Characterising undergraduate mathematics teaching across settings and countries: an analytical framework

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This paper explores the characteristics of teaching of a sample of university mathematics teachers in two countries, Greece and Great Britain, and in two settings, lectures and tutorials, seeking to identify a common ground for undergraduate mathematics teaching. Our observations of teaching and our sociocultural perspectives enabled us to develop a framework for a detailed description of the observed teaching. The description reveals categories of teaching actions, and the associated tools teachers use in selecting tasks for their students, providing comprehensive explanations, extending students' mathematical thinking, or evaluating students' mathematical meaning. The findings are across settings and countries in the direction of a profound understanding of undergraduate mathematics teaching.

This study is situated in mathematics education research which observes and characterises mathematics teaching in terms of teacher's planning, enacting and reflecting on their practice (e.g. Drageset, 2014; Jaworski, 2003). The purpose of the study is to search for a possible common ground for university mathematics teaching between different settings (lectures and tutorials) in two different countries. Mathematics education research offers indications that the teaching of specific mathematical practices, such as those encountered in courses of undergraduate mathematics, can in fact be very similar in different institutions and across the world (Winsløw, Barquero, Vleeschouwer & Hardy, 2014). In our study, the different settings and countries contribute to a direction of synthesising empirical research towards the big picture of how university teaching happens, which is important to be addressed (Speer, Smith & Horvath, 2010) for the improvement of teaching.

Research into university mathematics teaching has begun to study university teaching in lectures (e.g. Petropoulou, Potari & Zachariades,

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2011; Sierpinska, Bobos & Pruncut, 2011; Treffert-Thomas, 2015) or in tutorials (e.g. Jaworski, 2003; Jaworski & Didis, 2014; Mali, 2015; Mali, Biza & Jaworski, 2014) with an increased interest. However, empirical research in this area is still limited (Weinberg, Wiesner & Fukawa-Connelly, 2016). Also, we are not aware of research which explores commonalities in teaching in these two different settings, which are prevalent in undergraduate teaching (Pritchard, 2010; Viirman, 2014). In our work, we are interested in contributing to the aforementioned scarcity in extant research in order to understand and improve teaching. Improving teaching at this level can only come after developing a profound understanding of what constitutes teaching in the first place. In doing so, we searched for patterns in observed teaching and then for commonalities and differences across patterns in two settings, lectures and tutorials, and two countries, Greece and Great Britain. In particular, we addressed the following research question about teaching: What are the common patterns of teaching across teachers in our sample?

We consider teaching in terms of actions and tools. Actions include what the teacher does and says to attain teaching goals. Tools specify what the teacher uses to perform actions. Recognition of tools within actions added an analytical layer to our understanding of teaching. This additional layer offered more depth in the characterisation of teaching. We refer to teaching and teachers, rather than lecturers or tutors, to indicate our focus on gaining insights into a possible common ground of teaching across settings and countries.

Setting the scene of settings and countries

Our study involved 32 teachers (six in Greece and 26 in Great Britain) and six university researchers (including ourselves). Before we start a discussion on teaching, we offer insights into the lecture and tutorial settings in the two countries. These insights are indicative of the breadth of the social frame in which the observed teaching is situated.

All university teachers in Greek mathematics departments hold a PhD in mathematics (e.g. Functional Analysis, Topology). They are research mathematicians who are required to teach along with their research. As it is the case in nearly all countries, the predominant instructional activity in mathematics departments is the lecture. Specifically in Greece, every mathematics module lasts for a 13-week semester and includes four-hour lectures on theory and two-hour tutorials on exercises per week. The tutorials are in the same lecture theatres as the lectures and for the same number of students. However, the teacher and the content usually differ in the tutorial setting. A PhD student in mathematics usually demonstrates exercises in tutorials, while research mathematicians teach theory in lectures. Students' attendance is not compulsory in either lectures or tutorials. Also, only graduates from mathematics departments are allowed to teach mathematics at secondary level (i.e., grades 7 to 12). So, some students are prospective secondary mathematics teachers.

