Clements & Sarama, 2007, 2011; Mulligan & Mitchelmore, 2015; Verschaffel, Greer & De Corte, 2007). While some studies on more general characteristics of young children's mathematical development have started to emerge (e.g. Mulligan & Mitchelmore, 2015), we still need more knowledge of how significant mathematical ideas are expressed in children's activities (Jacobs, Lamb & Philipp, 2010; Sumpter & Hedefalk, 2015).

Between teacher controlled direct instruction and children's free play, lies a spectrum of play-based pedagogical strategies where the teacher guides children's learning by asking questions and responding to their actions. In this kind of guided play, the underlying activity can be either teacher or child initiated (Wager, 2015). The key elements are that the children engage in discovery-learning in a practice that makes sense to them, and that the teachers are co-playing with the children, encouraging the children's natural curiosity by articulating their actions and building on their understanding, interests and cultural practices (Fisher, Hirsh-Pasek, Newcombe & Golinkoff, 2013; van Oers, 2010; Wager, 2015). While free play is self-directed, guided play is guided by an adult, who may have created an activity or utilized a situation of free play. Previous studies have shown that guided play can support the learning of early arithmetic (Wager & Parks, 2016; van Oers, 2010) and is superior to direct instruction and free play for developing geometrical knowledge (Fisher et al., 2013). When teachers draw attention to the mathematical ideas emerging in play, they create a link between children's intuitive ideas and formal mathematics (Fisher et al., 2013; Moss, Bruce & Bobis, 2016; Samuelsson & Carlsson, 2008; Wager, 2015; van Oers, 2010). Such strategies are also in line with the Swedish curriculum for the preschool, stating that the preschool activities should be based on children's experiences, interests, needs and views, and that children should get stimulation and guidance from adults in order to increase their competence and acquire new knowledge and insights through their own activity (Skolverket, 2016). However, in order to notice mathematically important ideas in children's activities, teachers need deep knowledge of mathematical concepts and principles (Ginsburg & Seo, 1999; Moss et al., 2016).

As mentioned above, there is a vast field of research on early arithmetic and the understanding of number, which have also been shown to be a good predictor of overall mathematical achievement (Clements & Sarama, 2007; Gersten, Jordan & Flojo, 2005; Jordan, Glutting & Ramineni, 2010). However, while not as extensively studied, there is also a possibly causal relation between spatial thinking and general mathematical ability (van Nes & de Lange, 2007; Moss et al., 2016; Verdine et al., 2014). Spatial structuring – the mental operation of constructing an organization or form for an object or set of objects (Battista et al., 1998) – may be closely related to subitizing and the early conception of number, as these processes are conceptually close and seem to engage the same areas in the brain (van Nes & de Lange, 2007; Verdine et al., 2014). From very early ages children recognize symmetry and pattern, and more abstract understandings of these concepts develop over time (Clements & Sarama, 2007). Mulligan and Mitchelmore (2015) separate the notions of pattern, a predictable regularity in numbers, space or measure; and structure, which is the way the elements of a pattern are organized and related. They have found awareness of pattern and structure to be critical and salient to mathematical development at large, and Mason, Stephens and Watson (2009) claim that "appreciation of mathematical structure is vital for understanding, and well within grasp of learners of all ages, even if it is not explicit or articulated" (p. 12). Spatial structuring can form a base for structural thinking in other topic areas, as structuring two- and three-dimensional space has been found to contribute to students' understanding of multiplication and algebra (Mulligan & Mitchelmore, 2015). Still, geometry and spatial skills have not received the same attention and time in early education as arithmetic and number sense, and teachers are often less educated, comfortable and interested in geometry than in other mathematical areas (Clements & Sarama, 2011; Moss et al., 2016; Moss, Hawes, Naqvi & Caswell, 2015).

Mathematics is by its nature connected, without clear boundaries between topic areas. This is reflected in mathematical practice, and children's activities are not an exception. For example, playing with interlocking blocks requires counting skills as well as measurement concepts (Verdine et al., 2014). In order to link children's intuitive ideas to formal mathematics, teachers need to notice key ideas in mathematical practice. It follows that the study of children's mathematical practice requires analytical tools that focus on generic aspects overarching mathematical areas – sometimes called proficiencies, processes or competencies (Kilpatrick, Swafford & Findell, 2001; NCTM, 2000; Niss & Jensen, 2002). Insights in how children exercise such competencies would further the understanding of the complexity of early mathematics, and potentially form a ground for professional development for preschool teachers. This is especially needed in the intersection of early arithmetic and early geometry – an important but overlooked area in early childhood education.

