Tölur og mengi – Numbers and sets

A New Math textbook in Iceland in the 1960s

KRISTÍN BJARNADÓTTIR

Only few mathematics textbooks were developed in Icelandic during 1930–1966, and the model textbooks in use were originally developed in the 1920s. In 1966, Iceland was hit by an international school-mathematics reform movement, the New Math. The learning material, which was introduced at the arrival of the movement until the 1970s, was all of foreign origin except one textbook. The author of that textbook, denoted *Tölur og mengi* [*Numbers and sets*], was main actor in the introduction of the New Math, Guðmundur Arnlaugsson, duly supported by his colleague, Björn Bjarnason. In the paper, their way of introducing the new ideas and the methods they used will be described. Among their tools is the textbook which later influenced other mathematics textbooks for adolescents after the peak of the New Math wave.

The aim of the paper is to investigate an Icelandic textbook written in the spirit of the reform movement, generally denoted New Math, and the role of its entrepreneurs, Guðmundur Arnlaugsson and Björn Bjarnason. For that purpose we shall recount how Arnlaugsson and Bjarnason became acquainted with the New Math ideology; to what sources they had access; the channels and tools they used in promoting implementation of New Math in Iceland; how the New Math ideas were reflected in Arnlaugsson's textbook, *Tölur og mengi* [*Numbers and sets*], in articles written by him and in a draft national curriculum document, written by Bjarnason; and what impact the textbook had on later textbook writing, if any.

Research method

The research method is historical: i.e. a careful analysis of a range of documents. The history is told within the framework of the history of

Kristín Bjarnadóttir
University of Iceland

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education and schools, and the general history of Iceland. The history was traced by referring to scholars' published work, legislation, regulations and reports. The textbook, *Numbers and sets*, was analysed, its foreword as well as its mathematical content. Information about its lifetime was sought in reports and official circulars and it was carefully compared to recognizable models. Biographical information about Björn Bjarnason and Guðmundur Arnlaugsson was accessed in the biographical lexicon *Kennaratal* [*Teachers' biographical lexicon*] (Kristjánsson, 1958–65; Kristjánsson and Harðardóttir, 1985–88) and *Dansk biografisk leksikon*, in addition to personal sources, such as an interview with Arnlaugsson's son and a convenience sample of four persons who studied the textbook *Numbers and sets*.

The proviso has to be taken that sources are limited and the history told is therefore adapted to what was available and the author of this paper found relevant. All quotes, originally in Icelandic, were translated into English by the author.

Iceland and its school system

Iceland was settled in the 9th and 10th century, mainly from Norway. It came under the Danish king by the Kalmar union in 1397. Cathedral schools were attached to the two bishoprics. Both the schools and the bishoprics were united into one in the early 1800s. The school belonged under the *Royal directorate of the university and the learned schools* in Denmark until Iceland gained sovereignty in 1918. The Icelandic school was, however, run by own regulations due to shorter academic year than in Denmark and underdeveloped public education. Students generally graduated from the Icelandic school at the age of 20. It was an elite six-year school with own admission requirements, unrelated to public education, and in 1928 admission to it was restricted to 25 new pupils a year. Another more rural high school was established in 1930, enrolling a similar number of pupils.

In 1903, the former seven year Danish learned school was split into a four year middle school and a three year high school (gymnasium). A parallel division was only implemented in Iceland by legislation of 1946 (Lög no. 58/1946; Lög no. 22/1946), two years after Iceland became republic in 1944. Since then, the high schools have been four-year schools from which pupils graduate at the age of 20. Regulations following the 1946 education acts prescribed a national entrance examination to the high schools in eight school subjects to be taken by pupils at the age of 16. Its goal was to ensure three factors: certain and standardized minimum knowledge; the selection of the fittest with respect to certain attributes, considered necessary for university preparation; and equity,

that is assessment by impartial persons of examination papers, and that the examinations were the same for all pupils (Pálsson & Ólafsson, 1961).

The high schools were obliged to accept all those who passed the national examination as opposed to their earlier own selection of pupils. As a compromise with the schools, the syllabus of the new centralized entrance examination was adopted as the former lower division examination in of the Reykjavík School according to regulations of 1937 (Reglugerð fyrir Menntaskólann í Reykjavík, no. 3/1937), in which the entrance examination stagnated until 1968. The 1946 education acts (Lög no. 34/1946; Lög no. 48/1946) also extended the age limits of compulsory education to 7–15 years. For that reason and others of economic nature, the education system suffered from lack of facilities and trained teachers, textbooks and curricula in mathematics as well as in other school subjects. The economic situation was reflected in restrictions of access to foreign currency and therefore limited mobility of educators, and it kept the country isolated from many influences from abroad. These factors, in addition to population growth and demands for "education for all", had stretched the system to a breaking point by the 1960s. During the 1950s, about 20 % or less of each cohort attempted the national entrance examination and around two thirds of that group earned right to enter the high schools. This ratio rose rapidly in the 1960s. The introduction of the New Math was a part of a revision of the school system in the 1960s, characterized by opening up the high schools and revising the content and teaching methods at all school levels (Bjarnadóttir, 2007, pp. 179–292, 421).

The tradition of adapting education to Danish requirements remained for a long time after Iceland's independence, as further education, such as engineering and other kinds of technical education, had to be acquired abroad, and was traditionally pursued in Denmark. Danish textbooks were in use in high schools, both due to the small market and for preparation for further education in Denmark, even if Danish was quite different from the mother tongue of the pupils.