In Great Britain, the qualifications of teachers and the instructional activities vary among universities. At the university of this study, the teachers hold a PhD in mathematics, mathematics education or physics. Their teaching workload includes one to three modules per year, a weekly tutorial and hours for one to one support to students at the mathematics learning support centre of the university. The lectures in each module last for three hours per week over a 13-week semester. Tutorials are an additional hour of teaching per week only to a small group of first year students. Work in tutorials is on the material of lectures in Calculus and Linear Algebra, which are compulsory modules. As tutorial work is with respect to students' difficulties, students' attendance is strongly encouraged although not compulsory. Students are expected to work on the material of the lectures beforehand and bring their questions to the tutorial. After graduation, students who decide to become schoolteachers register for teacher training.

An activity theory perspective of teaching

Taking a sociocultural view, we use activity theory (Leont'ev, 1979) to investigate teaching in a university environment. Although Leont'ev's work on activity theory is not about education, it has been used to analyse research studies in mathematics teaching and learning (e.g. Jaworski & Potari, 2009; Treffert-Thomas, 2015), and it is also used in our study. We employed the language of activity theory to make sense of actions which are goal-directed and mediated by tools. In particular, we studied the instructional activities of lecture teaching and tutorial teaching through teaching actions. Following von Cranach and Valach (1984), we consider that *action* is the behaviour of a human actor which is consciously and purposefully aimed towards a goal. According to Leont'ev (1979), actions are always directed towards a conscious goal.

Leont'ev's (1979) notion of action is compatible with Vygotsky's action which is mediated by tools. Following Vygotsky (1978), we consider that the teacher acts with *tools* for teaching in order to mediate mathematical meaning to students. Cole and Gajdamaschko (2007) suggest that Leont'ev's and Vygotsky's contributions of the notion of action can be considered as complementary to a common theme; the common theme in this study is the commonalities in teaching in the lecture and tutorial settings. We agree with Morgan (2014) that the study of university teacher's conscious teaching actions makes more sense when these actions are interpreted in the light of the broader context within which the individual teacher is situated. Activity theory offers a lens for the analysis of teaching, taking into account that broader context. In our study, the broader context includes the settings and the countries, and the *common* teaching actions and associa-ted tools across them. The teaching tools specify what the teacher uses to perform actions. They are also relevant to the broader context in terms of their development and use in certain cultures and communities where teachers belong. According to Vygotsky (1978), the tools influence the mind and behaviour of learners. In other words, the tools develop the students' mathematical cognition and enculturate them into mathematical practices, such as justifying or generalising.

Literature informing the study

Undergraduate mathematics teaching has been studied with an increasing interest from mathematics education researchers (Biza et al., 2016). Here, we particularly focus on *research* literature, which is indicated by Treffert-Thomas and Jaworski (2015) in their current literature review at university level. This is the sort of literature that, for example, investigates the teaching/learning space through researchers' observations of teaching. We took into account that one of the criticisms of educational research is that it is non-cumulative (Wellington, 2000); so, after our grounded identification of actions we considered how they relate to established concepts that describe teaching at the tertiary or secondary level. We were concomitantly aware that the established concepts at the secondary level may gain a different meaning at the tertiary level. In the literature review of this section, we account for those established concepts which proved to be useful for our understanding of the actions we identified in our data.

An important part of university teachers' work, which is rather unexamined by the literature in post-secondary mathematics education, is selecting and using tasks in the classroom (Olson & Knott, 2013). Olson and Knott studied teacher problem posing; they investigated mathematical tasks which a college instructor selected and posed to engage students in meaningful mathematical activity. They named the use of tasks that required students to explore relationships as "doing mathematical tasks", and the tasks that suggested links between procedures and underlying conceptual ideas as "procedures and connection tasks". In our study, we recognised that teachers selected tasks similar to this kind.

