Norwegian prospective teachers' MKT when interpreting pupils' productions on a fraction task

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This paper focuses on Norwegian prospective primary teachers' mathematical knowledge for teaching (MKT) when interpreting and making sense of pupils' answers. We named such knowledge *interpretative knowledge* and we consider it to be linked with common content knowledge and specialized content knowledge. In order to deepen these links and to access and develop such knowledge in prospective teachers, we designed a suitable set of tasks on a problem concerning fractions in order to investigate this particular kind of knowledge and clarify its features and dimensions. The results reveal the importance of developing such types of knowledge as a basis for teachers to effectively make sense and interpret pupils' productions and to make it possible to provide effective and meaningful feedback.

One of the important aspects of the complex work of teaching mathematics concerns the ability of promoting pupils' reflection upon the effectiveness of their own (and others') reasoning and representations chosen when solving a mathematical problem. This task requires, among other things, a teacher's ability to make sense of and provide effective feedback to pupils' solution processes as well as to support them in developing their mathematics knowledge. Such an ability is related to a particular kind of knowledge that differs from the one associated with a traditional view of teaching mathematics – as merely presenting a set of definitions and rules. This particular kind of knowledge and sensitivity should allow teachers to make sense of and support mathematical thinking; it is the mathematical knowledge needed for teaching and for the mathematical work that teachers have to do (Ball, 2003).

Although the uniqueness of teachers' knowledge is being recognized, and a large amount of research has been produced in mathematics

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education with this focus, there is still a need to develop a deeper and more ample understanding of the content of such knowledge.

In the context of the work presented here, we perceive teachers' knowledge in the sense of the *Mathematical knowledge for teaching* conceptualization (Ball, Thames & Phelps, 2008). We consider such conceptualization able to capture some of the most specific features of mathematical knowledge needed for teaching. Indeed we believe that teachers' knowledge should include aspects/dimensions that would allow teachers to understand the largest possible variety of pupils' answers, also the ones that could be simply labelled as incorrect or non standard at a first glance (for a particular perspective on this topic see Borasi, 1994). For such, teachers are required to have a mathematical vision and understanding of the different mathematical topics, possible different ways of using representations and elaborate connections. Indeed the understanding of pupils' answers also implies being able to evaluate and recognize the mathematical potentiality of pupils' solutions in order to work and explore them (even the incorrect ones, or the ones that differ from the ones teachers could provide) in a mathematically valid and significant way (Ribeiro, Mellone & Jakobsen, 2013b), and to provide a constructive feedback to pupils (e.g. Santos & Pinto, 2009).

We report on a specific part of a work aimed at identifying, discussing, and reflecting upon some particular features and dimensions involved in prospective teachers' MKT while interpreting and giving sense to pupils' productions. To do so, a particular kind of task [1] that can be perceived as a professional learning task (PLT; Smith, 2001) has been elaborated upon. As we will show in the following, the kind of task we designed could be adapted to different mathematical topics and they are essentially built on the request to interpret different pupils' solutions. For this study we contextualized the tasks in a problem involving fractions. We choose to work with rational numbers and their fractional representation (we will call them only fractions), because they represent one of the most critical topics in primary school mathematical learning (e.g. Behr, Lesh, Post & Silver, 1983; Newstead & Murray, 1998). Moreover a great part of research on fractions has focused more on pupils, leaving aside the teachers and their knowledge and beliefs on this topic (Ribeiro & Jakobsen, 2012). We believe that more research focusing on how teachers' training is implemented regarding specific mathematical critical topics is needed, and having this in mind we address the following research question:

> What knowledge is revealed by prospective teachers when they are confronted with PLTs involving pupils' productions on a fraction problem, and what can we learn from this in order to improve (prospective) teachers' training?