The present age limits of compulsory education are 6–16 year age after which pupils can enter high school level without entrance examination. The norm for graduation age is still 20 years in 2015, while it is planned to become 19 in the near future (*Hvítbók*, 2014). The four-year high school has been considered to reach one or one-and-a-half years into the Unites States' college level.

Guðmundur Arnlaugsson and Björn Bjarnason

Guðmundur Arnlaugsson (1913–1996) was enrolled in Reykjavík high school in 1927. That same year his master, Ólafur Daníelsson, completed

writing his textbook series in the vernacular, the first one in advanced mathematics. Arnlaugsson was an excellent student, and at his graduation he was granted a four-year stipend to study abroad. He chose to study mathematics at the University of Copenhagen. After three years he realized that he could not complete his studies in the fourth year so he went home to teach for three years at Akureyri high school. In 1939, he left again for Copenhagen to complete his studies in 1942. All connection between Iceland and Denmark was then broken due to the World war II, so he had to stay on in Denmark. He worked as a mathematics teacher at Danish high schools until the end of the war. He was appointed mathematics teacher at the Reykjavík high school in 1945 and he taught simultaneously at the University of Iceland from 1947. Arnlaugsson became a member of the national examination board in 1946, in charge of the physics examination. Privately, he was a champion chess player and ran regular radio programs on chess.

Björn Bjarnason (1919–1999) graduated in 1939 from Akureyri high school where Arnlaugsson was his mathematics teacher. He sailed that same year to study mathematics at the University of Copenhagen from where he graduated in 1945. He was appointed mathematics teacher at Reykjavík high school in 1948 and at the University of Iceland in 1950. In 1963, he became a member of the national examination board in charge of the mathematics examination. Arnlaugsson and Bjarnason were there-

Figure 1. *Guðmundur Arnlaugsson (left) and Björn Bjarnason (right)*

fore close colleagues for decades until they were both appointed headmasters at new high schools in Reykjavík in 1965 and 1970 respectively.

The fact that Arnlaugsson and Bjarnason stayed in Denmark for 9 and 6 years each is important. During that period they became members of the Danish mathematical community, and the journal *Nordisk matematisk tidsskrift* became the thread that linked them to that community when they had returned home after the war. The main actors in implementing the New Math in Denmark, such as Svend Bundgaard (1912–1984) (*Dansk biografisk leksikon*), were Arnlaugsson's contemporaries and schoolmates and their acquaintanceships later channelled the New Math movement to Iceland.

The New Math movement

In the aftermath of World war II, many countries considered reforms of their mathematics and science teaching. An influential arena was the *Commission internationale pour l'étude et l'amélioration de l'enseignement des mathématiques*, CIEAEM, The international commission for the study and improvement of mathematics teaching, founded in 1950. Among its members were the Swiss psychologist Jean Piaget, the French mathematician Jean Dieudonné, and outstanding secondary school teachers. The main concern of the CIEAEM was a growing attention to the student and the process of teaching, the relevance of psychology in mathematics education, the key role of concrete materials and active pedagogy, and Piaget's research of the relation between mental and mathematical structures as introduced by the French Bourbaki group of mathematicians, including Dieudonné, called *mathématique moderne*, modern mathematics (Furinghetti, Menghini, Arzarello & Giacardi, 2008).

By the end of the 1950s, widening discontinuities between the mathematics taught at universities and that taught in lower schools were beginning to give rise to curriculum reform projects in various countries that collectively became known as the New Math. Mathematicians and psychologists were brought together in curriculum development projects and studies were undertaken that drew upon both perspectives. A revival of interests in issues such as learning by discovery, readiness for learning, processes of learning and aptitude for learning, helped people from different disciplines to see some common ground (Kilpatrick, 1992).

The actions by CIEAEM and the New Math movement had roots in common with the Bourbaki School: set theory, functions, relations and logic were to find their places in the new curricula, supported by the methodology of discovery. Conferences were held and the various reform movements gathered at a seminar on school mathematics reform

in November 1959, held by the Organisation for European economic co-operation, OEEC, at Royaumont, France, denoted the *Royaumont seminar*. The member countries were invited to send three delegates and the seminar was attended by representatives from all the invited countries except Portugal, Spain and Iceland (OEEC, 1961, pp. 7, 213–219). A questionnaire was sent out before the seminar and replies were reported from Iceland (OEEC, 1961, pp. 129–206). Among its guest speakers were Dieudonné from the CIEAEM, Edward Begle, leader of the *American school mathematics study group project*, SMSG, and Svend Bundgaard, mathematics professor at Aarhus university, Denmark.

In the Seminar's conclusions, arithmetic – or rather computations – were claimed to have traditionally been considered a tool needed in life and business affairs (OEEC, 1961, p. 108). Therefore, most of the teaching of this subject had been a mechanical rote-learning of facts and algorithms. Psychological implications of learning procedures used in primary schools, and the shift of school aims to developing concepts and modes of thinking (as well as skills), were conceived to necessitate a corresponding change in arithmetic instruction. The learning should be the result of understandings arising from guided experimentation and discovery, with the use of physical objects of one sort or another. In this way, the child must be led to the abstraction of the quality of a set called its number. In getting to this abstraction, it was considered necessary to use the ideas – but not necessarily the language – of sets, sub-sets, correspondence, and order. The concepts must be correctly developed right from the start (OEEC, 1961, pp. 108–109).