In university mathematics classrooms, where advanced mathematical concepts and processes are introduced, students cannot be expected to reinvent entire bodies of mathematics by themselves; someone has to explain them the new material. For this reason, "telling" from the part of the teacher is instructionally important (Lobato, Clarke & Ellis, 2005). Explaining is dominant in conventional instruction, and it has been attributed the purpose of "telling". We also see explaining students what they are expected to learn and know (Larsson, 2015) as supportive to them and thus as an "expanded pedagogical telling practice" (Chazan & Ball, 1999). In research literature, we found several teaching actions that we regarded to be of explanatory nature such as: "describing" (e.g. a new concept, a representation; the meaning of a symbol; why something works) (Lobato et al., 2005); "reminding" students of a conclusion on which students and teacher have already agreed (Chazan & Ball, 1999); demonstrating several "steps" of the solution process (Drageset, 2014); "summarising" the idea (Baxter & Williams, 2010); "highlighting" the importance of the idea (Grandi & Rowland, 2013); "developing representational tools" (e.g. words, symbols, informal language, graphical representations) for students' meaning (Anghileri, 2006); and "parallel modelling" of an idea to a similar but simpler problem (Anghileri, 2006; Grandi & Rowland, 2013). Teachers sometimes ask rhetorical questions while explaining; these are questions posed without the teacher's requirement of an answer. That is intentional from the part of the teacher. According to Viirman (2015), by using rhetorical questions teachers can give an indication of how mathematics is actually done or they can highlight missing pieces of mathematical information. We consider that the above uses of rhetorical questions are of explanatory nature.

At university level, teachers may challenge and extend students' thinking. Fraivillig, Murphy and Fuson (1999) initially proposed three teaching actions for advancing (elementary) children's thinking, with "extending children's mathematical thinking" to be one of them. Extending included the encouragement of students' mathematical reflection through "analysing", "comparing" and "generalising" mathematical concepts and "considering interrelationships" among concepts. Later, Cengiz, Kline and Grant (2011) recognised actions, which were used by teachers to extend students' mathematical thinking. They indicated that such extending actions should include "providing reasoning" and "providing counter-speculation" for a claim. For Ponte and Quaresma (2015), "reasoning" is a fundamental aspect of mathematical practice. It includes "producing a statement", "generalising" (a definition, a procedure or a statement) and "justifying" (ranging from informal to more formal in relation to the context of the situation). Ponte and Quaresma also indicated that "interpreting" (e.g. a task and its different elements) is a fundamental aspect of mathematical practice, because it includes the establishment of connections. In our study, we identified aspects of mathematical practices in the actions which teachers used to extend students' mathematical thinking.

We consider that an integral part of the teaching activity is evaluating; an action found in literature to be of supporting or of informing/suggesting nature to students' contributions. For example, Conner et al. (2014) indicate that evaluating actions are centred on the correctness of the mathematics, and are in support of collective argumentation. Evaluating may include the teaching actions "validating" (e.g. "Now, I like that definition") or "verifying" the correctness of a contribution to the class. The latter action is also discussed by other studies in the mathematics education literature. For Grandi and Rowland (2013), indicating the correctness of a student answer functions as "confirming". Moreover, for Ponte and Quaresma (2015) validating students' answers (e.g. when the teacher supports a promising idea and s/he revoices it formally in more appropriate terms) is included in "informing/suggesting" actions. Likewise, in our study, we identified evaluating actions which referred to students' contributions but also included making judgments about a concept, representation or solution.

Our study also relates two types of teacher's questions to evaluating actions: "inviting questions" which are "general" or "direct" to students (Jaworski & Didis, 2014); and "control questions" (Viirman, 2015). Jaworski and Didis determine "inviting questions" to be questions with which the tutor asks "students to respond" (2014, p. 380). The role of "inviting questions" is to seek students' articulation of mathematical meaning. In our study, students' articulations were recognised from our participants as a basis for a teacher to evaluate if the audience follows the lesson. Examples of "control questions" are: "Do you follow?", "Do you understand?" and "Are there any questions?" (Viirman, 2015). In Viirman's data, the context from which "control questions" arose is "when a particularly important or complicated piece of mathematics has been presented" or "when the teacher is about to move on from one topic to another" (Viirman, 2015, p. 1175). We regarded that context as particularly useful for our recognition of "control questions". In our study, the role of "control questions" is to ask the students whether the mathematics makes sense to them, and to evaluate the students' mathematical meaning *regardless* that the students may not respond or express difficulties.