Theoretical framework

Teachers' knowledge is perceived in many different ways and with different interpretations and focuses. An important trend of reflections and studies, grounded in the work of Shulman and colleagues (e.g. Shulman, 1986; Wilson, Shulman & Richert, 1987), increased the attention on mathematics teachers' knowledge and its importance in and for practice. Along these lines, different conceptualisations of teachers' knowledge have emerged such as the *Knowledge quartet* (Rowland, Huckstep & Thwaites, 2005); *Knowledge for teaching* (Davis & Simmt, 2006); conceptualizations focusing on *Pedagogical content knowledge* (Baumert et al., 2010); *Mathematical knowledge for teaching* (Ball et al., 2008); and *Mathematics teachers' specialised knowledge* (Carrillo, Climent, Contreras & MuñozCatalán, 2013).

Among these we consider the Mathematical knowledge for teaching (MKT) as most suitable for our aim (Ball et al., 2008). Firstly, MKT effectively captures the particular features of mathematical knowledge needed for teaching that aim at developing pupils' mathematical knowledge and understanding. It attributes a specific orientation to teachers' mathematical knowledge, placing emphasis on the mathematical work they are immersed in (Ball et al., 2008). Secondly, the content of the subdomains of MKT are considered a relevant starting point for designing tasks for the mathematical preparation of teachers and for doing research on what inputs to teacher training and teacher knowledge have an impact on teaching practice and pupils' outcomes. Although the effectiveness of teacher education centred on this specific kind of mathematical knowledge has been the focus of attention from several researches (e.g. Baumert et al., 2010; Kersting, Givvin, Sotelo & Stigler, 2011), the understanding of what comprises teachers' knowledge is still limited.

Ball and colleagues (2008) present a list of recurrent tasks of teaching mathematics. This list includes, among others, the tasks of "evaluating mathematical explanations" and "evaluate the plausibility of student's claim" (p. 400). These two tasks are closely linked and relate to our research question where prospective teachers are supposed to make sense of pupils' solutions and use them to develop their MKT. We consider that teachers' mathematical knowledge involved in interpreting pupils' solutions and giving constructive feedback to them (in the sense of Santos & Pinto, 2009) needs a more special focus of attention in teacher training. Developing skills to work with such tasks implies a core element of teachers' knowledge, which we label *interpretative knowledge*.

Considering the lack of understanding about the specificities of this type of teacher knowledge, we designed tasks that can promote the awareness of its need and development in teachers. In our inquiry of interpretative knowledge, we focus mainly on the Common content knowledge (CCK) and the Specialized content knowledge (SCK) subdomains of MKT. The CCK is linked directly to the topics involved in the task and is a requirement for being able to solve it, and it is used both in teaching and in other professions that use the mathematics involved. However, complementary to this, teachers' interpretative/evaluative work needs a type of knowledge that we consider as a specific part of SCK. This type of knowledge complements CCK, yet little is known about it. Besides knowing a definition of a concept, how to solve a problem, or how to perform a certain calculation, it is essential that teachers also possess knowledge that allows them to understand the mathematical rationale behind calculations, definitions, or problem-solving processes (SCK; Thames & Ball, 2010). Teachers should possess a rich and ample knowledge of examples, strategies, and representations for problem solving that allows them to make sense not only to solutions similar to their own, but also make sense of pupils' answers, reasoning, and strategies, even those different from their own solution and thus reasoning and strategies that could be outside of their own space of solutions. Such space of solutions is considered complementary to the notion of space of solutions presented by Leikin (2007), as it does not refer to *expert solution space* nor individual or collective (collective here understood as solutions found from the same expert group – e.g. teachers, researchers).

Another reason to consider what we call own space of solutions as somewhat different from Leikin's space of solutions is the fact that we consider such space of solutions in tasks that might have one single solution, but possible different approaches to find and represent it. It does not involve what Leikin and Lev (2007) consider multiple-solution tasks, referring to tasks that contain an explicit requirement for solving a problem in multiple ways. Thus, when we mention own space of solutions, we consider not only the different approaches and mathematical considerations and representations used in finding a solution to the same problem, but we also link it with teacher knowledge, allowing them to understand and give sense to others' solutions (interpretative knowledge). This implies a somewhat more complex, deeper, and more ample mathematical knowledge than knowing for oneself, or even thinking in multiple possible solutions/approaches. Besides considering possible multiple solutions to a given problem, teachers' interpretative/evaluative work demands an interpretative knowledge that extends the frontiers of teachers' own space of solutions.

