A tool for understanding pupils' mathematical thinking

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This article provides a tool for studying pupils' mathematical thinking. Mathematical thinking is seen as a cognitive function that is highly influenced by affect and meta-level of mind. The situational problem solving behaviour is studied together with metacognition and affect which together with pupils' view of mathematics form a dynamic construct that reveals pupils' mathematical thinking. The case of Daniel is introduced to illustrate the dynamic nature of the framework.

Understanding and developing pupils' mathematical thinking are key issues in mathematics education. The new Finnish curriculum states that the task for mathematics instruction is to develop pupils' logical, precise and creative mathematical thinking which creates a basis for understanding mathematical concepts and constructs and develops pupils' ability to handle information and solve problems (FNBE, 2014, p. 429). The development of mathematical thinking has been evaluated with school tests at local (class), national (e.g. Rautopuro, 2013; Hirvonen, 2012) and international levels (e.g. OECD, 2014; Mullis, Martin, Foy & Arora, 2012). The focus of local and national tests is on evaluating how well the learning objectives written in the curriculum are reached (e.g. Hirvonen, 2012) whereas international assessments such as PISA aim to assess education systems worldwide irrespective of national curriculums (e.g. OECD, 2013). In Finland, the results seem to be similar in all assessments: pupils' performance in mathematics is declining (Välijärvi, 2014; Rautopuro, 2013; Hirvonen, 2012).

Standard and standardised tests have been criticised for testing pupils with short answer questions on low-level facts and skills (Lesh & Clarke, 2000) that don't provide insight into pupils' abilities (Iversen & Larson,

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2006; Niss, 1999). Nevertheless, teachers are resistant to other kinds of (formal or informal) assessment due to their subjective nature (Watt, 2005; Watson, 2000). Mathematical thinking is a cognitive process that teachers ought to be able to evaluate, and paper tests reveal only the end-product of the thinking process. Without a closer look at pupils' mathematical thinking and aspects that influence it (e.g. metacognition), the teacher has fewer tools to help pupils to develop their mathematical thinking.

The purpose of this paper is to answer the following research question: *Is it possible to construct a tool for understanding pupils' mathematical thinking that shows the dynamic process of problem solving, metacognition and affect in their thinking*? To answer this question, a theoretical framework for studying pupils' mathematical thinking is formed. Problem solving is studied with metacognition and affect as a situational process (state) that is influenced and guided by pupils' view of mathematics (trait; cf. Hannula, 2011). After forming the theoretical framework, it is tested with an example case to see if it can be used as a tool for studying pupils' mathematical thinking, and more importantly, if it actually shows the dynamic process of problem solving, metacognition and affect in mathematical thinking. Even though this study was initially built on the Finnish curriculum (see Viitala, 2015a), the present framework is adaptable to research in different countries since its theoretical building is based on international research.

The interpretation of the results is tightly connected to the example case. Thus, while discussing and summarising the results for the research question, the mathematical thinking of Daniel will also be summarised by answering the question: *What characterises Daniel's mathematical thinking and the opportunities to develop it when studied with this tool?*

Finally, one purpose of the research study on pupils' mathematical thinking was to find a tool that not only researchers, but also mathematics teachers can use during their ordinary classroom activities or as part of pupil assessment. Hence, before summarising and concluding the article, an example of how teachers can use the tool in the *Finnish context* is presented.

Theoretical framework

In spite of the wide use of the term "mathematical thinking" in mathematics education, there is no common understanding of the meaning of mathematical thinking or even a consensus on the abilities or predispositions that underlie it (see e.g. Sternberg, 1996). Studies are influenced by the underlying learning theory, the specific mathematical domain in which the study is conducted, the special viewpoint to the issue and related literature around these issues. The focus of the study can be for instance different thinking skills or styles (e.g. creative and critical thinking, McGregor, 2007; visual, analytic and conceptual thinking, Burton, 1999), problem solving (e.g. Mason, Burton & Stacey 1982; Polya, 1957; Schoenfeld, 1985), or issues that has an effect on mathematical thinking such as research on metacognition (e.g. Stillman & Mevarech, 2010; Schoenfeld, 1987; Flavell, 1979) and mathematics related affect (e.g. Pepin & Rösken-Winter, 2015; Hannula 2012).

In 1992, after a literature review, Schoenfeld recognised five aspects that are important in a study on mathematical thinking. These are the knowledge base, problem solving strategies, monitoring and control, beliefs and affects, and practices. Similar findings have also been found in connection to literature on problem-solving performance (Lester, 1994), and are also listed as part of final-assessment criteria in the upcoming Finnish curriculum (see FNBE, 2014, pp. 433–434).

There have been some attempts to connect the abovementioned attributes in problem solving. One example is Carlson and Bloom's (2005) multidimensional problem-solving framework for individual problem solvers. They studied professional mathematicians and detailed observations were done on how resources and heuristics interact with problem solving behaviour as well as how monitoring and affect were expressed during four problem solving phases (orienting, planning, executing and checking). Their analysis showed how all of the attributes (resources, heuristics, affect and monitoring) are present in every behavioural phase of problem solving.

Together with many other studies on problem solving, the multidimensional framework of Carlson and Bloom (2005) studies problem solving from a situational and contextual viewpoint. These situational and contextual processes of problem solving, metacognition and affect are called the *states* (cf. Hannula, 2011). The affective *trait* directs pupil's engagement and success in mathematics. Affective trait is a stable pattern of "how an individual feels and thinks in these different contexts and situations" (ibid., p. 44). For instance, pupils' belief systems (traits) have been found to have an influence on their problem-solving approaches (e.g. Callejo & Vila, 2009). The two different temporal aspects reveal different competencies in pupils: the state guiding pupils' thinking and actions in a contextual problem-solving situation, whereas the trait explains pupils' learning in mathematics (cf. Bailey, Watts, Littlefield & Geary, 2014).