Finally, Jaworski, Mali and Petropoulou (2016) studied what university teachers do when they perform their teaching in both lectures and tutorials. They started to look at differences and similarities between the two settings of teaching, and found some similarities of the teaching approach used. However, there has been no research specifically on common teaching actions in these two settings; that is important for understanding what is invariant in university mathematics teaching. Understanding invariance can contribute to a later identification of areas for improvement. Our study starts to address this new area of interest.

Methodology

In this study, we used data from a study of cases of lecture teaching in Greece, and a study of cases of tutorial teaching in Great Britain. It was important for us to collect a data set from lectures and tutorials in different countries in order to reveal teaching actions, which are not imposed by specificities within a country or a setting. We chose universities where the settings of lectures and tutorials were established. Besides that, in both countries the teachers were in the communities of research mathematicians or research mathematics educators; no one was a graduate student.

The study of lecturing to large cohorts of 100–200+ students was conducted in two geographically diverse mathematics departments in Greece. The data collection took place over five academic semesters. The data of the study derived from the second author's observations of first year lectures on Calculus. It also included reflective discussions with each teacher right after each observed lecture and group discussions between some of the teachers and mathematics education researchers, where issues from the observed teaching were discussed. The teachers were six experienced research mathematicians. Although they were selected due to their availability they were typical cases of university teachers in this country. Four of these teaching cases have been analysed in depth to characterise the observed teaching; these were the cases with the richest pool of data. Details about the process of data analysis for this study can be found in Petropoulou et al. (2015) and Jaworski, Mali and Petropoulou (2016).

The study of tutorial teaching to a small group of 2-8 students was at a British mathematics department. The data collection took place over three academic semesters. The tutorials of 26 teachers were observed (due to their availability) and the teachers also discussed with the first author their underlying considerations of the observed teaching. A detailed account of the process of data analysis is in Jaworski et al. (2016). Three cases of teaching were studied in depth. The three teachers were experienced in lecture and tutorial teaching at university level and active researchers in mathematics or mathematics education. The three teachers were selected due to differing levels of their familiarity with mathematics education research. One case was typical of university teaching in this country. The remaining two teachers implemented some innovations along with the rigorous presentation of the abstract mathematics in their effort to promote their students' mathematical meaning.

We conducted two initial studies: one for the characterisation of teaching in lectures, and one for the characterisation of teaching in tutorials. In both settings, we took a grounded theory (Glaser & Strauss, 1967) analytical approach to observational data of teaching in order to characterise undergraduate mathematics teaching in terms of teaching actions and the associated tools. However, we do not agree with the uncontaminated view according to which a researcher should not review the research literature; various researchers embrace that opposition (e.g. Dunne, 2011; Treffert-Thomas, 2015). For instance, we were both informed by research in mathematics teaching and other areas before embarking on our initial studies. In our studies, we first generated grounded categories of teaching actions, and made sense of those categories through various instances in our data. Then, we examined an extensive literature in order to identify established concepts that describe teaching in terms of teaching actions. Following Glaser's (1978) notion of theoretical sensitivity, we related the established concepts to our grounded categories in the data in order to identify actions and tools. In this way, we were sensitive to extant research in teaching.

In our cross-case study, a finely grained analysis revealed subtleties that offered a characterisation of the invariance of the observed undergraduate mathematics teaching. The cross-case analysis restricted the number of actions and tools, which we took from the initial studies; particularly, we kept only the actions and the tools, which formed patterns of teaching in the two settings. Thus, these actions and tools came out of commonalities across the two settings. As such, we consider that they are indicative of invariant concepts of undergraduate mathematics teaching. We then grouped thematically connected teaching actions into four categories. In this way, we explored the common patterns of teaching in our sample of teachers, ultimately developing an analytical framework of mathematics teaching at this level.