The understanding and use of a decimal-place system of numeration were considered necessary components of early instruction. With this place system – and the intuitive use of the laws of commutativity, associativity, and distributivity – all operations on whole numbers, common fractions, and fractions in decimal notation could be developed reasonably rather than seem like a set of magical tricks (OEEC, 1961, p. 109).

Children were to learn to calculate with reasonable speed and accuracy, as was demanded in everyday adult life. Beginning in the fifth school year and for the next few years, brighter children could be introduced to the study of number relations, involving odd and even numbers, primes, factorization, greatest common factor, least common multiple, and place-numeration systems to bases other than 10. Generalisations of arithmetical relations through the use of literal symbols could serve as an informal introduction to algebra (OEEC, 1961, pp. 109–110).

Ideas on pedagogical research were presented at the Royaumont meeting. The report of the meeting (OEEC, 1961, pp. 101–103) mentions briefly outlining of a research project of an enormous scope, proposing

a continued testing and evaluating of learning from age 5 to 18. The dimensions of the proposal were apparently highly ambitious (p. 101). An experimental study to test the envisaged curricular reforms before a general introduction was proposed; a pilot study run on a limited scale and under carefully controlled conditions before attempt were made to persuade colleagues in schools to adopt new ideas on a large scale (p. 103). Probably due to overwhelming dominance of secondary-school and university mathematicians, however, the work and the discussions of the Seminar concentrated on the reform of contents of mathematics teaching, and the pedagogical research proposals were not mentioned in the final recommendations (Schubring, 2014). One of the few mathematics educators present was Kay Piene (1904–1968) who wrote on the Seminar in the journal *Nordisk matematisk tidsskrift* (Piene, 1960a; 1960b). Piene died relatively early and is not known to have been in contact with Icelandic reform actors.

One of the final recommendations of the Seminar was that the member countries proceeded to reform mathematics teaching according to their needs, and it was recommended to establish as much cooperation as possible (OEEC, 1961, p. 125). Three Nordic countries were represented in Royaumont: Norway, Sweden and Denmark. The Nordic participants at the seminar agreed upon organizing a Nordic cooperation on the reform of mathematics teaching. The ideas were presented to governmental bodies and the issue was taken up in the Nordic council, which decided to set up a committee under its Culture commission, *Nordiska kommittén för modernisering af matematik-undervisningen* [The Nordic committee for modernizing mathematics teaching], abbreviated as NKMM (Bjarnadóttir, 2007, pp. 243–268). Each of the four Nordic countries – Denmark, Finland, Norway and Sweden – appointed four persons to the committee, which dominated mathematics instruction in the Nordic countries for most of the 1960s (Gjone, vol. 2, p. 78). Guðmundur Arnlaugsson knew Bundgaard from his Copenhagen years. Bundgaard was therefore Iceland's natural contact to the Royaumont seminar and the NKMM where it was not represented.

The Nordic committee issued a report (Nordisk råd, 1967a, 1967b). The programme for the Nordic reform was to work out curriculum plans, and to write experimental texts. The committee appointed teams of writers. The focus was on the mathematical content, and the teaching of grades 7 to 12 was its main object. However, mathematics courses throughout the school were to be handled, and the committee contacted for that purpose experts for grades 1 to 6. The NKMM primary-level material was written by primary teacher Agnete Bundgaard, sister of Svend Bundgaard. The Finnish Eeva Kyttä was her co-author of the books for the two first

years. Persons from each country would translate and adapt the joint publications to each language (Gjone, 1983, vol. 2, pp. 78–80).

New Math in Iceland – actions in the early 1960s

In the early 1960s the population of Iceland was 176.000 with 2 % increase p.a. Each cohort in schools was about 4000 pupils. Arnlaugsson and Bjarnason had slowly begun to modernize the syllabus of the mathematics stream of Reykjavík high school. Even if Iceland was not a tributary of Denmark any more, the mathematics syllabus of the high schools was still adapted to the Danish system as Icelandic students might eventually seek entrance to Denmark's technical university in Copenhagen. Arnlaugsson (1961) wrote an article on changes in the Danish high-school system where he obviously looked to similar changes as feasible in Iceland. In the autumn term 1963 he was granted sabbatical leave which he used to visit Denmark and the United States (Arnlaugur Guðmundsson, personal source). On his return, the Reykjavík high school took up the textbook *Principles of mathematics* by Allendoerfer and Oakley (1963) for the mathematics stream. The book is an offspring of the New Math ideology with emphasis on logic and set-theoretical approach to topics on the border of high-school and college mathematics. Allendoerfer was registered as a member of a conference held in Woods Hole, Massachusetts, in September 1959, where some thirty-five scientists, scholars and educators, gathered to discuss how education in science could be improved in our primary and secondary schools. Members of the Woods Hole conference, such as Edward Begle, leader of the SMSG-project also spoke at Royaumont, which illustrates the link between the conferences and the reform groups in Europe and the United States (Bruner, 1960, p. vi; OEEC, 1961, p. 217; Bjarnadóttir, 2014, p. 450–451). The *Principles* did not last long as a textbook, pupils were not used to reading English textbooks, and the *Principles* was even more formal than the traditional Danish high-school textbooks were, and during the following years, Danish and Swedish NKMM-books as well as the English SMP-series were tried out (Bjarnadóttir, 2007, p. 434).