In the following, I will draw on existing literature and form a framework for studying pupils' mathematical thinking. The framework is new in the sense that it asks explicitly for both trait and state data. This has seldom been the case in affect research (Hannula, 2011). The affective trait is studied for two reasons: First, it is used as background information for describing a pupil (cf. Pehkonen, 1995). Similar explanations are given by teachers describing their pupils. Second, it might have an explanatory value for direct aspects uncovered from problem solving (e.g. uncertainty in problem solving, see Viitala, 2015b). From the state aspect, problem solving, metacognition and affect are considered to form a dynamic construct that (together with the knowledge base and problem solving strategies, or resources and heuristics) reveal pupils' mathematical thinking.

The knowledge base and problem solving strategies are not given emphasis in this framework since these are aspects that, according to the Finnish curriculum (FNBE, 2004), should be evaluated with ordinary school tests. The purpose of the framework is to go beyond the information gained with ordinary school tests and offer a tool that can help teachers and researchers to evaluate pupils' mathematical thinking, and more importantly, to recognize the aspects that can help pupils to develop their mathematical thinking. In the following, the different concepts of the study are introduced following the trait (pupil profile and view of mathematics) and state (mathematical thinking, problem solving, metacognition and affect) aspects of the study.

Trait – Pupil profile and view of mathematics

The role of affect in mathematical thinking is largely recognised (e.g. Zan, Brown, Evans & Hannula, 2006; DeBellis & Goldin, 2006; Vinner, 2004; Schoenfeld, 1992; also FNBE, 2014, pp. 15, 429). However, theory around affect, its concepts and their connections have been used in very diverse ways both in Finnish and international research (see e.g. Hannula, 2007; Furinghetti & Pehkonen, 2002; Pepin & Rösken-Winter, 2015). The most current theorising of affect aims to dynamic representations or systems of affect in mathematics education (see Hannula, 2011; Hannula, 2012; Pepin & Rösken-Winter, 2015). Following this line of study, the psychological phenomenon of affect is seen here as a mixture of cognitive, motivational and emotional processes (Hannula, 2011).

The term affect is used as ”an umbrella concept for those aspects of human thought which are other than cold cognition, such as emotions, beliefs, attitudes, motivation, values, moods, norms, feelings and goals” (Hannula, 2012, p. 138). The cognitive domain includes mental representations that have a truth value of some kind to the individual, for instance knowledge, beliefs and memories (e.g. Goldin, 2002). Motivation reflects personal preferences and explains choices, and emotions are different feelings, moods and emotional reactions (Hannula, 2011). How these components are studied in connection to affective trait is explained below.

The affective trait is studied through pupils' view of mathematics. Unlike its origin in beliefs-research, pupils' view of mathematics is considered to include all the affective processes (cognitive, motivational and emotional processes; thus the word "view", see Rösken et al., 2011). It has four components: mathematics (as science and as a school subject), oneself as a learner and user of mathematics, learning mathematics, and teaching mathematics (Pehkonen, 1995). Similar categories have also been found in many other studies (see e.g. Op't Eynde, de Corte & Verschaffel, 2002).

Pupils' view of mathematics is a stable construct that influences the development of mathematical thinking both on a trait and a state level. On trait level, it influences the learning of mathematics (e.g. through motivation to learn mathematics, or confidence in school mathematics). On a state level, it can influence, for instance, how a pupil approaches new mathematical content or a problem (e.g. through a belief that a mathematics task should be solved in five minutes, which might limit pupil's effort to solve a task). The categorisation of the components in view of mathematics helps a researcher, or a teacher, to direct attention to the different aspects of view of mathematics.

Pehkonen's (1995) model of pupils' view of mathematics can be criticised from not considering social aspects of pupil's view of mathematics (social and socio-mathematical norms in mathematics classroom, Op't Eynde et al., 2002). Even though social aspects are not studied explicitly, they play an important role in this study. For instance, the problem solving processes in this study are influenced by the classroom culture and norms since the tasks were solved in an ordinary classroom situation. However, from a researcher's or a teacher's point of view, social aspects arise only if they are taken forward by the pupil.

Pupils' answers to questions about his/her view of mathematics might also raise metacognitive and meta-affective issues. These are considered as traits when the answers are based on memories of experiences from mathematics classes, for instance explanations about self-regulation in mathematics learning (metacognition) or how the feeling of anxiety towards a word problem is handled (meta-affect). These terms are defined later in connection to state aspects of the study.

The pupil profile is formed for background information (cf. Pehkonen, 1995). It is a short description of the pupil that is constructed using the information arising from his/her view of mathematics. A teacher forms a pupil profile while he/she is describing the pupil as a mathematics learner. Ability, difficulty of mathematics, success, and enjoyment of mathematics has been shown to constitute the core of pupil's view of him/herself as a learner of mathematics in different age groups (Hannula & Laakso, 2011; Rösken, Hannula & Pehkonen, 2011). Ability and success relate to personal beliefs and contain statements such as "math is hard for me" (ability) and "I am sure I can learn math" (cf. beliefs about oneself as a learner and a user of mathematics, Pehkonen, 1995). Difficulty of mathematics refers to mathematics as a subject (cf. beliefs about mathematics, ibid.) and enjoyment of mathematics to emotions. Even though motivation did not result as its own component in Rösken et al.'s study (2011), it is one of the main aspects of affect (Hannula, 2011) and considered as an important factor directing pupils' problem solving and mathematics learning. Thus, pupil profile contains descriptions of ability, difficulty of mathematics, success, enjoyment of mathematics and motivation to learn mathematics.