Results

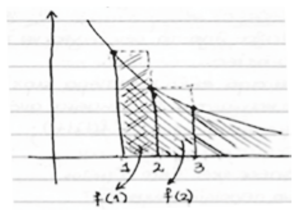
In this section, we first present an exemplification of our analytical approach to teaching actions, through two episodes of observed teaching. We selected the two episodes, because they are typical of the patterns of teaching we identified for our sample of teachers; they include several teaching actions and tools which are common in the two settings. Episode 1 was observed in a lecture and lasted for three minutes; Lect is the teacher and St is the student. Episode 2 occupied one minute in a tutorial; Tut is the teacher. We subsequently discuss the groupings of teaching actions, and we give substance to four identified categories: *Selecting*, *Explaining*, *Extending*, and *Evaluating* actions.

Example 1

This example comes from the study of university lectures in first year Calculus teaching, and concerns a part of a teaching episode where the teacher formulates and proves the theorem:

> For a decreasing, non-negative function which is defined in the interval [1, +∞], the existence of the integral $\int_1^{+\infty} f(t)dt$ is equivalent to the convergence of the series $\sum_{n=1}^{+\infty} f(n)$.

At the beginning of the episode the teacher formulated the theorem, and he prompted the students to consider ”how the series of the values of *f* is different from the integral of *f*”, aiming at students’ discovery of the relation $\sum_{k=1}^{n} f(k) \geq \int_1^{n} f(t)dt$ which is needed for the proof. The students did not respond and thus he said: ”how do we understand a theorem? Let’s draw a figure!” The excerpt below took place during the next three minutes. Figure 1 is a reproduction of the graphical representation the teacher sketched on the board.

Excerpt 1. *Negotiation of the relation between the series of values and the integral of a function f*	*Teaching actions (associated tools)*
1. Lect: We have a function *f*. And what does it do? It decreases. Good.	Highlighting (Rhetorical question)
2. Let’s say that such a function is somewhat like this. [He sketches the graph of figure 1 without the rectangles]. It is a decreasing function.  Figure 1: Reproduction of graph on board.	Connecting (Symbolic & Graphical representations) (Heuristics ‘draw a figure’ & ‘specialisation’)
3. Good. Ok. So, here [at point (1,0)] we got this arithmetical value [$y=f(1)$], at (2,0), at (3,0), etc. [He sketches the rectangles]. And what do we want? We want to sum up all these arithmetic values of the function *f*.	Interpreting (Formal language), Highlighting (Rhetorical question)
4. This arithmetic value [$f(1)$], when I get one as a step, what does it sum? This area [the first rectangle].	Reasoning (Formal language), Highlighting (Rhetorical question)
5. If I get the next arithmetic value [$f(2)$], what does it sum?	Evaluating (Inviting question-general)
6. St1: The second rectangle.	Incorrect input
7. Lect: Both [the 1st and the 2nd rectangle]. And if I go on for *n*, which is this sum? It is $\sum_{k=1}^{n} f(k)$.	Reasoning/generalising (Formal language), Highlighting (Rhetorical question)
8. But all these areas that we have summed up, they overtake - what?	Evaluating (Inviting question-general)

9. St2: The area under the graph of *f*.	Correct input
10. Lect: The integral of *f*: $\int_1^n f(t)dt$. Ok? This is the area we have drawn.	Refining / Rephrasing (Formal language), Evaluating (Control question)

In the above excerpt, the teacher employs *Selecting*, *Explaining*, *Extending* and *Evaluating* actions. He employs *Selecting* actions at the beginning, when he poses the problem of what differentiates a series from an integral (on given conditions), and when he brings the example of a specific graphical representation to help students make sense of the theorem. *Selecting* a problem or an example to initiate students into theory and foster their mathematical meaning was a typical teaching action in the observed lectures.

The teacher also develops and describes a graphical representation in order to start *Explaining* the theorem. In this way he helps students to get a sense of the symbols in an attempt to make the theorem more relevant to them. In turn 3 he describes the construction of the graph and highlights how students should think about what is given and what is asked by using rhetorical questions. (In turns 1, 3, 4, and 7 he makes four questions to students without expecting a response.)

