Aspects of a teacher's mathematical knowledge in his orchestration of a discussion about rational numbers

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In this article we discuss how aspects of a mathematics teacher's knowledge surfaced in a whole class discussion about decimal numbers, percentages and fractions. Our focus is the teacher's orchestration of the discussion in order to unpack the mathematical content for the students. His interactive teaching which included questioning and probing students' contributions in order to make the students take part in the discussion, were important features of this lesson. A range of aspects of the teacher's mathematical knowledge was revealed in studying the teacher's pedagogical moves, and we suggest that the interplay between the aspects of his knowledge was crucial in this lesson.

In mathematics, argumentation and reasoning are important skills. When students discuss a mathematical idea, they come to know that mathematics is more than a collection of rules and methods (Boaler & Greeno, 2000; Boaler & Humphreys, 2005). It is important for students to realise that mathematics is a subject in which they can have ideas and when students are incorporated in mathematical discussions, they feel that they are engaged in an intellectual act (Boaler & Humphreys, 2005). Oral skills are emphasised in the Norwegian curriculum as one of five basic skills:

> Oral skills in mathematics involve creating meaning through listening, speaking and conversing about mathematics. This includes finding one's views, asking questions and reasoning by means of informal language, precise terminology and the use of concepts. It means to be in conversations, communicating ideas and discussing mathematical problems, solutions and strategies with others.
>
> (Utdanningsdirektoratet, 2012, p. 5)

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We analyse a sequence from a lesson in 7th grade in which the goal was to "place fractions, percentage and decimal numbers on a number line". A whole class discussion which lasted for 50 minutes, took place. In the research literature, whole class discussions are reported as challenging for the teacher because they involve a "range of important and subtle pedagogical moves" (Boaler & Humphrey, 2005, p. 49). Our focus in this study is the teacher's orchestration of the discussion in order to unpack the mathematical content for the students. A classroom is complex and the complexity may be a constraint which influence teachers' mathematical teaching (Kleve, 2007). A teacher has to deal with many situations, both expected and unexpected, during a mathematics lesson. We wanted to identify aspects of the teacher's mathematical knowledge and how he used his knowledge to facilitate conditions for mathematical discussion that made the dynamics and interactions in the classroom possible. Our specific research questions became: *What aspects of the teacher's mathematical knowledge did the teacher draw on in order to support discussion*?

In order to investigate teachers' mathematical knowledge, several frameworks have been developed (Ball, Thames & Phelps, 2008; Fennema & Franke, 1992; Rowland, Huckstep & Thwaites, 2005a). Both Ball et al. and Rowland et al. based their work on Shulman's categories of knowledge (Shulman, 1986, 1987). In order to develop a framework for Mathematical knowledge *for* teaching (MKT), Ball et al. refined Shulman's categories of knowledge[1]. Since we have studied a teacher in a classroom setting, we have based our work on the *Knowledge quartet*, KQ, which is a framework for Mathematical knowledge *in* teaching (Rowland, 2008; Rowland et al., 2005a; Rowland, Turner, Thwaites & Huckstep, 2009). Rowland and Turner (2008) argue that in the Knowledge quartet as a theoretical framework "the distinction between different kinds of mathematical knowledge is of lesser significance than the classification of the situations in which mathematical knowledge surfaces in teaching" (p. 2).

The KQ provides us with tools for analysing how teachers draw on different kinds of mathematical knowledge in order to support learners in the classroom situations. We will use these tools for highlighting how an experienced teacher used his mathematical knowledge to support and structure a classroom discussion.

Theoretical background

Rowland et al. (2005a) based their work on Shulman's categories of knowledge. Through a grounded approach to data from video studies from lessons conducted by pre-service elementary teachers, 18 codes were grouped into 4 broad dimensions which formed the KQ: Foundation, Transformation, Connection and Contingency.

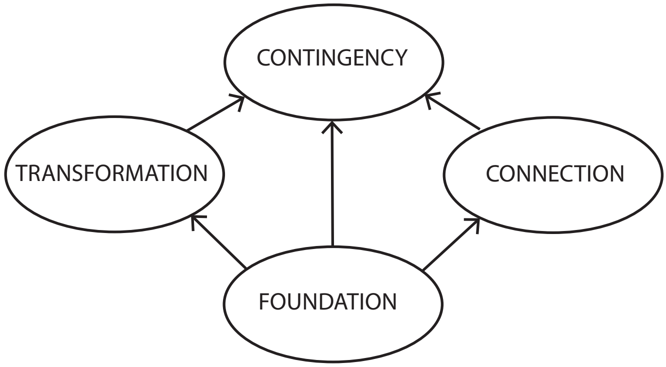

Figure 1. *The relationships of the four dimensions that comprise the Knowledge quartet (Rowland et al., 2009)*