We consider interpretative knowledge as a part of SCK. The content of interpretative knowledge shapes teachers' ability to make informed choices in contingency moments (as defined by Rowland et al., 2005) and thus how they respond to and deal with non-planned situations (improvisations) as they emerge in ways that supply sustainable mathematical knowledge to pupils.

The tasks were designed by combining the perceived potentialities of the sub-domains of MKT for designing tasks for teachers' training with practicebased approaches (Thames & Van-Zoest, 2013) perceived as PLTs (Smith, 2001). The assumption behind the use of PLTs is that they deal with the mathematical content, the pedagogy in play, and the pupils' thinking processes, simulating in this way real teaching work. By considering the importance of a practice-based approach, we also find it essential that prospective teachers experience the same kind of situations they will encounter in practice (e.g. Magiera, van den Kieboom & Moyer, 2011) as well as expected situations they will explore with their pupils.

In this paper, we pay attention to the specificity and importance of accessing and developing prospective teachers' interpretative knowledge in a problem consisting of sharing and fractions in primary schools. Fractions are among the most complex mathematical concepts that children encounter in their primary education years (e.g. Newstead & Murray, 1998), and many of their difficulties in working with this mathematical tool derive from the fact that fractions comprise a multifaceted construct and, as particular representations of rational numbers, they present many links with other ways of representing the same entities (Kieren, 1995). Here we focus on the rational number as quantity and on the complex relationships among different possible ways to express the same quantity by adding different rational numbers – that is the operational composition of a number. The reason to focus on such aspects is the belief that working on several representations of the same rational number is a possible threshold to appreciate some aspects of these mathematical entities. As underlined by Subramaniam and Banerjee (2011, p. 100):

> The expression reveals how the number or quantity that is represented is built up from other numbers and quantities using the familiar operations on numbers. This interpretation embodies a more explicit reification of operations and has a greater potential to make connections between symbols and their semantic referents. The idea of operational composition of a number, we suggest, is one of the key ideas marking the transition from arithmetic to algebra.

The well-known difficulties about rational numbers, in particular their fractional representation, are obviously also linked with the teachers' knowledge and perception of the topic (Hill, Rowan & Ball, 2005). This justifies our choice to place our inquiry on teachers' interpretative knowledge in the specific topic of fractions.

Method

We designed an open questionnaire with a set of tasks built for discussing (prospective) primary teachers' knowledge involved in making sense of some non-standard pupils' productions, namely, interpretative knowledge. It was translated into the three languages – Portuguese, Norwegian, and Italian – and we used the questionnaire in mathematics courses for prospective primary teachers in our respective countries (November 2012 to September 2013). The purpose of this paper is not to address issues related to cultural differences, but instead use the diversity of the contexts as one element that can contribute to a richer understanding of prospective teachers' knowledge of the subject at hand.

In this paper we focus only on data gathered in Norway. Two classes with prospective primary teachers enrolled in the Norwegian teacher education program ("Grunnskolelærerutdanning") for grade 5–10 mathematics teachers were asked to answer the questionnaire. The prospective teachers were in the first year of the four-year teacher education program and had been taught from half to one semester of a mathematics and method courses (15 ECTS[2]). It was voluntary for the prospective teachers to include their anonymous answers to be part of this study, but all of the 49 prospective teachers present in the two classes handed in their answers.

The tasks presented in the questionnaire started by stating a problem that supposedly a student in primary school (at least in grade 6) would be able to solve, involving rational numbers: *If we divide five chocolate bars equally among six children, what amount of chocolate would each child get?* In order to allow a space for reflection, the first question asked the prospective teachers to solve the problem for themselves. Afterwards, the prospective teachers were asked to give sense to seven different pupil's productions to the same problem, by this request:

> Take into account the following pupils' solutions. For each of them say if you consider it mathematically adequate or not by arguing your idea, and, in the cases you think the student's solutions is not adequate, think of possible feedback to give to that student in order to help him/her to develop his/her mathematical knowledge.