The aim of this study is to gain understanding of children's mathematical practices by describing the interplay of key mathematical ideas. More specifically, the study was set to answer the question: *How do young children exercise mathematical competencies in the intersection of early arithmetic and geometry?*

Theoretical framework

In the present study, a framework developed especially for analysing the exercising of mathematical competence is used (Säfström, 2013a, 2013b). The framework is an adaptation of the research framework described by Lithner et al. (2010), and the terminology used is inherited, via Lithner et al. (2010), from the Danish KOM-project (Niss, 2003; Niss & Jensen, 2002;). Mathematical competence is defined as the ability to handle essential elements of mathematical practice (cf. Niss, 2003; Säfström; 2013a). Whenever this ability is applied it is also sustained and improved (Säfström, 2013a). The phrase "exercising competence" is meant to cover both applying and developing ability. In accordance with Niss (2003), mathematical competence is seen as multi-dimensional, with the range of mathematical topic areas as one dimension. Each topic area considers specific abstract mathematical entities (AMEs) – which may of course overlap – for example numbers are central to arithmetic, and points and lines are central to Euclidean geometry. Lithner et al. (2010) include a large variety of mathematical concepts as AMEs, but here they are restricted to self-sufficient, abstract concepts treated as objects within a mathematical practice.

Besides topic areas, this framework describes two other dimensions of mathematical competence, named *competencies* and *aspects* (Lithner et al., 2010; Niss, 2003; Säfström, 2013a). The competencies are major, distinct constituents of mathematical competence, each concerning the ability to handle one family of essential elements, namely representations, procedures, connections, reasoning and communication. The aspects capture the dual nature of each competency (Niss, 2003), which is described below. Since the AMEs are central within each mathematical practice, the competencies are defined by their relation to AMEs.

The *representation competency* is the ability to handle representations of AMEs. A representation can be mental or real; it can take the form of e.g. a spoken word, a gesture, a written symbol or a material object (Goldin, 2003; Hiebert & Carpenter, 1992; Kilhamn, 2011; Lithner et al., 2010; Sfard, 2009). The NCTM standards express the twofold nature of a representation as both: "a process and a product – in other words, to the act of capturing a mathematical concept [...] in some form and to the form itself" (NCTM, 2000, p. 67). In mathematics, representations can be constructed and used to understand and record information, to facilitate the exploration of a problem, and to monitor and evaluate a problem solving process (Stylianou, 2011). Speiser, Walter and Sullivan (2007) showed that students not only use standard representations to anchor arguments, but also develop their own, and evaluate representations. More generally, representations can serve as sharable objects of thought – in other words as means of communication – and can help coordinate thought (Kirsh, 2010), serving as a means for reflection on other elements.

The *procedure competency* is the ability to handle mathematical procedures, which have been described as sequences of mathematical actions solving a task (Lithner et al., 2010). More generally, procedures are sequences of manipulations of representations, e.g. balancing equations or calculating and marking simple points in order to draw a curve. Within early arithmetic, conceptual subitizing and object counting, are important procedures (Clements & Sarama, 2007). It is thereby clear that procedures always are linked to specific representations of specific AMEs. Since the term procedure sometimes carries negative connotations (cf. Star, 2005), it is important to stress that both imitating others' procedures and constructing one's own are included in this competency. Both types of use have their place in mathematics: while construction and adaptation stimulate creativity, repetition and imitation can lead to revelations if one notices patterns and connections in the process (Handa, 2012).

The *connection competency* is the ability to handle mathematical connections. Connections form the texture of mathematical practice. Lithner et al. (2010) list five types of connections: between two AMEs, between parts of one AME, between two representations of an AME, between parts of one representation, and between representations of different AMEs. In this framework connections between procedures and parts of procedures are also included, following Hiebert and Carpenter (1992) and Ellis (2007). Connections form one's network of knowledge (Hiebert & Carpenter, 1992), and the richness of connections is thereby a quality of mathematical knowledge (Baroody, Feil & Johnsson, 2007; Star, 2005, 2007). Johanning (2008) especially stresses the importance of connections across contexts, since stepping outside one topic area into another, comparing the two, may lead to new appreciation of both areas.