State – problem solving, metacognition and affect

The purpose of the framework is to help researchers to understand and evaluate pupils' mathematical thinking and to develop it further. Problem solving is used as a tool to reach this aim. Thinking is situational, a state, and pupil's activities, actions and explanations during problem solving are interpreted as visible signs or expressions of his/her mathematical thinking. Thinking is considered being mathematical when it relies on operations that are mathematical in separation of thinking about the subject matter of mathematics (Burton, 1984). In problem solving, the cognitive and affective processes are intertwined (see e.g. Hannula, 2011; Zan et al., 2006; DeBellis & Goldin, 2006; Vinner, 2004) and directed by metacognition (e.g. Schoenfeld, 1992, 1987). Also meta-affect is seen to direct pupils' problem solving (DeBellis & Goldin, 2006). These issues are discussed next.

Problem solving. In the current curriculum in Finland, learning problem solving is one of the three tasks for mathematics instruction together with developing mathematical thinking and learning of mathematical concepts (FNBE, 2004). According to the final-assessment criteria, teachers should evaluate problem solving from two perspectives: problem-solving heuristics (e.g. "formulat[ing] a simple equation concerning a problem connected to day-to-day life and solve it either algebraically or by deduction", ibid., p. 166), and problem-solving phases as a thinking method. In the latter category, pupils are expected to "know how to transform a simple problem in text form to a mathematical form of presentation, make a plan to solve the problem, solve it, and check the correctness of the result" (ibid., p. 166).

The abovementioned four phases of problem solving are very similar to Polya's problem solving phases: understanding the problem, devising a plan, carrying out the plan and looking back (Polya, 1957; see table 1). Transforming a problem to a mathematical presentation requires understanding the problem. The second and third phases are the same in both descriptions. Checking the result is part of looking back. These behavioural steps in problem solving offer a framework for looking at pupils' cognitive processes, that is, mathematical thinking in problem solving. The steps are not understood to happen linearly (see e.g. Schoenfeld, 1985; Mason et al., 1982; from metacognitive research e.g. Stillman & Galbraith, 1998) and going back and forth between the steps is a natural part of problem solving processes (see e.g. Mason et al. 1982; Viitala, 2015a).

Table 1. *Problem solving (PS) phases of Polya (1957) and Finnish curriculum (FNBE, 2004)*

Polya's PS model	Finnish curriculum
Understanding the problem	Transforming a problem to a mathematical presentation
Devising a plan	Making a plan to solve the problem
Carrying out the plan	Solving the problem
Looking back	Checking the correctness of the result

In this study, a mathematical task is called a problem if the solver has to combine previously known data in a new way to her to solve a task (e.g. Kantowski, 1980). Given this definition for a "problem" we need to recognise that a task can be a routine task for one pupil and a problem to another (cf. Lester, 1994; Schoenfeld, 1992). Thus, with problem solving we refer to the activities and actions pupils perform while solving a given mathematical task *or* a problem. Problem solving is directed cognitive processing that requires mathematical reasoning (Mayer, 2003).

When pupils' cognitive processes are studied, their activities, actions and explanations during problem solving are interpreted as visible signs or expressions of their mathematical thinking. These explanations and the researcher's interpretations of the problem solving process are then complemented with explanations and interpretations of metacognitive and affective processes.

Metacognition. Metacognition is an inseparable part of mathematical thinking and problem solving. Even though a pupil might have the knowledge and skills for solving a problem, inefficient control mechanisms can be a major obstacle in solving problems (Carlson, 1999). Also metacognition has many different meanings in educational research, however, a majority of the researchers have returned to Flavell's early definition (Stillman & Mevarech, 2010).

> Metacognition refers to one's knowledge concerning one's own cognitive processes and products or anything related to them [...] Metacognition refers, among other things, to the active monitoring and consequent regulation and orchestration of these processes in relation to the cognitive objects or data on which they bear, usually in service of some concrete goal or objective. (Flavell, 1976, p. 232)

Metacognition can be categorised as metacognitive knowledge and metacognitive skills (Flavell, 1979). In Flavell's model, metacognitive knowledge refers to the interplay between person characteristics, task characteristics and strategy. Person characteristics refer to beliefs about individual and others as cognitive processors, task characteristics refer to task management and confidence for achieving the goal, and strategy refers to evaluations of the effectiveness of chosen strategies to achieve the goal. Evaluation of the effectiveness of the chosen strategy in problem solving is studied as part of metacognition in this paper. However, the two other aspects referring to beliefs and estimation of confidence will be discussed as part of affect (cf. view of mathematics, Pehkonen, 1995).

Metacognitive skills refer to control and self-regulation (Schoenfeld, 1987; Veenman, Elshout & Meijer, 1997). Metacognitive control during problem solving includes monitoring problem solving progress, deciding on the next step, and directing resources (Schoenfeld, 1987). Van der Stel, Veenman, Deelen & Haenen (2010) studied metacognitive skills in problem solving through four mathematics specific metacognitive activities that can be studied from pupil's overt behaviour. These activities are orientation, planning, evaluation, and elaboration (see example questions and a simplified connection between these activities and Polya's (1957) problem solving phases in table 2). In this study, the focus is on the quality

Table 2. *Polya's (1957) problem solving (PS) phases and metacognitive activities in PS with examples (van der Stel et al., 2010, p. 220)*

Polya's PS model	Metacognitive activities	Examples of metacognitive activities
Understanding the problem	Orientation	Estimating the answer / Making a sketch of the problem to represent the problem
Devising a plan	Planning	Designing a step-by-step action plan, instead of working by trial and error / Writing down calculations step-by-step
Carrying out the plan	Evaluation	Monitoring action plan / Checking an answer by recalculating
Looking back	Elaboration	Paraphrasing the problem / Drawing conclusions while referring to the problem statement

of metacognitive skilfulness with only a little attention to the *quantity* of these skills (cf. van der Stel et al., 2010).