Upon Arnlaugsson's return, in 1964, he offered himself as consultant in mathematics teaching for the Ministry of education in a half-time position. He had learned about the reform movements in Denmark and the United States and realized that school mathematics in Iceland had lagged behind other countries. The most influential arithmetic textbooks for primary and lower secondary school levels were originally written in the 1920s. The arithmetic textbook by Daníelsson (1920), Arnlaugsson's master, composed for the then lower department of the

six-year high schools, was on the reading list for the national entrance examination to the four-year high schools from 1946. It was still there in 1968 (Menntamálaráðuneytið, 1968, p. 57). Daníelsson's teacher student, Elías Bjarnason (1939–1941), published the original version of his arithmetic textbook for primary level in 1927–1929. As textbooks for compulsory level were provided free by the State textbook publishing house, only one textbook in each subject was provided for the tiny Icelandic market, and E. Bjarnason's (1939–1941) textbook was still the sole book in use in the mid-1960s.

During Arnlaugsson's term in the ministry in 1964–1966, he made a survey among schools on lower secondary compulsory level, gave in-service courses for teachers at lower secondary level and wrote a textbook, *Tölur og mengi* [*Numbers and sets*] (Arnlaugsson, 1966), the only New Math textbook of Icelandic origin until the 1970s. Furthermore, he learned through his acquaintance with Svend Bundgaard since his Copenhagen years, about the NKMM textbook series for primary school level by Agnete Bundgaard. Arnlaugsson recommended it for translation to succeed E. Bjarnason´s series. In the early 1970s, the Bundgaard series had reached nearly half the cohorts born during 1962–1965. In connection with that and Arnlaugsson's own textbook, he ran a television series of 17 episodes on the national television station about mathematics teaching and the New Math (Bjarnadóttir, 2011).

The content of the Bundgaard series was highly theoretical. The first volume introduced the number concept and counting based on the abstraction of the quality of a set, one to one correspondence, ordering, addition and its properties, and subtraction as an inverse operation to addition. The commutative and associative laws, Roman numerals and place-value notation to the base five, prime numbers, permutation of three digits, the transverse sum and its relation to the nine times table were all introduced before the close of the third grade. In the fourth year the whole set theory was introduced, with pairing, subsets, intersection and union, in addition to various place-value systems and geometry with points, lines and planes introduced in a set-theoretical framework (Bundgaard and Kyttä, 1967–68; Bundgaard, 1969–72). This syllabus was run for ten cohorts while teachers and parents were not prepared for so alien presentation of primary level topics. The Ministry of Education arranged in the early 1970s to have a new textbook series written, a kind of synthesis of new and old ideas (Bjarnadóttir, 2007, pp. 293–305).

Arnlaugsson was appointed headmaster in a new high school with its first pupils enrolled in 1966. The new school was organized on a modular system, similar to American high schools but different from the former four-grade system in older schools. Arnlaugsson had left the ministry by

then, but continued to write notable articles about the aims of the curriculum changes. In the mathematics stream of his school, he introduced Danish and Swedish textbooks produced on the basis of the NKMM work as well as the English SMP-series (Bjarnadóttir, 2007, p. 434). The search for suitable texts for university preparation, also accessible for adolescents to read in foreign languages, continued for quite a while.

A draft curriculum for the national examination

The national examination board was a small group of people working closely together. Both Arnlaugsson and B. Bjarnason belonged to it in the 1960s, and they also met daily at Reykjavík high school. Their cooperation was close. In September 1968 the paper *Drög að námskrá í landsprófsdeildum miðskóla* [*A draft curriculum for the national examination*] (Menntamálaráðuneytið, 1968) was published. Bjarnason wrote the section on mathematics but there is reason to believe that he consulted with Arnlaugsson. The document acted as regulations until 1974. In the introduction to the mathematics curriculum, the reader was informed that changes according to the New Math movement began four years earlier, which must have been when Arnlaugsson and Bjarnason introduced Allendoerfer and Oakley's (1963) *Principles of mathematics* as a textbook for the mathematics stream of the Reykjavík high school in 1964. No suitable textbooks for the lower secondary level had been available until Arnlaugsson's *Number and sets* had been published and been subject to trial for two years (Menntamálaráðuneytið, 1968). At that time around 1000–1200 pupils, 25–30 % of the year cohort, and 1400 or 34 % in 1969, attempted the national examination (Bjarnadóttir, 2007, p. 421):

> A good many years ago the curriculum of the secondary schools was subjected to revision under the leadership of the O.E.C.D., the Organization for economic cooperation and development in Paris. This revision has led in most places to radical changes taking place in mathematics curriculum and instruction. The aim is
>
> 1. to base school mathematics on the basic concepts of the set theory, which simultaneously are simple and general
>
> 2. to put more emphasis on the meaning and the nature of numbers and of number computations than has been customary.
>
> Four years ago changes in the [mathematics] curriculum in Icelandic high schools were initiated in accordance with these new aims. At once it became clear that these changes could not be successfully