The *reasoning competency* is the ability to reason mathematically. Reasoning is defined as the explicit act of justifying choices and conclusions by mathematical arguments (Lithner, 2008). Arguments can take verbal, written or mental form, as well as the form of gestures. There is a whole spectrum of ways to build on others' arguments, from repeating to creating novel ideas. Mueller (2009) found examples of reiterating, redefining and expanding arguments in students' peer discussions. When reiterating, you confirm someone else's argument and ground it in your own thinking by restating it. When redefining, you turn the argument around, using different words, and thereby you add to the ways to think about it, both ascertaining it and opening up new ways of further reasoning. When expanding, you put the argument into a wider context and continue the line of thought.

The *communication competency* is the ability to communicate within a mathematical practice. Lithner et al. (2010) define communication as the engagement in ”a process where information is exchanged between individuals through a common system of symbols, signs or behavior” (p. 165). In this study, any attempt to exchange information is included, even if the means are not agreed upon. It is also argued that communication aims higher than at mere exchange of information, namely at construction of shared meaning (Shein, 2012; Truxaw & DeFranco, 2008). While we all have experienced exercising the other competencies without communicating, communication is problematic as an analytical construct. Reasoning can be hard to separate from communication, since internal reasoning evades observation. The same is true for the analytic aspect (see below). Nonetheless, the ability to communicate is central for mathematical practice (Truxaw & DeFranco, 2008), and reflection on communication can reveal how modes of communication can aid or hinder negotiation of meaning (Shein, 2012).

The *productive aspect* of a competency is exercised when carrying out processes with various degree of adaptation, from using established representations, procedures, connections, and ways of reasoning and communicating, to constructing one's own (Niss, 2003; Säfström, 2013a). The productive aspect therefore spans from pure imitation to innovation and creation. The *analytic aspect* involves meta-reflection, such as describing and evaluating usefulness and correctness of representations, procedures, connections, reasoning and communication, but also more general reflection on, and monitoring of, processes carried out (Niss, 2003; Säfström, 2013a).

Method

To ensure that the data contained mathematical practices, I chose to design a series of activities related to numbers, numerals and interlocking plastic bricks. I developed a script, comprising questions and cues, to guide the sessions. After a short introduction and familiarisation phase, which ended in bringing the children's attention to the studs of the bricks, the first item in the script was: ”Can you find some bricks with *diffe-rent* numbers of studs?”[1]. Questions were chosen to create opportunities for constructing and connecting representations of numbers, and to elicit verbal reasoning. For example, one item asked the children to find bricks for numeral cards, and several items in the script served to remind myself to ask the children of how and why they knew things they stated. Furthermore, the questions dealt with both arithmetic and geometric concepts and skills, enabling intermingling of the two topic areas. The choice of material was in line with the idea that ”practice with blocks may provide an early analogue for learning explicit measurement concepts and for understanding discrete units, helping build a more concrete link between number magnitudes and number language" (Verdine et al., 2014). In addition, the bricks were assumed to be well known by the children, thus needing less introduction, but not generally used for representing numbers, thus leaving room for creativity. The script was tested with two children before the actual data sessions, and assessed appropriate for children of four to five years of age.

The activities prompted by the script can be seen as guided play, since they are activities with well-planned curricular materials created by an adult, who comments on discoveries and coplay with the children (Fischer et al., 2013). The children sometimes initiated other activities, which they were allowed to pursue. Such activities were seen as free play, as they were self-directed, fun and voluntary (Fischer et al., 2013; Sumpter & Hedefalk, 2015).

A total of eight children participated in pairs, and all four sessions were video-recorded. The first session was conducted in the early summer 2011, and the other three in the spring 2012. All children were, at the time of their session, in their last year of preschool. The parents of the participating children had given their written consent before the date of the data session. The children were selected by the preschool staff, and asked if they wanted to participate and if they agreed to be filmed. The session was ended if the children seemed tired or uninterested, or if they expressed a will to stop. They were also given the opportunity to watch their own film after the session. The children were given fictitious names retaining their gender, and for the sake of anonymity, no distinction was made between the children in the first and the second round of sessions. In order to visualise the children's actions without use of photos, schematic pictures of how the children interacted with the material accompany the transcripts.