Affect. The affective state follows the same structure as the affective trait: affect is seen as a mixture of cognitive, motivational and emotional processes. Affective state is situational and contextual and the task related beliefs, changing emotions, feeling of confidence and task motivation are studied together with pupils problem solving and metacognitive processes (cf. task characteristics of metacognition, Flavell, 1979).

One aspect closely connected to affect is meta-affect. Meta-affect can be seen as ”standing in relation to affect much as metacognition stands in relation to cognition, and powerfully transforming individuals' emotional feelings" (DeBellis & Goldin, 2006, p. 132). Carlson and Bloom (2005) emphasized the role of effective management of frustration and anxiety in problem solving that were shown to be an important factor in their participants' persistent pursuit of solutions to complex problems. Recognising the different feelings in problem solving might help teachers to enhance their pupils' problem solving behaviour, especially in the case of negative emotions. In this study, meta-affect is studied together with the emotional states.

Summary of the framework

In order to understand and develop pupils' mathematical thinking, teachers and researchers need a tool that goes beyond ordinary mathematics tests. The present framework recognises all the five aspects influencing mathematical thinking that were found to be important in studies on mathematical thinking: the knowledge base, problem solving strategies, monitoring and control, beliefs and affects, and practices (Schoenfeld, 1992). The knowledge base and problem solving strategies are already tested with ordinary mathematics tests. Hence, they are not the main focus of the present study. Metacognition (monitoring and control) is influencing pupils' problem solving and present in pupils' explanations about learning mathematics. Affect (beliefs and affects) is guiding the problem solving process both from state and trait (view of mathematics) levels. Practices are not studied explicitly but they are present both in state (e.g. metacognitive decision to draw a picture of a problem in an aspiration to understand it, if it is usually done in mathematics lessons) and in trait (e.g. explanations about teaching mathematics).

Problem solving, metacognition and affect are highly connected. As the literature review showed, it is often difficult to differentiate between knowledge and metacognition, or metacognition and affect. Since problem solving is studied as a dynamic process, the somewhat unclear categorising does not limit the study on mathematical thinking. On the contrary, seeing problem solving, metacognition and affect as highly interrelated can give us a more informed picture of pupil's mathematical thinking than studying these aspects separately in problem solving. The framework is built to direct our attention to different aspects that influence pupil's mathematical thinking. However, this interpretive study is open to all results arising from the data (such as social aspects of view of mathematics).

The structure of the framework is shown in figure 1. The structure is not meant to be exhaustive in respect to different aspects influencing mathematical thinking and their connections. It is a simplistic representation of the framework that shows the tools with which mathematical thinking is studied, and how trait and state are present in the study.

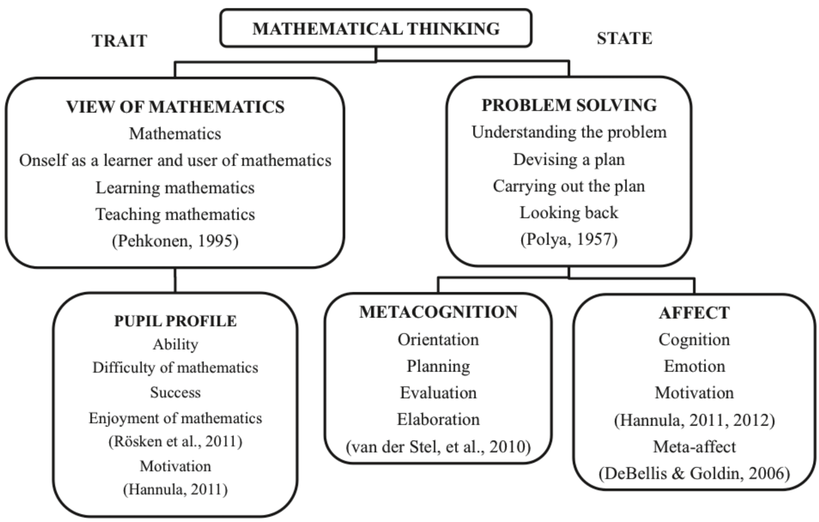

Figure 1. *A simple structure of the framework*

Methods

In this section, the phases of data collection and data analysis are explained. How teachers can use the framework in their work will be explained later in this article.

The purpose of the case of Daniel is to test the framework and to illustrate its dynamic nature. Daniel participated in a research study on pupils' mathematical thinking, and his thinking was analysed using the tool presented in this article (for previous results from the project see Viitala, 2013, 2015a, 2015b). At the time of the data collection, Daniel was at the final grade of comprehensive school (age 15).

The data used to analyse Daniel's mathematical thinking was collected from mathematics lessons and interviews in three cycles over the course of three months. In each cycle, both trait and state data were collected. The trait was about Daniel's view of mathematics and the state about problem solving. The data analysis followed the state and trait structure introduced in figure 1. Further elaborations on data analysis are given below.

Trait – pupil profile and view of mathematics

The trait data were collected through interviews. The questions treated the cognitive, emotional and motivational (Hannula, 2011) aspects of affect and followed the following themes: Daniel's background, mathematical thinking, and Daniel's view of mathematics (Pehkonen, 1995). The interviews were semi-structured and focused (Kvale & Brinkmann, 2009) and included both open and closed questions. The closed questions were either taken from large-scale studies on mathematics related beliefs (e.g. KIM-study), and/or they were asked as a follow-up question to another question (see example questions in table 3).

Table 3. *Interview themes and example questions*

Theme	Example questions
Background	Tell me about your family.
Mathematical thinking	What does mathematical thinking mean? / How do you recognise it?
Mathematics	What is mathematics as a science? / Does it exist outside of school? (How? Where?)
Oneself and mathematics	Is mathematics important to you? / Does it help you think logically? (How?)
Learning mathematics	How do you learn mathematics? / Is it most important to get a correct answer?
Teaching mathematics	Does teaching matter to your learning? (How?) / What is good teaching?