This request has a twofold aim – firstly, to catch prospective teachers' attention by proposing something practical linked with what we expect they will meet in their practice (Smith, 2001). Secondly, it aims also to work on their beliefs and knowledge about mathematics and pedagogy. After applying these tasks, one to two lessons (audio recorded) were dedicated to discussing and working on the prompts coming from the questionnaires. Here we will focus mainly on data gathered from the prospective teachers while solving the tasks, although sometimes we use comments from the discussion afterwards to reinforce the analysis to their productions.

The seven pupil's productions were taken from previous research (Ribeiro, Mellone, Jakobsen, 2013a)[3] and were selected in order to explore rational numbers in an algebraic perspective and navigate between different representations of it, as for example their operational composition (Subramaniam & Banerjee, 2011). They all refer to different operational compositions of the quantity expressed by the number $\frac{5}{6}$.

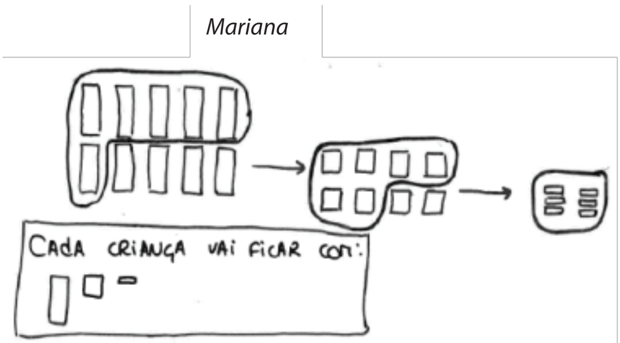

Figure 1. *A pupil's solution, stating "Each child gets:"*

All the given pupils' solutions could be considered to have a correct reasoning supported by division represented by drawings, but some of them stay only in the drawing register like Mariana's (figure 1). In her drawing, the five chocolate bars were represented by rectangles and are considered divided in ten equal parts. The first encircled amount shows the possibility for distributing that amount among the six children (now each child gets half bar), leaving the four halves. Through an arrow, the four pieces left outside are subjected to a new division in halves (eight equal parts), six of them being distributed as earlier (so each child gets half of the divided amount – or one quarter of bar). On the last part of the representation, the pupil divides the remaining two amounts (halves of halves) into six equal pieces – encircling in a closed line to indicate that it is possible to distribute them among the six children (so at this stage each child gets one third of the two previous pieces, $\frac{1}{12}$ of a bar). In a sketching box, the three different parts of a chocolate bar that each child gets are drawn again. Although the reasoning leading to the division is correct, no numerical representation of the made division is present. A numerical representation corresponding to the drawing would be: $\frac{1}{2} + \frac{1}{4} + \frac{1}{12}$ of a bar.

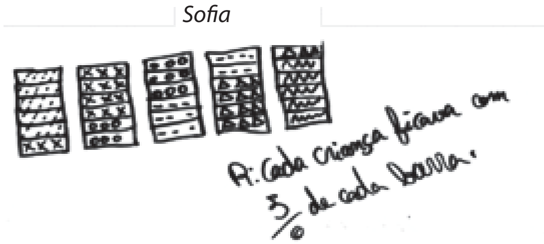

Figure 2. *Example of a pupil's solution stating: "Each child gets $\frac{5}{6}$ of each bar."*