The empirical data

In all four sessions the setting was the same: the two children and the author sat together around a small table in a secluded room in the children's preschool. The activities were structured by the script, but adapted to the children and the order of events, allowing for free play if initiated by the children. After the first session, some questions that were considered too easy or too hard were removed and a new section concerning numerals was added. The questions were also modified to be more balanced with respect to the competencies, especially to further elicit verbal reasoning. Nevertheless, the overall theme was kept the same. The first session was, in total, 1h 15min and the subsequent sessions ranged between 25–40 min.

Two sets of bricks were used during the session: one mixed set (borrowed from the preschool) and one set of four different types of bricks, each in their own bag and their own colour, as seen in figure 1. These bricks, with one, two, four and eight studs respectively, were used to provide opportunity to discuss the doubling pattern: 1, 2, 4, 8. In addition, a set of cardboard tiles with numerals 1–15 was used in the last three sessions. The numerals, carrying a decimal structure, and the bricks in figure 1, carrying a binary structure (each brick corresponds to one of the first four positions of the binary number system), offered opportunities to make nontrivial connections between two representations of number.

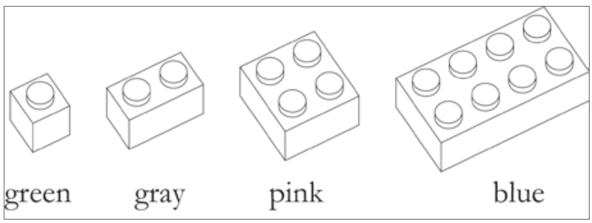


Figure 1. *The bricks in the second set*

Before entering the analysis, episodes where the children did things without connection to mathematics were set aside from the data. Next, the videos were carefully watched several times and utterances transcribed to get a good overview of the material. In a separate round, the children's activities with the bricks and tiles were documented in the transcript. The data, comprised by both video and transcript, was divided into short episodes where one or two children dealt with a particular issue. The episodes were then sorted with respect to issues dealt with, and one or a few examples of episodes were chosen to represent the handling of each issue. The choice was based on the quality of the data, so that examples where actions were visible and utterances frequent and audible were chosen as representatives.

The analysis process

This section describes the main structure of the iterative process leading to the empirical results. The analysis can be described as a form of deductive content analysis guided by the theoretical framework. The aim of the analysis was to categorise and describe the children's actions in terms of competencies and their aspects. The focus lies on the interplay of ideas, rather than individual competencies. To help this process, a series of questions with auxiliary explanations was developed, called the analysis guide. As the first session was analysed, each main question was given a number of subquestions to enable a detailed explanation of the phenomenon categorised. The full analysis guide, as well as a more detailed description of this process, is presented by Säfström (2013a).

The analysis has three main steps, starting with determining the content of the episode by detecting and naming which objects and properties the children deal with, and what actions they perform or discuss. This initial step results in a list of items, which are then characterised as AMEs, representations, procedures or connections. The representations found are organised and classified, in order to determine what abstract notion they are examples of, such as quantity, shape, or length. These notions are the AMEs, while the concrete examples and objects are representations. Actions or manipulations of representations are categorised as procedures. Properties arising from comparisons of objects, such as ”longer than”, are examples of connections. Additional connections are found by looking for relations between the AMEs, representations and procedures that are used in the episode.

Each time a representation, procedure or connection is used, it is established whether it has been previously introduced by someone else, and if so, is adapted by the user, in order to investigate the productive aspect of the competencies. Since the prior knowledge and experience of the children cannot be fully known, the analysis needs to be restrictive in labelling uses as construction. The analytic aspect is considered by searching for examples of meta-reflection, such as description or evaluation of representations, procedures or connections.

The second step is to consider the presence of reasoning, in other words looking for examples of justification and argument. The object as well as the means of justification should be related to the result of the first step: it must be a representation, procedure or connection that is justified, and arguments are often in the form of connections or procedures, or aided by representations. If arguments are used, adapted or built upon by others, the productive aspect is exercised, while the analytic aspect requires presence of evaluation of argument or general discussions on reasoning.

The third step is to look for intentional mathematical communication, i.e. to determine whether the participants try to exchange mathematical information or co-construct mathematical meaning in any way, and thereby exercise the productive aspect of communication. If communication in itself is discussed or evaluated, then the analytic aspect is exercised. This step makes use of the previous steps, as the information exchanged should refer to AMEs, representations, procedures, connections or reasoning.

As seen above, previous research has often focused on individual competencies, but few studies have considered how the competencies relate to one another. The analysis intends to relate the competencies, since the aim of this study is to describe the interplay of key mathematical ideas. As a consequence, this analysis cannot go into the same depth regarding every competency, as the studies focusing on merely one at a time. Rather, this study draws on the previous studies of specific aspects of mathematical practice, aiming to understand how these aspects interplay in children's mathematical practice.