The analysis of Daniel's view of mathematics followed the same categorisation as the data collection, emphasising the connection between school mathematics and real life (also emphasised in the Finnish curriculum, FNBE, 2004) and the emergent issues from the abovementioned categories. The analysis was done one theme at a time (mathematical thinking, mathematics, oneself and mathematics, learning mathematics and teaching mathematics). After the first description about the theme issue, data reduction was executed allowing emergent and repeatedly referred issues to be highlighted. The final and condensed description was an interpretation of these results.

The pupil profile was also derived from the interview data following the descriptions of Rösken et al. (2011) for ability, success, difficulty of mathematics, and enjoyment of mathematics (cf. "Oneself as a learner and user of mathematics", Pehkonen, 1995). Pupil profile also contained Daniel's most recent mathematics grade and his motivation to learn mathematics.

State – problem solving, metacognition and affect

The state data was collected from mathematics lessons and interviews. In each cycle, Daniel solved a real-life based mathematics task in an ordinary classroom situation (see an example task below, School Excursion, OECD, 2006, p. 87; cf. real-life connections in mathematics education in the Finnish curriculum, FNBE, 2004).

> A school class wants to rent a coach for an excursion, and three companies are contacted for information about prices.
>
> Company A charges an initial rate of 375 zed plus 0.5 zed per kilometre driven. Company B charges an initial rate of 250 zed plus 0.75 zed per kilometre driven. Company C charges a flat rate of 350 zed up to 200 kilometres, plus 1.02 zed per kilometre beyond 200 km.
>
> Which company should the class choose, if the excursion involves a total travel distance of somewhere between 400 and 600 km?

The task solution was then further discussed in a stimulated-recall interview in which Daniel also assessed his confidence before, during and after solving the problem, as well as his confidence in school mathematics using a 10 cm line segment (scale from "I couldn't do it at all" to "I could do it perfectly"; see about estimation of certainty e.g. in Hannula, Maijala, Pehkonen & Soro, 2002; considerations on when to estimate confidence in problem solving, see e.g. Morselli & Sabena, 2015). Also some additional tasks were solved in the interviews. All interviews were video recorded.

The state data was analysed first by going through the problem solving phases (Polya, 1957) for all the tasks. Then, metacognitive decisions (van der Stel et al., 2010) and affective states (cognition, emotion, motivation, Hannula, 2011; meta-affect, DeBellis & Goldin, 2006) emerging in problem solving processes were investigated and descriptions of them were given. Finally, connections between problem solving (state) and view of mathematics (trait) were studied. The descriptive results are introduced in the following section of the paper.

Analysis and results: the case of Daniel

The purpose of this part of the article is to answer the first research question: *Is it possible to construct a tool for understanding pupils' mathematical thinking that shows the dynamic process of problem solving, metacognition and affect in their thinking?* The thinking is studied with one representative problem solving process, and only selected parts of the view of mathematics. The state and trait results are presented first separately and then brought together in discussion and summary of the results.

The results are supported with excerpts taken from the interview data. In these excerpts, the question Daniel is answering to or words completing the sentences are written in parentheses. The translation has been done by the researcher and critical words have been checked by an experienced researcher in mathematics education.

Trait – pupil profile and view of mathematics

Pupil profile. The pupil profile is a short description of Daniel's mathematics grade, motivation to learn mathematics, and view of himself as a learner of mathematics (Rösken et al., 2011; cf. oneself as a learner and user of mathematics, Pehkonen, 1995):

> Daniel is very confident and successful in mathematics. He has the highest grade in mathematics and he is very aware of his success. He likes mathematics, it is easy for him and he is motivated to learn it. He values mathematics and it is one of his favourite subjects.

The following excerpts support this description:

Ability and success (personal beliefs)

> (Does your grade describe your know-how?) Yes. (How would you justify your grade in mathematics?) Well, activity during lessons, test grades, the eagerness to study, how much I study and, then, how well I comprehend matters.

> (How confident are you about you and your skills in mathematics?) Very confident. 100 %.

> Learning (mathematics) is easy [...] if you know (something) from the beginning, then new things are easy to understand, and then it's easy.

> (If you should learn mathematics on your own, would you learn it?) Yes. (Without teaching?) Yes.

Difficulty of mathematics

> (Does learning mathematics require a lot of work?) Not necessarily, if you have been listening well in lessons.

> (Does learning mathematics take time?) It takes some time. It might even take a day or a week to comprehend it well. But for me, it has never taken a week. Usually it takes one lesson to learn. Or then it might be a day or so: after learning something for one lesson, and then on the next day there is another lesson on the same topic, you can comprehend it just then.

Enjoyment of mathematics

> (When I learn mathematics) the kind of like good feeling comes. When you learn something, or for instance if you don't get it at first and finally you do understand it, then you get a nice feeling.

> At times, (learning mathematics) it is also quite fun. [...] It is not that serious [...] even though of course it is important. (For you it is laid-back?) Yes, it is. [...]

View of mathematics. Daniel thinks that mathematics is the most important school subject and it is needed everywhere through life (cf. real-life connections in the curriculum, FNBE, 2004). For these reasons he claims to be motivated to learn mathematics. Like mathematics, also mathematical thinking can exist anywhere. For Daniel, mathematical thinking is "thinking mathematically about some calculations or matters". It is not just calculating something, but also models for thinking. Daniel has a perception that if you are good in mathematics, you are able to think faster.