> implemented unless a different approach was applied in the national examination classes, where the basis is laid for algebra, one of the most important basic topics of high school mathematics. But as no textbooks of an appropriate form existed in Icelandic and only very few teachers had had an opportunity to study the new views, it proved completely impossible to implement such changes. By the publication of Guðmundur Arnlaugsson's textbook, *Numbers and sets*, and with in-service courses for mathematics teachers, the attitude changed so much for the better, that it proved, two years ago, possible to modify the syllabus partly into the new directions, and the individual school authorities were offered the option of introducing a new syllabus, but were not instructed to do so. This was evidently not a long-term solution and it was only done in order to give the mathematics teachers time to familiarize themselves with the new ideas. – Last winter [1967–1968] the majority (more than 2/3) of the national examination pupils benefited from instruction in the new material, and as it now can be expected that the mathematics teachers have had time to adjust to changed customs, the implementation of two kinds of syllabuses will now come to an end and the same syllabus will apply to all. However, it was not considered possible to avoid adapting the syllabus almost wholly to those textbooks already available in Icelandic, even if some of them are rather old and are not suited to the desired requirements.
>
> (Menntamálaráðuneytið, 1968, pp. 56–57)

The draft curriculum document continued with guidelines to teachers on how to arrange the syllabus, such as to begin by the algebra of sets before the conventional algebra studied from Daníelsson's (1927) old textbook:

> It is very important that the pupils obtain clear ideas about the basic concepts of the set theory and their relations and acquire a mastery of its symbolic language. These basic concepts emerge in the basis of every branch of mathematics, and therefore they must often be referred to and used. The symbolic language allows ideas and their relations to be expressed in an exact and clear way. It is desirable not to begin working with the algebra of numbers (i.e. the conventional algebra) until the pupils have acquired mastery of the relations of sets and the introduction to set theory, to be found [...] in the textbook.
>
> (Menntamálaráðuneytið, 1968, p. 59)

This can be interpreted as top-down structure, to begin by the general theory and proceed down to the specific. The set-theoretical concepts were to be introduced in the first term of the academic year, before the introduction to the algebra of numbers.

Articles by Arnlaugsson

Simultaneously with and after writing the textbook *Numbers and sets*, Arnlaugsson worked at presenting and promoting the new ideas at in-service courses for lower-secondary school teachers and writing articles in the teacher journal and in a handbook on teaching. He wrote an article in 1967, "New perspectives in arithmetic teaching" which may be considered his agenda. He tried to guide the reader on how to look upon the New Math:

> One should not focus too much on new words and concepts. The value of mathematics teaching is not solely a matter of whether the syllabus offered is new or old. It depends first and foremost on to what degree understanding follows. The value of the new concepts lies more than anything in facilitating the *understanding* of matters that previously have been unclear and hidden.
>
> (Arnlaugsson, 1967, pp. 41–42)

He then considered the current situation and said:

> The main drawback in arithmetic and mathematics teaching in this country I think has been that it has been too mechanical, too much aimed at a certain kind of skills and not much enough emphasis on overview and understanding. Procedures which are not backed up by enough understanding are quickly forgotten and of little use [...] It seems to me [...] that the curriculum has been too much partitioned into definitely separated groups. The pupils have been taught a separate method for each group, while the connection between different groups has been totally invisible, even if often the subjects are closely related; the wood has not been seen for the trees [...] views, which have been considered practical, have been too dominating. Arithmetic seems to have mainly been taught for the purpose that people will not let themselves be cheated in business, can measure their own vegetable garden if needed [...] and not because it has its own value. (Arnlaugsson, 1967, p. 42)

One notices that Arnlaugsson was concerned with overview and generalization, extracting the structure, rather than practical things which he claimed that too often became quickly obsolete. He rather looked at mathematics as an art of beauty and with its own intrinsic values:

> [...] people have largely overlooked the inner value of mathematics. Opportunities to introduce to pupils the beauty hidden in mathematical thinking, even when it concerns the simplest basic things, have been neglected. Children have sometimes been tired out with endless repeated struggles with lifeless numbers, instead of making

> new and good friends of living numbers, learning to know the characteristics of individual numbers, seeing what lies behind the computation methods that they are learning. The examinations are probably partly to blame. Instead of learning "for life", the emphasis has been placed on things that look good in an examination.
>
> (Arnlaugsson, 1967, pp. 42–43)

Here, Arnlaugsson mentioned examinations, which were an important target of learning and teaching at the point of pupils' lives when they were seeking entrance to the highly selective high school. The examinations were of great consequences to them. Arnlaugsson was concerned about this attitude and stressed the view that doing mathematics is a mental training in logical thinking:

> The role of arithmetic teaching should be to a considerable degree from the start to teach the child to think. [...] the arithmetic and the mathematics must not part from another kind of logical thinking, they should precisely be the tool to train the child in logical thinking. If this is clear to the teacher, and he/she has the overview of the coherence of the topics of arithmetic that he/she is teaching, he/she could doubtless achieve a better result than many do now, even if there were few actual changes in the syllabus.
>
> (Arnlaugsson, 1967, pp. 43–44)

The focus was on overview and structure and there were many intriguing structures in numbers, but there was more to it:

> One still cannot hide the fact that the changes that are now going on, shift in some ways the very basis of arithmetic teaching. Previously, arithmetic was mainly concerned with numbers, but now arithmetic teaching is increasingly focussed on the *structure* of mathematics [...] the numbers are surely still important, as number systems are intriguing and interesting in many ways, but numbers are not the only things that are now discussed.
>
> (Arnlaugsson, 1967, p. 44)

Arnlaugsson wrote a chapter in a book on learning and teaching (Arnlaugsson, 1971) where he emphasized again that teaching must not become too mechanical, exercises must not turn into a routine, and practical problems became quickly obsolete. The chapter contains guidelines to teachers about the art of inquiry and search for patterns with references to Sawyer (1964b, p. 169) (Arnlaugsson, 1971, p. 306) and to *The process of education* by Jerome Bruner (1960), in addition to a list of books with recommended reading for teachers.