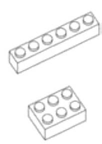
When the episodes had been analysed, further reduction of examples was made, in order to present diverse and rich examples of how competencies are exercised by young children. Four examples were selected to represent both guided and free play, and both interaction and individual endeavours. To capture the children's use of utterances, gestures and material objects, the data was described by use of a combination of excerpts, schematic pictures and accounts of events.

Results

Four examples are provided: Nina handling doubles, Olle's ship, the interaction between Simon and Tom, and Lea's brick collections. The author's first name is abbreviated to AI. The four examples demonstrate the diversity in children's exercising of mathematical competencies, and describe the interplay of mathematical ideas in each case. Each example begins with a description of the data, which is followed by the result of the competency analysis of the example. For clarity, AMEs are set in **bold**, representations in *italics* and procedures within "quotation marks". The names for representations and procedures are chosen to describe the children's individual actions in detail.

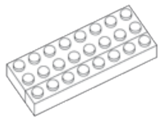
Nina handling doubles

Nina has taken on the task to find bricks in the mixed set relating to the different numeral cards. She had already placed the 2 x 3-brick on the 6-card, when she found the 1 x 6-brick.

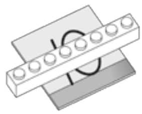


Nina: Hm. Six again. One two three. Six. 'Cause if you break it apart, it becomes like this [claps hands together and points]

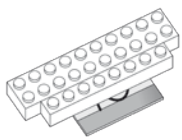
Later, she looked for a brick with ten studs. She had previously found and tallied a brick with two rows of ten studs, but since there was no card with 20, she put it aside. There were also bricks with two rows of eight studs present on the table.



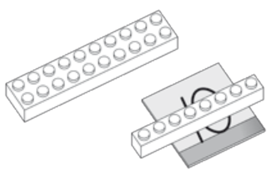
Nina: Same length. Ten … I know that ten plus ten is twenty, so it's ten



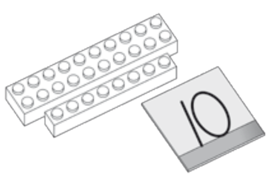
AI: Mhm



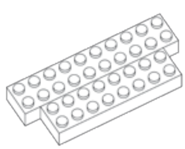
AI: That's right … hm



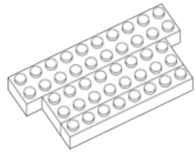
Nina: Ten plus ten is twenty



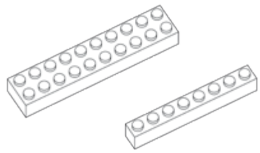
Nina: Though it's probably te …



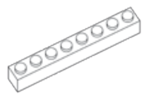
AI: No, oh, this one, aha, it was maybe shorter than this



Nina: Mm. What about that one then?



Nina: [inaudible] twenty



Nina: Eight is this one. One ... look here now: one two three four five six seven eight

The specific numbers Nina deals with (*three*, *six*, *eight*, *ten* and *twenty*) are examples of the more abstract mathematical entity **quantity**. Nina does, however, also deal with the notions of **shape** and **length** of various bricks. The specific bricks can then be seen as representations of both **quantity** and **shape**.

In the first episode, Nina establishes a connection between two brick representations of *six*: the *2x3-brick* and the *1x6-brick*, by means of the connection between *three* and *six* – she knows that *six* is the double of *three*, and she "verbally counts" to *three* two times on the *1x6-brick*. She then argues for this connection by describing how to "reshape" the *1x6-brick* into the *2x3-brick* verbally, as well as with gestures.

In the second episode, Nina makes use of the double connection once more. She knows that *twenty* is the double of *ten*, and she uses this to establish the *1x8-brick* as a representation of *ten*. However, she then evaluates her representations and thereby put her procedure into question. At this point, Nina and the author communicate about "measuring". This leads Nina to "measure" the *1x8-brick*, the *2x8-brick*, and the *2x10-brick* altogether and thereby revising the connections between the bricks (which are shorter, longer, or of the same length). She then reasons for the *1x8-bric*k not being a representation of *ten* by showing that it is shorter than *2x10-brick*, and then that it is a representation of *eight* by "verbally counting" the studs of the brick. Altogether Nina exercises both the productive and the analytic aspects of the representation, procedure and connection competencies, since she uses, as well as evaluates representations, procedures and connections. In addition, she reasons and communicates, in other words exercise the productive aspects of the corresponding competencies.