Moreover, in this episode, the teacher employs *Extending* actions such as ”connecting” the property of decreasing and the graphical representation of f (in turn 2); ”interpreting” the arithmetical values of $f(k)$ on the graph (in turn 3); ”reasoning” for the sums of $f(k)$ (in turn 4); and ”generalising” for $\sum_{k=1}^{n} f(k)$ (in turn 7). His actions of connecting, interpreting, reasoning and generalising are seen to extend students’ thinking, because they initiate students into fundamental mathematical practices.

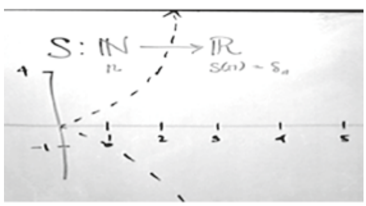
Regarding his *Evaluating* actions, the teacher invites the students to offer input by asking inviting questions (Jaworski & Didis, 2014) general to all students, because he does not ask a particular student in the lecture theatre (in turns 5 and 8). Jaworski and Didis (2014) stress that the role of inviting questions is to seek students’ articulation of mathematical meaning. We thus consider that the role of such questions is to seek students’ articulation of mathematics, and to ”evaluate” their mathematical meaning. For example, after the second inviting question (in turn 8) the teacher evaluates that the student’s contribution is of relevance by rephrasing it in formal language (in turn 10). He then asks a control question (Viirman, 2015) before moving on.

Example 2

This example is part of a tutorial for which the students suggested work on the mathematical content of ”*max*, *min*, *sup* and *inf*” of sets and sequences. Excerpt 2 below starts with the teacher introducing to the students some of her thoughts about the task:

> Determine whether the sequence $s^n = n^2(-1)^n$ is bounded or unbounded.

The analysis of the excerpt provides exemplification of *Selecting*, *Explaining*, *Extending*, and *Evaluating* actions.

Excerpt 2. *Negotiation of whether the sequence $n^2(-1)^n$ is bounded.*	*Teaching actions (associated tools)*
1. Tut: You do know what the graph of x^2 looks like, and what the graph of $(-x)^2$ looks like.	Reminding
2. And this is really a restriction of points from one or the other of these two curves [figure 2].  Figure 2: Reproduction of graph on the board.	Connecting (Graphical representations), Simplifying (Heuristic ‘draw a figure’), Interpreting (Formal language)
3. Do you see that?	Evaluating (Control question)
4. So, on the bottom, you’ve got that downward parabola. And over here, you’ve got this upward parabola. Now obviously, they should be symmetric [about the x-axis].	Interpreting (Formal language)
5. And you’re bouncing back and forth here,	Reasoning (Informal/ natural language)
6. and you can start to see that this is not going to be a bounded sequence.	Concluding/ Formulating

Excerpt 2 is part of a tutorial for which the teacher selected, from the lecture material, the task ”Determine whether the sequence $s_n = n^2(-1)^n$ is bounded or unbounded.” For this task, the tutorial group connected conceptual ideas (e.g. functions and sequences) and procedures (e.g. checking whether the sequence is bounded or unbounded with the use of definitions). We thus identified that the above task is a ”procedures and connection task” (Olson & Knott, 2013) as well as a tool associated with the practice *Selecting*.

Our analysis of *Explaining* and *Extending* (students' thinking) indicates the actions of reminding, connecting, interpreting, reasoning and concluding/formulating a conjecture. So, the teacher "reminds" the students of the graphical representations of $f(x) = x^2$ and $f(x) = (-x)^2$ (*Explaining* in turn 1); "connects" the graphical representations of functions and sequences (*Extending* in turn 2); "interprets" with formal language the aforementioned connection as well as the graphical representation of $s_n = n^2(-1)^n$ [Figure 2] (*Extending* in turn 4); and "reasons" with informal/natural language the "concluding" remark about "formulating" the conjecture that s_n is not a bounded sequence (*Extending* in turns 5 and 6). Extending also includes the teaching action "simplifying" which enculturates the students into the advanced mathematical practice of heuristic reasoning (in turn 2). In particular, "simplifying" is carried out with the heuristic "draw a figure" for the sequence $s_n = n^2(-1)^n$. The figure is the graphical representation of s_n (Figure 2), and the heuristic enables the teacher to make the conjecture.