The textbook *Numbers and sets*

The textbook *Tölur og mengi – Numbers and sets* by Guðmundur Arnlaugsson (1966) was published by the state textbook publishing house. It is clear from the author's forewords that it was intended for the preparation of the national entrance examination. It was therefore not provided to pupils for free as the examination was one year above compulsory level. The textbook was divided into two sections. The first one was on numbers and their properties. As it was intended for pupils aged 15, they had long ago had their introduction to the number concept. The Bundgaard series was first implemented on experimental basis for the 7 year-age group that same year when *Numbers and sets* was published, in 1966, so those pupils had missed the set-theoretical introduction. The second part of *Numbers and sets* was an introduction to the properties of sets. From the forewords it may clearly be understood that the book was written under the influence of the New Math movement:

> The emphasis on skills and mechanical ways of work has moved aside for demands for increased understanding. This development has pushed several basic concepts from logic, set theory and algebra down to primary level. The experience from many places indicates that children – even very young children – can easily adopt these concepts, which previously were only introduced at university level, and enjoy them. Furthermore, they seem to be conducive to increased clarity and exactness in thinking and computations.
>
> (Arnlaugsson, 1966, p. 4)

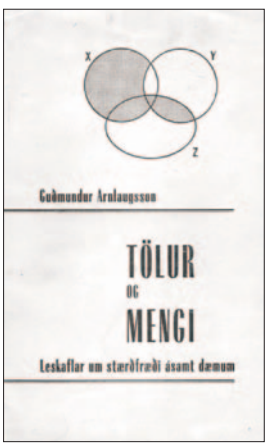

Figure 2. *Tölur og mengi – Numbers and sets*

Here Guðmundur Arnlaugsson suggested that the basic concepts of logic and set theory would facilitate understanding, even for small children. These words indicate that he was aware of theories by the Swiss psychologist Jean Piaget, on which the idea of implementing abstract algebra into school mathematics was based. Piaget wrote in his "Comments on mathematical education", that:

> [...] having established the continuity between the spontaneous actions of the child and his reflexive thought, it can be seen from this that the essential notions which characterize modern mathematics are much closer to the structures of "natural" thought than are the concepts used in traditional mathematics.
>
> (Piaget, 1973, p. 82–83)

In an address to the reader, the author emphasized that the book was intended for reading, not exclusively as a collection of exercises as often was customary (Arnlaugsson, 1966, p. 6). The table of content of *Numbers and sets* was as follows (p. 3):

1. Numbers
    - 1.1. What is mathematics?
    - 1.2. Examples on patterns
    - 1.3. What is the rule?
    - 1.4. Even and odd numbers
    - 1.5. Other categorizations
    - 1.6. Division
    - 1.7. Prime numbers and other numbers
    - 1.8. Number notation
    - 1.9. Other bases than ten

2. Sets
    - 2.1. What is a set?
    - 2.2. Elements of sets
    - 2.3. Statements and open statements
    - 2.4. Equality on sets – Implications
    - 2.5. One-to-one correspondence of sets
    - 2.6. Subsets – The empty set – The universal set
    - 2.7. Venn-diagrams
    - 2.8. Intersection – Union – Set difference
    - 2.9. Large and small numbers – Exponents – Scientific notation
    - 2.10. Number of elements in sets
    - 2.11. Overview of sets.

The section on numbers concerned topics that had only been discussed lightly in Icelandic textbooks earlier in the twentieth century (Gíslason, 1911–1914). It covered prime numbers and factorization, patterns, square numbers, divisibility by nine, two, four and eight, the decimal place-value system, and the binary, octal and hexadecimal systems. Notably, these are the topics counted as suitable for brighter children in the report from the Royaumont seminar, mentioned earlier (OEEC, 1961, p. 109). Arnlaugsson did not have opportunity to alter the compulsory syllabus except through pointing out the Bundgaard's textbook series for primary level. However, better was late than never; he decided to introduce the topics for the bright adolescents preparing for the national examination into the elite high school.

Arnlaugsson emphasized visualization which was a novelty compared to other textbooks in use at that time. Looking at *Vision in elementary mathematics* by W. W. Sawyer (1964b) and *Mathematician's delight* (Sawyer, 1964a), first published in 1943, it is clear that Arnlaugsson sought many ideas on visual presentation in Sawyer's books which target teachers and others on the outlook for illuminating presentations of mathematical ideas. Number patterns (Arnlaugsson, 1966, pp. 8–12) take a few examples from Sawyer (1964b, pp. 78–81, 167–175; 1964a, pp. 94–95) but rather as ideas than being copied. In the section of even and odd numbers and further classifications (Arnlaugsson, 1966, pp. 12–19; Sawyer, 1964b, pp. 8–19), diagrams have been simulated, as well as are examples about divisibility (Arnlaugsson, 1966, pp. 19–24; Sawyer, 1964b, pp. 28–38), and place numeration systems in other bases than ten (Arnlaugsson, 1966, pp. 30–45; Sawyer, 1964b, pp. 20–27, 38–39). Another section concerns primes and prime factoring, greatest common factor and least common multiple deduced from prime factoring, and Roman numeration (Arnlaugsson, 1966, p. 24–30). None of these topics had been treated so thoroughly in Icelandic school mathematics, at least not for half a century. But, as Arnlaugsson stressed in his paper, the question was not if the ideas were old or new, the main concern was promoting understanding.