Olle's ship

Early during his session, Olle initiated his own construction activity, unrelated to the script. This activity is seen as free play. The construction engaged him for the major part of the session, and resulted in the structure shown in figure 2.

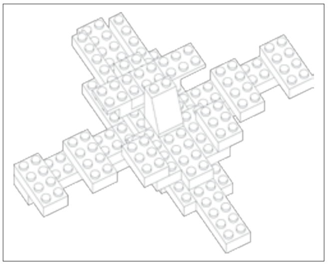
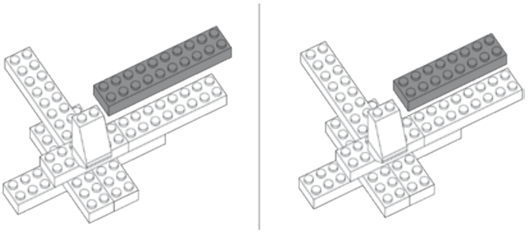
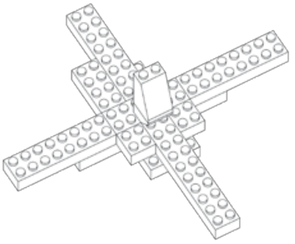


Figure 2. *Olle's construction*

At several occasions, Olle talked about what he was doing.

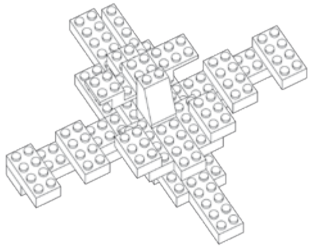


Olle: I'm looking for lego bricks that I need. But then, I need to measure, They have to be the same length.



Olle: Look ... THIS is what I've built! I mean, I've never built anything like this.

He also involved Nina in the search for specific bricks:



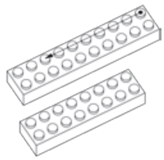
Olle: No! Where's one of those "tvåöring"? I need one of those "tvåöring" now! Thanks! Thanks, Nina! How did you know that I needed a blue one?

The central AMEs in this example are **shape**, **length** and **symmetry**. Olle is engaged in "symmetric construction" with bricks, which involves "measuring" in order to "balance the construction". The focus on bricks with the same shape implies an intermediate level of representations between **shape** and specific bricks: an *ideal brick* or mental image of the brick. He makes use of the connection between **shape** and **length**: in order for two bricks to be of the same **shape**, one needs to "measure" the **length**, while the width is seen directly. The "measuring" then becomes a means for concluding whether the *2 x 10-brick* and the *2 x 8-brick* are representations of the *ideal* 2 x 8-brick, i.e. "measuring" is a tool for reasoning. His frequent descriptions of his actions show that he is monitoring and reflecting on his procedures.

Olle is communicating at several occasions during his construction. When he urges the author to look at what he has built he does not explicitly attend to mathematical aspects of his construction, but that may be due to a lack of words to express these aspects. When he expresses his need for a specific brick he invents a verbal representation of this brick: *tvåöring* (a word for a coin, similar to a tuppence, no longer used in Sweden), which Nina understands and therefore proves useful. In other parts of the session, he also uses the words *ettöring* and *åttaöring* for bricks with one and eight studs respectively. In summary, Olle exercises the productive aspect of all five competencies: he communicates arguments for the procedures he performs in order to connect different representations. Besides being an instance of reasoning, the justification of measuring is also an example of exercising the analytic aspect of the procedure competency.

The negotiation between Simon and Tom

Simon and Tom often interacted during their session. In the episode presented below they are both looking for bricks connected to the number eight.



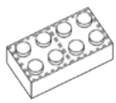
Tom: Okay. [looking at the pile of bricks] Eight. [grabs and counts one row] One two three four five six seven eight ...

AI: It was really big.

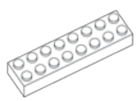
Tom: [starts counting quietly on a brick on his plate]

Simon: This one has eight, I can see that.

AI: Ah, that one has eight. How can you see that, then?



Simon: [showing on the brick with his thimbs] 'Cause four and ... 'cause there there is ... there is four and there is four – eight.