While *Explaining* and *Extending* students' thinking, the teacher uses a control question for *Evaluating* the students' meaning of a sequence as "a restriction of points" of a function. She asks the students "Do you see that?" (in turn 3). The students do not respond verbally, but in this tutorial the teacher says to the first author that she also looks at their faces to access whether they make mathematical meaning. Our interpretation is that this is a control question because the teacher viewed that the definition of a sequence as a restriction is "particularly important or complicated" (Viirman, 2015, p. 1175) for the students, who come to the class to resolve difficulties. Also, in that tutorial the students asked for work on sequences; so, she makes the control question to evaluate their mathematical meaning.

Four categories of teaching actions at university level

Zooming out of the two episodes, we see how the teachers address teaching in similar ways across the two settings, despite the differences in class size, time allocation and breadth of mathematical content. We see that both teachers select problems and examples on which they build the theory and the mathematical observations they intend to teach. We also see that the teachers use a vast amount of time to "tell" and to explain (thereby supporting students' thinking), rather than to elicit students' thinking, both in lecture and tutorial. Furthermore, we identify various advanced mathematical practices in both teachers' teaching, such as heuristic reasoning and connections within and between mathematical areas, which expand students' spectrum of mathematical experiences, and thus extend students' thinking. Finally, we see that in both cases, teachers evaluate their students' meaning of mathematics by asking, for instance, specific types of questions.

We grouped the thematically connected teaching actions, which we found, into four categories. The four categories of teaching actions offer an analytical framework of mathematics teaching at university level. We now describe the categories to give them substance.

Selecting actions

Selecting refers to the actions of posing a problem or bringing a specific example to teach the mathematical content of the lecture or the tutorial. We further associate to *Selecting* actions: a) a task either selected by the teacher to initiate students into theory, or included in the lecture material for students' work; b) a mathematical example of a concept or of a procedure, selected to foster students' meaning in mathematics. For instance, a task could be a "procedures and connection task" (Olson & Knott, 2013), and an example could be even a "counterexample" to refute students' invalid arguments (e.g. Giannakoulias, Mastorides, Potari & Zachariades, 2010). Teachers' *Selecting* tasks and examples is crucial for students' mathematical meaning. In our study, *Selecting* actions aimed to include the students in the lecture or in the tutorial; and to help them realise, through the selected task, the need for the mathematics being taught (rather than to develop a fragmented collection of concepts, theorems and mathematical areas).

Explaining actions

In this category we consider all actions that a teacher employed when s/he aimed to make the advanced content relevant to students. In our study *Explaining* actions could be: "describing" (e.g. a new concept; a representation; the meaning of a symbol; why something works); splitting a complex process into "steps" to make it more comprehensible; "developing representational tools" to explain mathematical ideas further; reminding, highlighting and repeating; inferring, summarising, refining and concluding; and "parallel modelling" of an idea to a similar, simpler problem. We also recognised that teachers aimed to explain the mathematical content by using mathematical representations (e.g. symbolic, tabular and graphical), by asking rhetorical questions, and by using informal/natural language which was familiar to students.

Extending actions

Our analysis of episodes of university mathematics teaching triggered actions that teachers employ to initiate students into advanced *mathematical thinking and practices*. We used the concept *Extending* to denote such actions. Interpreting a task; asking a challenging question; analysing; formulating a conjecture; and providing heuristics were considered to promote mathematical reflection, and thus to extend students' thinking (Fraivillig et al., 1999). We grouped all these actions in the category *Extending*, and recognised fundamental aspects of *mathematical practices* within them. These practices included "establishing mathematical connections" and "providing reasoning" for a claim. We further recognised that teachers' use of teaching tools, such as challenging questions, different types of proofs (e.g. proof by contradiction as an alternative to existence proof), and various heuristics (Polya, 1971), extended students' thinking. For example, the teachers often used heuristics labelled as: "work section-formal write up", "draw a figure" and "specialisation". The heuristic "work section-formal write up" refers to the division of the work sheet or the board into a draft section for work and a section for the formal write up. "Draw a figure" is about sketching a graph or a diagram. "Specialisation" refers to the consideration of special case(s) of a given set of objects. Using heuristics was indicated by the teachers of our sample as a way for the students to be initiated into how mathematicians think and work.