The second part of *Numbers and sets* was an introduction to logic and the properties of sets. True and false statements, equivalence, implications and one-to-one correspondence were introduced to prepare concepts such as subsets, the empty set and universal set, Venn-diagrams, intersection and union of sets, set difference and counting the elements in the various combinations of sets (Arnlaugsson, 1966, pp. 46–81). It is possible that Arnlaugsson made use of Allendoerfer and Oakley's (1963) textbook here. However, the concepts are presented in a less formal way and the exercised are adapted to local circumstances. That book is not listed in Arnlaugsson's (1971) list of recommended readings for teachers,

but another possible source, *Basic concepts of elementary mathematics* is by W. L. Schaaf (1965) is on that list.

In 1967, the year after the publication of *Numbers and sets*, Arnlaugsson began a 17 episode television series on the national television station. *Numbers and sets* were then being tested in two thirds of the national examination classes as reported in the *Draft curriculum* (Menntamálaráðuneytið, 1968, p. 56). Four pupils were chosen for convenience as a sample. They reported that the series was largely built on the textbook. The target group of the television series was mainly parents of children in primary schools, studying the Bundgaard textbook series, but the series was also suitable for pupils studying the textbook. The four students remembered, 40 years later, that they had enjoyed the TV-series, and one of them remarked especially that the logic-part of the second section of the book had been a revelation to him (Bjarnadóttir, 2011). This provides some indication that the novelties in the textbook *Numbers and sets* had something to offer to the future intellectuals who were seeking entrance to the high schools in the late 1960s.

The textbook *Numbers and sets* was in use for ten years. During the two academic years 1966–1968 it was subject to trial (Menntamálaráðuneytið, 1968). The following four years, 1968–1972, it was compulsory reading for the national examination. In 1972–1974, a translated NKMM teaching material was on trial in several schools. In 1974, new legislation on nine year compulsory education (Lög no. 63/1974) was enacted to enter into full effect in 1976. In 1974–1976, the last years of the national examination, three options were offered for the examination (Menntamálaráðuneytið, 1974, pp. 2–4; Bjarnadóttir, 2007, p. 427). After that, a new Icelandic New Math series (Lárusson, 1974–75), a complete series for all the three grades of lower secondary level of the new nine-year compulsory school, a follow-up of the Bundgaard series, was the prescribed reading. The textbook *Numbers and sets* was written as a one-semester preparation for high school mathematics through the national entrance examination. That era was over by the 1974 act, and there was no use for *Numbers and sets* any more on lower secondary level.

Interaction with the national examination

Björn Bjarnason, who introduced the *Numbers and sets* as compulsory reading for the national examination, also composed the examination in 1963–1971. When reviewing the examination papers, the remarkable fact appears that the test items became increasingly detailed. During the conventional syllabus period up to 1966 the "word problems", reasonably connected with real life, were about or above 60 %. Also, half part of

the examination was solving problems that the pupils had seen before. It now became difficult to decide if a problem was a "word problem" or not. By the introduction of New Math the word problems became fewer, and more abstract. They might be in words, but there was no story. They were increasingly short, and the number of problems increased inversely with the length of the problems. The ratio of word problems to the total number of items in the examination dropped below 40 % from 1967 when the New Math was introduced, i.e. from up to two-thirds of the examination, sometimes down to one-third, even as little as one-fourth (Bjarnadóttir, 2007, pp. 426–427).

Simultaneously, the number of pupils attempting the examination had grown from 19 % of the 16-year old national cohort in 1959 to 36 % in 1973 (Bjarnadóttir, 2007, pp. 286, 421). There was growing demand for access to education for all. Economy was boosting and parents could afford after-compulsory-level education for their children. The ratio of the year cohort who wished to take the entrance examination to the high schools increased rapidly, and other options in education were only few and underdeveloped. Some measures had to be taken for one person to be able to be an external examiner in the whole capital area as had been the role of the national examination board member. Fill-in sheets, and right/wrong answers were introduced along with multiple-choice items for one year. Furthermore, in the 1970s, only the solutions of pupils with grades in an interval critical for passing the examination were marked by an external examiner. In order to ensure equality in grading, it had to be incontestable. So instead of training pupils to a test by revising old story problems, they were now trained to provide simple right/wrong answers. The problem was partly built in the New Math. New and foreign concepts and procedures demanded learning efforts on behalf of the students and had therefore to be taken to the test for themselves, not only act as aids for deeper understanding. These circumstances caused that the national examination may be interpreted as working against the emphasis on understanding in Arnlaugsson's writings. The national examination terminated in 1976 and the structure of the examination remained as described until then.