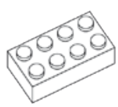


Tom: There is eight here as well!

AI: Aaa ... a little more than eight, perhaps ...

Simon: Yea! [reaches for a brick on Tom's plate]

Tom: No.



Simon: [pointing at a brick on Tom's plate] That's eight!

Tom: Yes, but if you don't count one of the rows it's eight.

The example of Simon and Tom circles around the AME **quantity**, with the example representations *four* and *eight*. They evaluate the *2 x 8-brick* and the *2 x 4-brick* as representations of *eight*, and both Simon and Tom argue for their representations with the help of procedures. Simon makes use of ”seeing and adding” the two *fours* on the *2 x 4-brick*. This implies an intermediate *mental image of four*, similar to the case of Olle. Tom makes use of ”counting one row” when reasoning for his representation of *eight*. It is clear that they exchange information, trying to convince each other and the author of the soundness of their representations. All in all, Simon and Tom exercise both aspects of the representation and procedure competency, as they both choose representations and justify their choices by explaining their procedures. Since they communicate their reasoning, establishing connections between representations, they also exercise the productive aspect of the connection, reasoning and communication competencies.

Lea's brick collections

Lea was finding combinations of bricks to put on the numeral cards. She has organized the cards in a row from 1 to 15, and is asked to use the second set of bricks. She placed bricks on the cards, counting studs only in the cases of seven and nine (see figure 3).

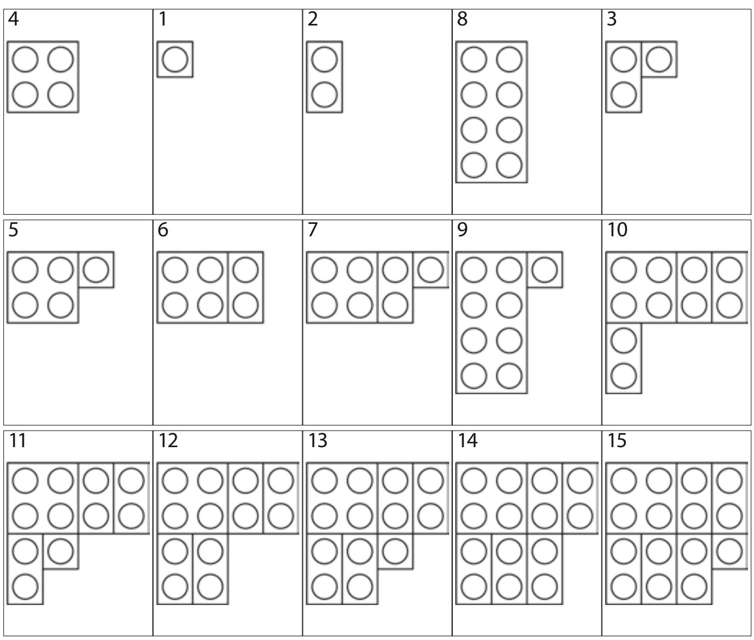


Figure 3. *Lea's brick collections*

Lea deals with **quantity**, represented by *numerals* and *bricks*. She uses the *numeral* representations given, and she constructs representations of the quantities in the form of *collections of bricks*. Her procedure ”forming brick collections” evolves over time, starting with ”recognizing single brick quantities”, to ”adding one to single brick quantities” and ”adding two to single brick quantities”, which is adapted further in the construction of the *7-collection*. Her counting of the studs in this case is interpreted as an evaluation of her procedure and its result. The construction of the *9-collection* causes her to slow down and hesitate, but she then continues to evolve her procedure to ”alternating between adding one or two to previous collections”. Her systematic approach implies monitoring the construction of representations. This procedure enables her to construct the *10-* to *15-collections* without counting all the studs. During this episode, Lea exercises both aspects of the representation and procedure competencies, as she construct and evaluate both procedures and representations. In her procedure, she also makes use of connections, exercising productive aspect of the connection competency. However, since she does not exercise the communication competency, there is no way of telling if she exercises the reasoning competency.

Conclusion and discussion

The results of this study show that preschoolers can use spatial representations when reasoning about number, and that they are able to connect spatial and numerical structure. The competency framework made it possible to focus on children’s use and evaluation of representations, procedures and connections, without restriction to one topic area. As a result, it was possible to detect the intermingling of arithmetic and geometric concepts and skills in the children’s practice. Furthermore, it is shown that children not only use and invent effective procedures, but also are able to explain, justify and evaluate such procedures.