It is difficult for Daniel to describe how he learns mathematics: he learns by listening in mathematics lessons and doing homework. He understands the cumulative nature of mathematics but he seems to connect new knowledge to the old one actively only when it is evident. For him, mathematics is "kind of becoming familiar maybe, somehow". Listening, focusing and thinking leads to the point where "pieces click together".

When Daniel talks about teaching mathematics, he says that good teaching contains describing mathematical things in detail and teaching in a very easy way so that you understand "it" for sure. He agrees that this means that the teaching starts "with the easiest" and teaching proceeds step-by-step. He thinks that his teacher is a good mathematics teacher, and most of Daniel's mathematics learning happens in the mathematics lessons, thus, this also might be understood to be the way how Daniel builds his knowledge and skills in mathematics.

State – problem solving, metacognition and affect

The state is discussed through one task, School Excursion (OECD, 2006, p. 87, see task description in Methods). The reporting follows Daniel's problem solving phases and includes metacognitive and affective (both state and trait) considerations to show the dynamic nature of the different processes in problem solving.

Researcher: How many times did you read the task?

Daniel: 3 times, I guess. [...] First I just read it and looked what has to be done.

Even though understanding the problem does not take long for Daniel, the phase is coloured with affect. Daniel estimates his confidence to be 4.7 after reading the task (on a scale 0–10, 10 being the positive end).

Daniel: Somehow, it felt at first a little like obscure because those zeds were there. Then, after I started thinking that it is probably the currency, or that it is definitely the currency, so it, the whole time, started to develop there (to a more confident direction).

Daniel feels unsure about the task description and his meta-affect directs his attention towards zeds. He then works towards understanding the meaning of zeds to become more confident to solve the task. Daniel also remembers facing a bit similar task earlier, in elementary school. Even though he does not remember any task in particular, the feeling of familiarity gives him confidence.

Daniel does not report on doing any metacognitive activities while he tries to understand the problem (e.g. estimating the answer) but at some point of solving the task, he pictured it in real life: he thought of a bus, a motorway, museums and an amusement park (cf. real-life connections in the curriculum, FNBE, 2004). Daniel *plans the task* for about 4 minutes before starting to solve it.

Daniel: [On second reading] I started looking at the numbers. [...]

Daniel: I start there, I write down those 400 and 600 km there first, for the fun of it. Then, I kind of take the intermediate result, or I mean the 500 km [...] because it's there, in between, so conveniently. So with that I try to calculate.

The interval of the travel distance disturbs Daniel "a little" but he feels confident that he can solve the task. This confidence can be traced to an affective trait: Daniel has always been able to solve the given tasks.

Researcher: Do you face challenging tasks that you don't understand right away?

Daniel: Not really because the teachers explain them well so you understand them immediately.

Researcher: There haven't been insurmountable tasks for you?

Daniel: No.

Daniel plans to write all the expressions to the prices step-by-step, for every company in the order they are written in the task description (metacognitive decision).

For the next almost 4.5 minutes Daniel *carries out his plan*: he writes down the expressions for the bus companies and then calculates all the prices with a calculator. At this point, he feels 100 % confident (10, on a scale of 0–10). While writing down the expressions, he occasionally adds units after numbers. He makes a decision to add units to all numbers (metacognitive planning).

After calculating the prices for all the companies, Daniel realises that two of the companies (A and B) are equally as cheap.

Daniel: Well, I had an initial plan already, how I, that I look at all the [...] prices (for all companies at 500 km). [...] After performing that, it came to mind there that (the prices might be different with other distances).

This realisation drives Daniel to *devise a new plan* for solving the task (metacognitive activity). He calculates the prizes in 600 km for the two remaining bus companies (A and B, *carrying out the plan*).

Daniel: [...] And then in the end, I read (the task) through one more time and I made sure I have used all the numbers from there.

Daniel's justification for reading the task description through one more time refers to a belief that all the numbers from a task have to be used. Hence, his belief (trait) guided his problem solving (state). The subsequent discussion showed, however, that even though this belief guides Daniel's problem solving, it does not necessarily determine it:

Researcher: Is it usually important to use all the numbers from a task?

Daniel: In most cases all the numbers have to be used, but in some (tasks) there can be trick numbers that you don't necessarily have to use.

Researcher: The extra numbers don't disturb you too much?

Daniel: No, not too much.

When writing down the answer, Daniel is both *looking back* to his solution (problem solving behaviour) and drawing conclusions (metacognitive activity). He recapitulates his work, relates the answer to the problem and draws conclusions while referring to the problem statement:

> The class should choose company A, because 600 km costs 675 z but otherwise A and B cost about the same amount.

After writing down his answer, Daniel starts to solve another task in the lesson. However, he quickly returns to the School Excursion task. This way of working with a task while doing something else was expressed also in the interviews when solving difficult tasks at home was discussed:

Daniel:	I start to think about it first on my own, and then if I cannot do it, but usually it is that I get it at some point of the day.
Researcher:	So you ”let it brew”?
Daniel:	Yes. If I cannot do it, I still think about it, maybe like quite a bit. If I don't kind of do it, for instance if I go and do something completely different, it can still circle in my head, I still think about it a little. [...]
Researcher:	You have the desire to continue with a task?
Daniel:	Yes.
Researcher:	You don't quit?
Daniel:	I never quit. I have to solve it.

Working with the task for a long time shows persistence, also with the task discussed here. On the other hand, returning to the task shows some uncertainty. Daniel decides to calculate the prices for companies A and B in 400 km (*devising a plan*), just to be sure. After solving them with a calculator (*carrying out the plan*) and not writing anything down (metacognitive decision), Daniel ”feels good” (emotion) and completes his written answer (*looking back* and elaborating, metacognitive action):

> But, if the distance is 400 km, B is better, because it costs 550 z, whereas A costs 575 z. So: 400 km -> B; 500 km -> A, B; 600 km -> A.