In other pupil's solutions, a mix of drawings and numerical and natural language is used, and the answers reach different degrees of precision. For example, in some of the given pupils' productions, the solution was given using the mixed expression "each child gets $\frac{5}{6}$ of *each* bar" which, if true, would mean that each child would get 5 bars and so clearly indicates a wrong amount of chocolate that each child gets, revealing problems with the role of the whole (Ribeiro & Jakobsen, 2012). An example of this is shown in figure 2, where Sofia filled each "small" rectangle series with different patterns, where each filling pattern is used to sign the quantity of chocolate for each child. Although her drawing presents a possible subdivision of the bar, the given answer is not correct. However, her drawing allows to explore, meaningfully (navigating among different representations) the equivalence among $\frac{5}{6}$ of one bar, $\frac{1}{6}+\frac{4}{6}$ of one bar, $\frac{2}{6}+\frac{3}{6}$ of one bar, $\frac{3}{6}+\frac{2}{6}$ of one bar, $\frac{4}{6}+\frac{1}{6}$ of one bar, $\frac{5}{6}$ of one bar that are the fractional representations of the quantities corresponding to the different filling patterns. It would also allow exploration of correspondences between the written and pictorial forms – how to make them consistent.

Results and discussion

We start by presenting and discussing the solutions given by the 49 prospective teachers concerning their own answers to the question (*what amount of chocolate would each child get?*). First we classify prospective teachers' answers in six categories according to the kind of number representation used (natural number, different fractions or decimal number). Such a first approach intends to obtain information from their own knowledge on how to solve the problem in terms of what can be perceived as CCK, thus it gives us rich evidence of their knowledge and understanding of fractions. Afterwards we address the issue of prospective teachers' interpretation of pupils' productions proposed in the task.

From the 49 Norwegian prospective teachers that participated, two of them did not answer the first question, but they made some comments about the second part of the task, when they were given the pupil's answers. This led us to a preliminary hypothesis that even those who did not answer the problem themselves were ready to comment on others' work – even in cases they might not understand at all. In the answers from the remaining 47 Norwegian prospective teachers', some of them presented more than one solution. The answers were grouped in six different types, arranged in descending order:

a) The solution stated that each child gets $\frac{5}{6}$ of a chocolate bar.

b) The solution states that each child gets 5 pieces of chocolate.

c) The solution states that each child gets 0.83.

d) The solution states that each child gets $\frac{1}{6}$ of the chocolate.

e) The solution states that each child gets $\frac{5}{6}$ of each chocolate bar.

f) Other solution attempts not fitting the categories above.

Solution a), stating that each child gets $\frac{5}{6}$ of a chocolate bar, was the most frequent solution. Twenty-two of the prospective teachers had this as a solution and among them, 14 had this as the only solution while eight complemented it with the decimal number 0.83 (solution c). Those eight just presented the fractional and the decimal representation without any explanation on how and why the two solutions represented the same quantity.

The second most frequent solution among the prospective teachers was alternative b), where they did not make any use of fractions, but only presented natural numbers as solutions. A typical solution strategy in this category was to divide each chocolate bar into six "equal" pieces, and then conclude that they would have a total of 30 pieces of chocolate, thus concluding that each child would then get $30:6 = 5$ pieces each. This is somewhat surprising, since the text explicitly said that the purpose of the problem (the context) was to get children to work with fractions. Despite this, the prospective teachers did not involve fractions in their own solution and remained in the domain of natural numbers only, revealing some of their difficulties and lack of familiarity with the mathematical notions of fraction.

Five had a solution that fitted with category d) – that each child would get $\frac{1}{6}$ of the chocolate. Of these, three of them had this as the only solution. We consider this solution as closely linked to the idea of using only natural numbers, but they moved (slightly) into the space of solutions using a fraction representation. The first part of the reasoning was presented in a similar way as the one that give as answer a natural number (each child get 5 pieces out of a total of 30 pieces of the bars). Instead of saying that each child gets $30:6=5$ pieces each, they consider the role of the whole but without articulating with the specific situation, presenting as a solution $\frac{5}{30}=\frac{1}{6}$ of the chocolate for each child – giving, thus, a correct solution grounded in an incorrect reasoning.

A promising finding was that only two of the prospective teachers had solution alternative e), that each child would get $\frac{5}{6}$ of each chocolate bar, as one of their solutions. This being clearly a wrong answer was also mentioned as a solution in two of the pupils' solutions presented in the second part of the task.