Evaluating actions

Evaluating indicates teacher actions of making judgments about students' contributions; their mathematical meaning; or the resources they use, such as a theorem or the lecture material. We group in this category the actions, validating students' contributions; confirming; and asking questions. Only two types of teacher questions are associated with *Evaluating* actions. This is because in both settings, the teachers acted with "control questions" (Viirman, 2015) of students' mathematical meaning, and "inviting questions" (Jaworski & Didis, 2014) to students to offer their ideas; with the latter being "direct" to a student or "general" to all students. In our study, *Evaluating* actions are of informing/suggesting nature; this is also identified by other studies (e.g. Ponte & Quaresma, 2015). That is to say, *Evaluating* actions inform students about the validity of their contributions, or suggest them the correct mathematics. Also, in our data, the teachers always employed those actions in a positive, affirming manner.

Concluding remarks

Our study goes beyond a characterisation of teaching in a particular context, to identify the common ground between teaching in lectures and teaching in tutorials in two different countries, albeit the obvious differences. The purpose of the study is to synthesise common characteristics of teaching, and thus to help us develop in the direction of a more profound understanding of what invariance in university mathematics teaching is across settings and countries.

The resilience of lectures as a standard component of most university mathematics courses and the open context of the tutorials, where students work on their difficulties in mathematics, make these instructional settings thought provoking, and trigger the interest for their exploration. This paper contributes to the dearth of research literature regarding observational studies of university mathematics teaching by offering an overview of how university mathematics teaching happens in certain ways across settings and countries. It proposes an analytical framework that may be of use to researchers for a detailed analysis of teaching at this level. The framework offers new concepts, relating to teaching actions and tools, which can be applied and developed to describe mathematics teaching within and between lectures and alternative settings, in a country and between countries.

Our categories of teaching actions conceptualised as *Selecting*, *Explaining*, *Extending* and *Evaluating*, blended teaching actions and tools, and enabled us to gain insight into both actions and tools in university mathematics teaching. We consider that these categories are concepts grounded on observational data, and they also synthesise the existing literature. We carefully described the meaning that these concepts take at the undergraduate level, and gave authentic citations of our data to illustrate them. Importantly, these concepts enabled us to analyse teaching at this level across settings and countries.

We note that the distinction between the categories was not always a clear cut. An example of this tension occurred while categorising the action "developing representational tools"; we found that it was of explanatory nature, and concomitantly a heuristic with which some teachers attempted to extend their students' thinking. An instance of the explanatory nature of the action was when the teachers used it to demonstrate a graphical representation of a concept in order to provide students with insights into what it means. However, considering such tensions was constructive, because it sharpened our conceptualisations, and enabled us to study more holistically the teaching we had observed.

Our account of commonalities in teaching of our sample of teachers in lectures and tutorials is meant to be neither exhaustive nor representative of all actions and tools that could be identified in these settings. Rather, it just starts to contribute to an emergent theory in terms of frameworks for a characterisation of university mathematics teaching. For example, we consider that *Selecting* includes a part of planning the lesson, and *Evaluating* includes a part of reflecting on the lesson because of the teacher-student interaction; however, it was in our design to look at both categories from the perspective of the teaching that happens in-the-moment in the classroom.

Finally, it is our understanding that in order to touch upon the important issue of diffusion of research findings in mathematics education, our community should move from collecting stories of particular teaching cases or practices to synthesising results. In this way, we start to get the big picture of how university teaching happens in certain ways, which is important to be addressed for Speer et al. (2010). Also, the pressure exercised on universities regarding the need for a scrutiny on their teaching practices calls for an awareness of what these practices are in an inclusive way. This is an initial step towards reform of teaching practices or professional development at the undergraduate level. We see potential for our study to contribute to such efforts in future.

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