In the procedures the children use, one recognizes well-known skills such as subitizing and object counting (Clements & Sarama, 2007). Both Tom and Nina seem to explain use of conceptual subitizing as they quickly determine number of studs on two halves of a brick, and then argue for the total number of studs. However, while Tom ”sees and adds” four and four to make eight, Nina is ”reshaping” the brick to conclude that it is the same as another one. The spatial structuring of the bricks thus aids both use and explanation of conceptual subitizing, in line with the relationship between early spatial sense and emerging number sense hypothesized by van Nes and de Lange (2007). Nina’s clapping and description of how the 1 x 6-brick can be reshaped into the 2 x 3-brick shows that she is not only capable of constructing an organization or form for an object (i.e. spatial structuring (Battista et al., 1998)), but also to imagine movements of that form (i.e. spatial visualization (van Nes & de Lange, 2007). These examples provide some further insight in how arithmetic and spatial skills are related, as called for by Verdine et al. (2014).

Verbal object counting is often used as a means to evaluate other procedures, as when Nina raises doubts about the representation of ten, and when Lea establishes the representations of seven and nine. They both invent their own, more effective, procedures for constructing brick representations – Nina by use of halves, and Lea by adding 1 x 1- and 1 x 2-bricks – but they use object counting to confirm the soundness of their procedures. Previous research has shown that preschoolers are able to invent sophisticated arithmetic strategies (Clements & Sarama, 2007), but this study also shows that they are able to reflect on and evaluate strategies.

Lea's alternating addition of bricks shows awareness of pattern (Mulligan & Mitchelmore, 2015). Her connection between the numerical pattern and the spatial pattern indicates a structural awareness, which could have been investigated further, if she had been asked to explain how the brick collections were related. Pattern and structure are also present in the case of Olle's ship. Even if there were elements of unpredictability in his construction, he gives accounts for the relation between the elements of his construction: "they have to be of the same length". Structural thinking is vital for mathematical understanding and development (Mason et al., 2009; Mulligan & Mitchelmore, 2015), and the results of this study show that it can emerge in both guided and free play.

The children in this study exercise mathematical competencies in flexible and proficient ways, and there is rich interplay of mathematical ideas in their practices. Though they have little experience of mathematical concepts and efficient procedures, they do justify their actions and conclusions, not only when asked, but also spontaneously. Naturally, these examples do not give a full picture of how competencies are exercised by young children; in deed not even for these particular children. Nonetheless does this study provide additional evidence that children engage in diverse types of mathematical thinking, as argued by Hachey (2013), and can construct mathematical concepts and strategies which are broad, complex and sophisticated, as stated by Lee and Ginsburg (2009).

Many researchers argue that mathematics teaching should build on children's informal ideas, and recognize and respond to the mathematics that emerges in play (Ginsburg & Seo, 1999; Lee & Ginsburg, 2009; van Oers, 2010; Samuelsson & Carlsson, 2008; Wager, 2015; Wager & Parks, 2016). This study has given examples of activities that provide rich opportunities for teachers to attend to key mathematical ideas, and to challenge and stimulate the children's exercising of mathematical competencies further. However, identifying and mathematizing children's activities requires nuanced knowledge of both mathematical content and children's understandings (Ginsburg & Seo, 1999; Moss et al., 2016), and might be one of the most challenging responsibilities for preschool teachers (Wager & Parks, 2016). This is especially true within geometry, since many teachers lack preparation, content knowledge and interest in this area (Clements & Sarama, 2011; Moss et al., 2015). The results of this study provide some examples of how spatial abilities can be exercised in activities concerning number, thus linking the – among teachers – less familiar field of geometry to the more familiar number sense and arithmetic. These connections between spatial and numerical structure could deepen the teacher's knowledge of mathematical concepts and principles, which is needed in order to notice key mathematical ideas in children's activities (Ginsburg & Seo, 1999; Moss et al., 2016). The results could therefore prove valuable for teacher's professional development, and by extension for children's spatial thinking and general mathematical ability. The competency framework could serve as a tool for professional noticing (Jacobs, Lamb & Philipp, 2010; Sherin, Jacobs & Phillip, 2011), aiding play-based pedagogical strategies. However, the practical value of competency analysis as a tool for teachers in their professional development and everyday practice needs to be studied further.

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Note

1 Swedish original: "Kan ni hitta några bitar med olika många knoppar på?"

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