When we looked at how the prospective teachers approached the problem – we found that 21 of them used only numbers in their solution methods, but a majority (24) used a combination of a graphical and numerical representation to solve the problem. A typical solution from this category would contain a drawing of chocolate bars, divided into small pieces (as if they were rectangles), using them as helpers for counting to get the total number of pieces – linking it, or not, with the considered whole.

After we got an overview of the prospective teachers' own answers and processes of solving the problem for themselves, we moved forward to analyze their comments and reasoning to the given pupils' answers to that same problem. By this we aimed at going deeper on understanding the content of prospective teachers' knowledge (our focus was never to reinforce the lack of knowledge), and on the possible links between solving the problem for oneself and interpreting others' solutions – what is involved in the interpretative knowledge. For the majority of the prospective teachers, their own answers to the problem involved mathematical aspects similar to those presented in the second part of the task (pupils' productions). These were aspects that we intended to discuss with them and that we wanted them to reflect and build their knowledge upon – allowing for increasing their knowledge level both in terms of CCK and SCK.

Although it was not part of the required task, prospective teachers' primary focus was to classify the correctness or incorrectness of the given pupils' productions. From the seven pupils' solutions they were asked to comment on, only two of the solutions were inarguably incorrect – the two stating that each child would get $\frac{5}{6}$ of each chocolate bar. Both of these solutions used a graphical approach that was correct, but the written conclusion that followed the solution was incorrect – stating that each child gets $\frac{5}{6}$ of each chocolate bar (e.g. Sofia's solution in figure 2) – there was a lack of correspondence between different forms of representation.

Only three of the prospective teachers commented on such errors in their written reflections. On the other hand, all the other solutions were identified as being wrong solutions that can be argued to be partly or completely correct. We use the notion of partly correct meaning that a solution could be made correct by adding some missing information (Santos & Pinto, 2009) that could be given to the pupils in order to explore with them the mathematical reasoning sustaining their solution (in the sense of a constructive feedback).

For Mariana's solution (figure 1), we got several different answers and interpretations concerning its correctness/adequacy as well as reasoning and understandings. Three prospective teachers consider this solution incorrect, as 13 others meant that the solution was incomplete – that it was not correct as it was, but if something more was added to the solution, it could be made correct (but without expressing what would have to be added).

Among the 47 prospective teachers, 12 of them mentioned they either did not understand what Mariana was doing, or that it was hard to understand what she had done. In one case, a prospective teacher said that "she does not understand fractions – she is just dividing the pieces". Such a comment reveals some particular beliefs concerning fractions and difficulties in going out of their own space of solutions and expanding it. Marianas approach is different from the ones they were used to – not focusing on a numerical approach and inclusively involving only a pictorial representation, but with a powerful representation for exploring a large set of content in mathematics in the specific domain of fractions.

Though the majority thought Mariana's solution was too complicated, one of the prospective teachers also meant that Mariana is showing "very good reflections" and another one wrote that her way of solving the problems was "smart". These different ways of perceiving Mariana's solution also provide us with some insights into where to focus more attention on teachers training, as well as on the need for future steps to improve our own perception of the hows and whys that may lead to these different views, interpretations, and development of interpretative knowledge from prospective teachers in order to enrich the content of such knowledge and improve the quality of training.

Similar comments are also made regarding Sofia's solution (figure 2). Sofia presented in her solution a representation that, by itself, gives us information on the correct amount of chocolate that each child gets, but by complementing it with the incorrect wording and mathematical expression "that each child gets $\frac{5}{6}$ of each chocolate bar," its inability to navigate between different representations does not allow her to present a correct answer. This fact was not noted by most of the prospective teachers, as they focused only on the graphical representation, saying for example "the graphical representation she made is nice."

For all solutions, we found that although most of them tried to interpret the students' solution, they just stayed at a descriptive and evaluative level. Such interpretations still involve only aspects of CCK and thus, in a completely different level than the one required to interpret, give sense and elaborate a constructive feedback for pupils that concerns a core element of teachers' interpretative knowledge.