Influences of *Numbers and sets*

The textbook Number and Sets had lasting influences. The Bundgaard primary level series turned out to be too theoretical for teachers as well for parents who all had grown up with the same conventional textbook by E. Bjarnason (1939–1941). The new textbooks for the primary level, succeeding the Bundgaard series, could be considered as a synthesis of the

old and the new. There, increased attention was devoted to the properties of numbers, and to pedagogical aspects recommended at the Royaumont seminar, such as guided experimentation and discovery. The New Math series by Lárusson (1974–75) for the lower secondary level, which lasted beyond the national-examination era, was partly replaced by a synthesis as well. To take an example, a one-term textbook for grade eight in that synthesis (Bjarnadóttir, 1979) was written under the strong influence of Arnlaugsson's *Numbers and sets*. It included number sequences (1979, p. 3), primes, (p. 9), square numbers (pp. 33–34), factoring and prime factoring in particular (pp. 6, 19), graphical representations of numbers (p. 9), divisibility and greatest common factor (pp. 25, 29), least common multiple (pp. 14–15, 23, 27–29), number patterns, and particularities like Russian/peasant multiplication, modular arithmetic, introduction to use of variables and the plotting functions into the coordinate system.

One may claim that the New Math was not altogether abandoned, but, as was suggested in the Royaumont report (OEEC, 1961, p. 109), the ideas – but not necessarily the language – of sets, sub-sets, correspondence, and order, continued to be used in the new synthesized material for the primary level, which remained in use until the turn of the century, and in various material used at secondary level. Other aspects, promoted by the reform movements, such as teaching methods, including guided experimentation and discovery, also gained increased attention.

Concluding remarks

The implementation of New Math reform movement in Iceland was led by two mathematicians, Guðmundur Arnlaugsson and Björn Bjarnason. Arnlaugsson was already a well-known person who enjoyed trust and admiration as a chess master and for his regular radio programs, but most of all as the head teacher of the mathematics stream at the old and respected Reykjavík High School. Björn Bjarnason, his collaborator, was less known but equally trustworthy. Their long stay in Copenhagen during the World war II, and earlier in the case of Arnlaugsson, made them belong to the Danish mathematical community and they gathered reform information and ideas from there, not the least through personal contact with Svend Bundgaard, a plenary speaker at the important Royaumont seminar.

However, Arnlaugsson and Bjarnason did not only seek information and influences from Denmark but also from other sources, such as in American books by Allendoerfer and Oakley, and by Sawyer, which certainly was not a New Math offspring; *The mathematician's delight* was originally written in 1943 but presenting unconventional

pedagogical aspects. It is also highly probable that they studied the report, *New thinking in school mathematics* (OEEC, 1961), of the Royaumont seminar. Arnlaugsson worked in the Ministry of Education during 1964–1966 as the only specialist and consultant in mathematics teaching. As such he probably had access to OEEC's report. Replies were reported from Iceland in the pre-seminar questionnaire (OEEC, 1961, pp. 135–140), so the report must have been available at the Ministry. The topics: number relations, involving odd and even numbers, primes, factorization, greatest common factor, least common multiple, and place-numeration systems to bases other than 10, listed in the Seminar Report (OEEC, 1961, p. 109) are all presented in the textbook *Numbers and sets* in the same friendly and confiding style as Arnlaugsson used to apply in his former very popular radio programs on chess.

Similar influences from the report may be sensed in the draft curriculum, written by B. Bjarnason. There, even a more theoretical approach may be detected, top-down from the general set-theoretical structure to the specific conventional number algebra. Both authors see the structure, the general pattern, of more paramount importance than the specific practical applications that so quickly become obsolete. Guðmundur Arnlaugsson and Björn Bjarnason were eagerly determined to use their opportunities, provided by their work for the Ministry of education, both as members of the national examination board and Arnlaugsson as mathematics teaching consultant, to reform mathematics teaching in the country. Their main tools in the reform process were precisely the textbook *Numbers and sets* and the articles by Arnlaugsson (1966; 1967; 1971) and the draft curriculum (Menntamálaráðuneytið, 1968), written by Bjarnason, in addition to the implementation of the Bundgaard primary school textbook series initiated by Arnlaugsson.

Arnlaugsson and Bjarnason were university educated mathematicians and their views of mathematics teaching were from that viewpoint. Mathematics had its own intrinsic values and beauties, and seeing the general structure was more important than counting details. Certainly, Arnlaugsson knew that reaching pupils' attentions and leading them to think was the most important thing, whether the terms used were new or old. There is reason to believe that he managed to realize that vision to a considerable degree in his textbook, *Numbers and sets*.

The general feeling is that the textbook *Numbers and sets* was as popular as a textbook can be. It was written in a lucid style in a handy format printed on a nice paper, and was in that respect more attractive than later textbooks. Being compulsory reading for the national entrance examination, sales numbers may not reveal how popular it was, but Bjarnason said in his draft curriculum that "the attitude changed so much for the better"

after it was presented on trial in 1966–1968 (Menntamálaráðuneytið, 1968, p. 56). The testimony of the four pupils supports the view that the attitude was positive, both among pupils and teachers.

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Kristín Bjarnadóttir

Kristín Bjarnadóttir is professor emeritus in mathematics education at the University of Iceland, School of Education. Kristín completed M.Sc. degree in mathematics at the University of Oregon in 1983 and Ph.D. degree in mathematics education at Roskilde Universitets Center in 2006, specializing in the history of mathematics education under the supervision of professor Mogens Niss. Kristín has taught mathematics and physics on lower secondary level of compulsory schools, and mathematics, physics and computing science at upper secondary school level. She has acted as expert in creating national curricula and been co-author of mathematics textbooks for lower secondary level, as well as written textbook on discrete mathematics for upper secondary level.

krisbj@hi.is