As in mathematics, Daniel is 100 % confident about his work at the end (10, on a scale 0–10). It took Daniel 16 minutes to complete the task.

Discussion and summary of results

The purpose of this part is to summarise what was found in relation to the research question about the dynamic process of problem solving, metacognition and affect, and to answer the question: *What characterises Daniel's mathematical thinking and the opportunities to develop it?*

As shown above, metacognition, affective state together with meta-affect and affective trait all have an important role in Daniel's problem solving. The relationships between problem-solving, metacognitive and affective processes *are found to be dynamic*: they have an effect on each other and Daniel moves naturally between these different processes. While metacognition and affect (both state and trait, also through meta-affect) has an effect Daniel's problem solving behaviour, his metacognitive decisions and problem solving behaviour (and success) has an effect on his affective state (emotions). The answer to the question about Daniel's mathematical thinking is described below.

Daniel can be characterised as a confident, successful and thorough problem solver. Many times he does not go into details while solving problems (e.g. the units were not all correct in School Excursion which might have reduced his scores in school tests) but he ends up with correct answers. In case of problems, Daniel asks for help from friends or the teacher. Daniel liked, perhaps even enjoyed, solving the given tasks.

When *understanding a problem* or *looking back*, the role of feelings, more precisely confidence, and the way Daniel handles these feelings by directing his problem solving (meta-affect, DeBellis & Goldin, 2006) are highlighted. For instance, with another task, Daniel sat quietly after solving a task. He explained this by saying:

> [...] Well, somehow I searched for the kind of confident feeling, like completely 100 % feeling of confidence, that those (calculations) are correct.

Daniel's metacognitive skills were highlighted in *planning* and *carrying out a plan*. For instance, with another task, Daniel went to an incorrect direction while solving the task but he had the metacognitive skills to monitor his work and direct his attention to a more productive direction. Additionally, while Daniel moves easily between different problem-solving phases, he might also move between different metacognitive phases within one problem solving phase.

> [...] At the same time (while solving a problem) I started thinking how it would be reasonable to continue and do them, or write them down [...].

When solving problems, Daniel says in the interview that he is "quite aware" of his own thinking all the time. However, this is not visible in the stimulated-recall data. In the interviews, when Daniel was asked to explain what he was thinking in the video, he could not recall his thoughts, only actions. Similarly, when explaining his learning of mathematics, Daniel refers to behavioural actions he goes through (through teaching), as well as refers to learning as feelings (becoming familiar with something). This might mean that thinking mathematically and learning mathematics are very automatic for Daniel. On the other hand, when tasks were solved in the interviews and why-questions were asked on the spot, Daniel was more able to answer them.

One reason for not being able to explain his thinking afterwards might be that, Daniel seems to be a bit unorganised as a problem solver.

He is jumping back and forth between different phases of his calculations (and between tasks) and it is hard even for Daniel to interpret what he is doing in the video. Additionally, his notes are messy (e.g. calculations are not necessarily written chronologically and one written expression might be used to calculate many calculations). Being able to return to different problem-solving or learning situations, and practising precise and focused problem solving could develop his problem-solving and learning skills, and consequently, mathematical thinking. Thus, Daniel might benefit from paying more conscious attention to his problem-solving and learning processes.

An example on how teachers can use the tool in the Finnish context

One purpose of forming the framework for studying pupils' mathematical thinking is that teachers can use it in their mathematics lessons and as part of pupil assessment. This is particularly relevant now when Finland is under curriculum reform.

According to the new curriculum (FNBE, 2014), the main part of pupil evaluation is formative assessment that happens as part of everyday teaching and working. It asks for observing pupils' learning processes and communicating with them. Feedback that advances learning is said to be qualitative and descriptive, and should help pupils to perceive and understand what they are supposed to learn, what they have learnt already and how they could advance their own learning and improve their performance (pp. 50–51).

The summative assessment can also include verbal evaluation. The verbal evaluation allows teachers to describe the level of a pupil's performance, but also to describe the pupil's strengths, progressions, and targets of development (ibid.). Below, there is an example of how the tool can be used to develop and evaluate pupils' mathematical thinking as part of mathematics teaching in the Finnish context.

Trait – pupil profile and view of mathematics

When a teacher is asked to give a short description of a pupil in his/her mathematics class, he/she quickly forms a first version of a *pupil profile*, for instance: "Sofia is an average pupil but does not bother to study mathematics and then underachieves in it". This can be used as a starting point for learning discussion many teachers in Finland are expected to have with their pupils as part of qualitative pupil assessment.

In a learning discussion, the teacher can talk with the pupil about the teacher's observations in connection to the pupil profile and ask possible reasons for the observed issues. This discussion can be short but informative enough to be used to set long-term goals for learning that both the teacher and the pupil agree, for instance: Sofia has a belief that she cannot do well in mathematics, and hence, does not study it. So, Sofia is asked to pay attention to the achievements that she did not believe she could accomplish (and perhaps write them down).

These learning goals can be supported by the teacher in everyday classroom situations when appropriate, and they will be taken forward in the following learning discussion (that might happen in a month or two). In the learning discussions, the pupil profile can be altered if there is a reason to do so and the long-term learning goal can be changed. If there is no obvious target for the long-term learning goal, the teacher can follow the themes behind the core of pupil's view of mathematics (ability, success, difficulty of mathematics, enjoyment of mathematics, and motivation to learn mathematics).

These observations and discussions about pupil profile open doors to pupil's *view of mathematics*. The teacher should recognise that pupil's view of mathematics can influence the development of mathematical thinking through cognitive, motivational and emotional processes. In Sofia's case, the cognitive belief that she is not good in mathematics affects her emotional and motivational bond to mathematics. Through positive experiences and supporting feedback this might change.