Concluding remarks and future perspective

In the first part of our research question, we asked what knowledge prospective teachers reveal when they are confronted with the PLTs involving the request to interpret and make sense of pupils' productions. First of all, we had the opportunity to access some of their revealed CCK when they worked on the problem by themselves. We noticed that these prospective teachers revealed in their own answers a mathematical knowledge which is, somehow at the same level of the pupils they might be teaching. This might negatively influence the starting level of discussions in teacher training, and further steps in and for developing their SCK as well as interpretative knowledge is thus needed.

These findings are aligned with earlier work we did with Italian and Portuguese prospective teachers (Ribeiro & Jakobsen, 2012; Ribeiro et al., 2013b). This again reveals that prospective teachers' knowledge on fractions needs an effective focus of attention on research, leading to reframe the focus of teacher training, and fulfilling the identified difficulties concerning fractions – and that it is not necessarily only an aspect of the context itself. These findings thus seem more problematic, as fractions is one of the topics in which a lot of research has been developed; some of the main difficulties in pupils' knowledge and understanding on fractions have been identified since the 1980's and 1990's (Behr et al., 1983; Kieren, 1995; Newstead & Murray, 1998). Moreover, the six different types of answers to the problem given by the prospective teachers reveal different aspects that need to be focused on at the beginning of their training in order to elevate their knowledge of how to do (CCK), grounding the development of their SCK. For example, the answers of the types b), c), d), and e) by not expressing the unit of measure of reference revealed a poor clarity in defining the role of the whole, which is one of the most problematic issues regarding the use of rational numbers, already mentioned by Kieren (1995).

While answering the second part of the questionnaire, prospective teachers reveal some difficulties in giving sense to pupils' solutions different from their own (which seems to be linked with their CCK); in other words, prospective teachers' own space of solutions has only a single element. This makes it harder to appreciate and understand different student solution strategies, and alternative representations, that may differ from their own (Ribeiro et al., 2013b) – even if sometimes the mathematical reasoning embedded in students' answers could be just slightly different from their own. In that sense, there is a need to deepen the links between CCK and SCK in teachers' interpretation knowledge/capacity in this same domain (fractions) in order to help them to develop their own knowledge for teaching, in particular their interpretative knowledge, contributing, in this way, to eradicate pupils' difficulties.

The nature of the results presented in the paper reinforces our conviction about the need for more research that would allow us to obtain deeper understanding on the nature and content of teachers' knowledge and in terms of their interpretative knowledge. Taking, for example, Mariana's solution and the prospective teachers' answers, discussions, and reflections, one way that could allow for even more discussion could be to ask a follow-up question on what could be Mariana's reasoning if instead of having six bars we had seven. In itself, this could be associated with a completely different approach and space of solutions, but can also be solved by using Mariana's first approach as a starting point. Such discussions grounded on presenting some possible pupils' solutions to the follow-up problem would be a way to complement not only the development of prospective teachers' knowledge (knowing how to perform – CCK) but also to give sense too – SCK. In addition, they would open up a space to broader discussion on teachers' and students' roles. All those with responsibility in teachers' training should be aware that the same demand we place on teachers should also be applied in our own context, where our students are the teachers, assuming thus, in an explicit manner, the need for the professional development of mathematics educators.

Finally we want to underline that, after administering the questionnaire, the designed tasks allowed for reflection in which we, as lecturers, worked with the prospective teachers through discussions (that we recorded) on their MKT and beliefs about mathematics and its teaching-learning processes. We believe that the proposed tasks could represent a good example of PLTs to explore with prospective teachers and further analysis on these recorded lessons will be conducted.

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Notes

1 Task in this context refer to a professional learning task given to prospective teachers, and not to a task of teaching, even though a learning task can be situated in a task of teaching.

2 European Credit Transfer and accumulation System – ECTS.

3 The productions were all in Portuguese, but were accompanied by a translation in the other countries.

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