As a summary, pupil's view of mathematics and the pupil profile can offer a way to describe and evaluate pupil's development through lower secondary school offering documentation from the pupil's development as a mathematics learner and thinker in a long-term sense. In this manner, the pupil profile can also be used as a part of summative evaluation as well as a starting point for pupil's self-evaluation.

State – problem solving, metacognition and affect

The state offers teachers information about the situational and contextual thinking processes. It can also reveal issues connected to pupil's view of mathematics.

Information about pupil's thinking processes are given in everyday classroom situations. The key is to observe pupil's problem solving, ask questions about it, and most importantly, listen to the answers. If the purpose is to learn about pupil's problem solving, metacognition or affect, the problem should be one that the pupil is competent enough to solve. Otherwise the focus might turn more towards mathematical knowledge and heuristics.

As an example, Sofia got stuck after reading the problem and performing a first calculation. As before, she asks help and repeats that she is not good with word-problems. In Sofia's case, there might be a problem with *affective trait* (a belief that "I am not good in word-problems") and meta-affective skills. On the other hand, if she is able to proceed with the problem after suggesting to draw a picture about the situation, the reason might also be in *metacognitive skills*. Furthermore, if the calculation does not make any sense to Sofia or the teacher in connection to the problem at hand (e.g. summing up all the numbers in the problem), the problem might be connected to *problem-solving behaviour* and insufficient planning of the problem.

The teacher gets sense of pupils' mathematical thinking while working with them in ordinary mathematics lessons. The purpose is not to understand pupils' mathematical thinking all at once, but to take small steps towards getting to know their thinking. Also with state, the discussion can continue in the learning discussions. The key for the teacher is to focus on one issue at a time (problem-solving behaviour, metacognition, or affect as in Sofia's case) so that the learning discussions can be kept short and include both short- and long-term goals for the pupils (short term goals being mathematical in most cases).

If the teacher has problems to interpret pupils' skills, the framework can offer him/her concrete tools to categorise pupils' answers so that the weak points could be recognised and the development of mathematical thinking supported. The key elements of the framework can also be developed into key questions that a teacher can use as an actual tool in his/her work. In connection to mathematical thinking, it is also important to remember that (unlike traits) the states are contextual, and in different situations the same pupil might need very different kind of help.

Summary and conclusion

This article endeavoured to answer the research question: *Is it possible to construct a tool for understanding pupils' mathematical thinking that shows the dynamic process of problem solving, metacognition and affect in their thinking?* To answer this question, a theoretical framework for studying pupils' mathematical thinking was formed based on research literature around mathematical thinking. After forming the theoretical framework, it was tested with an example case of Daniel to see if it can be used as a tool for studying pupils' mathematical thinking, and more importantly, if it actually shows the dynamic process of problem solving, metacognition and affect in mathematical thinking.

As a result, the tool for understanding pupils' mathematical thinking was found to successfully expose the dynamic processes of problem solving, metacognition and affect in Daniel's thinking. In fact, all of these aspects were an inseparable part of Daniel's thinking process. In addition, Daniel's view of mathematics (trait) supported the findings from problem solving (state). In spite of the similar results, the trait and state perspectives are important to study separately as they report from different competencies that influence pupils' mathematical thinking: the trait revealing more stable competencies affecting pupils' mathematics learning, and the state revealing the contextual and situational competencies influencing pupils' problem solving processes.

The question about the case of Daniel concerned the use of the created tool: *What characterises pupils' mathematical thinking and the opportunities to develop it when studied with this tool*? This question was answered by interpreting the results of Daniel's mathematical thinking revealed while answering the first research question. The results showed that Daniel's metacognitive skills in problem solving as well as the natural moving between different problem solving and metacognitive phases can be characterised to be the key in his success as a mathematical thinker. His metacognitive skills outpaced affect in planning and carrying out the plan, and he was fully confident throughout the study. On the other hand, his lack of ability to return to the thinking processes after solving a task or learning in mathematics, directs our attention towards a point where Daniel could be helped to become a more successful mathematical thinker: Daniel could benefit from paying more conscious attention to his processes of problem solving and learning mathematics.

One aim of the research study was also to present a tool for studying pupils' mathematical thinking that not only researchers, but also mathematics teachers can use during their ordinary classroom activities or as part of pupil assessment. In the latter part of the paper, an example is given on how this tool could be used during ordinary classroom situations and as part of pupil assessment in the Finnish context. The first task for the teacher is to recognise if a phenomenon is connected to a state or a trait. States are less stable and can be influenced more easily. Traits, on the other hand, are more difficult to change. Thus, instead of aiming to change pupils' (affective) traits directly, teachers should aim to recognize the aspects that might hinder pupils' learning and concentrate on helping them to work through these different feelings, attitudes or beliefs in a fruitful way. Some of these obstacles that influence mathematical thinking might also be uncovered in a problem solving situation.

In an earlier publication (Viitala, 2015b), another pupil's mathematical thinking was reported using an earlier version of the framework. Unlike Daniel, this pupil, Emma, was not very confident problem solver and her affect determined many activities and actions in her problem solving processes. Learning mathematics took time for her, she asked a lot of questions and she was quite aware of the processes needed for her to learn something. Emma was found to benefit from support to overcome her feelings of uncertainty. Thus, while both Daniel and Emma were successful problem solvers, they were found to need different support for learning mathematics and developing mathematical thinking.

Based on these two example cases, the tool can be said to successfully reveal different aspects that influence the development of individual pupil's mathematical thinking in different pupils. However, these two pupils are high achievers with a positive view of mathematics. The next step would be to adapt this framework to data from low achievers with a negative view of mathematics to see if the framework is fruitful also for understanding these pupils' mathematical thinking and for evaluating how they could be best assisted towards developing their mathematical thinking.

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