Foundation is the mathematical knowledge the teacher has gained through his/her own education, it is knowledge possessed. This knowledge can inform pedagogical choices and strategies. It is the reservoir of pedagogical content knowledge you draw from in planning and carrying out a lesson. In addition to the knowledge possessed, this dimension also includes teachers' held beliefs about mathematics and adherence to textbook. *Transformation* is informed by Foundation and focuses on the teacher's capacity to transform his or her foundational knowledge into forms which can help someone else to learn it. It is about examples and representations the teacher chooses to use. The third category, *Connection,* is also informed by Foundation, and is about connections between mathematical concepts which binds together distinct parts of the mathematics. It concerns the coherence in the teacher's planning of lessons and teaching over time and also coherence across single lessons. *Contingency* is about situations in mathematics classrooms which are not planned for. Identifying "contingent moments" in order to analyse aspects of a teacher's mathematical knowledge has proven to be helpful (Kleve, 2010b; Solem & Hovik, 2012). The teacher's choice whether to deviate from what s/he had planned and the teacher's readiness to respond to pupils' ideas are important classroom events within this category. The illustration in figure 1 shows how the Contingency dimension is informed by the three other dimensions. The Contingency dimension has also been expanded in a later research (Rowland, 2008, 2012; Rowland et al., 2009; Weston, Kleve & Rowland, 2012). One such expansion is "Teacher's insight" which is demonstrated when a teacher is realizing that children are constructing the mathematical ideas and something that sounds 'half baked' which means that they are in what Vygotsky (1978) termed Zone of proximal development (ZPD) where teacher can help with a scaffolding question or two (Rowland, 2012). Contingency indicates responses to students' unexpected ideas which are not planned for. But what about students' ideas which come up as responses to the teacher's deliberate questions? Or when so called "contingent moments" are planned which may be the case for experienced teachers? In the research literature several studies suggest that experienced teachers have developed a competence to take unexpected input from students into account (Ainley & Luntley, 2005; Kleve, 2010a; Rowland, Huckstep & Thwaites, 2005b). Also, they may have developed and established classroom norms which can promote productive classroom discussions (Boaler & Humphreys, 2005). Boaler wrote that when observing classroom discussions it may be easy to conclude "that the teacher is doing relatively little in the moment concluding that the students are self-motivated or the teacher has done all of the work in the past, establishing careful classrooms norms" (p. 49). However, she emphasised that a careful viewing reveals that the teacher may enact a number of pedagogical practices and moves which can enable such discussion.

Rowland and Zazkis (2013) related the Contingency dimension to what Mason described as "knowing-to act in the moment" (p. 139). According to Mason and Davis (2013) in-the-moment pedagogy is "the teacher's capacity to engage flexibly and productively with their students" (p. 184) and the major factor in this action is the teacher's own mathematics. This "includes the scope, and range of mathematical thinking, associated pedagogical strategies, and didactic tactics that are available to come to mind in the moment" (p. 184). Knowing to act in the moment is not only a question of what teachers know but how they are aware of, use and exemplify their knowledge (Watson & Barton, 2011). Mason (1998) introduced the term awareness about which Mason and Davis (2013) wrote: "A teacher who is aware [...] is in a position to direct student attention to what really matters" (p. 189). In this study we link the Knowledge quartet to an experienced teacher's pedagogical practices and his awareness in orchestrating a whole class discussion in mathematics.

Methodology

The KQ as an analytical framework was developed through a grounded approach to data analysis of lessons conducted by pre-service primary teachers. Although the KQ was developed in the pre-service context, here we use it as a tool in analysing the pedagogical moves made by an experienced teacher in orchestrating a classroom discussion. By experienced we refer to a teacher who had been teaching for 15 years and also that he had an established relationship with this group of students, having been their teacher for nearly 7 years, since grade 1. Thus the students were also experienced in the sense of their participation in the established classroom norms. In this class the teacher had established a classroom culture and norms which a pre-service teacher or a newly educated one has not had the possibility to do.

Our data are taken from an on-going action research project which is about classroom conversations in mathematics focusing on what questions teachers ask (Solem & Ulleberg, 2013). We have chosen one lesson with a teacher, here called Kim. The reason for choosing this lesson, was not because it was representative for the research project, but because we found it especially interesting with regard to the act of orchestrating a whole class discussion which is a relevant issue within mathematics education. Our data are field notes (including pictures), transcribed audio recordings from the lesson, and the teacher's prepared notes for the lesson. The lesson is from grade 7 (12–13 years old), and the task was to place the fractions, decimal numbers and percentages which were written on a flip over (see figure 2), on a number line. The session analysed in this article, lasted for 27 minutes.

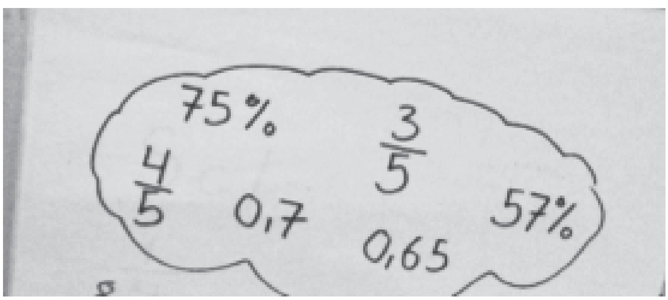

Figure 2. *From the flip over*

In the analysis we used the four dimensions of the Knowledge quartet – Foundation, Transformation, Connection and Contingency – with corresponding codes (Rowland et al., 2005a; Rowland et al., 2009)[2]. We have also used further elaborations of the four dimensions described in Rowland et al. (2009). In the first excerpt we have been focusing on the teacher's foundational knowledge based on the *codes adherence to textbook*, *overt display of mathematical knowledge* and *theoretical underpinning of pedagogy*. Also knowledge of factors which have been shown to be significant in the teaching of mathematics, as part of Foundation

(Rowland et al., 2009), surfaces here. In the second excerpt, our focus is on the *teacher's responses to students' ideas* (Contingency) and also his use of effective questioning as a teaching strategy and use of analogies, which in Rowland et al. (2009) is discussed as part of the Transformation dimension of KQ. In excerpt three, his choice of an empty number line as a representation reflects the Transformation dimension. Here we also discuss how the Contingency dimension (*responding to students' ideas*) is informed by the Transformation and Connection dimensions. In the last excerpt (excerpt 4) our main focus is on how students are connecting ideas, *making connections between concepts* (Connection) and the teacher's awareness of students' mathematical thinking, his *insight during instruction* (Contingency).

In the analysis of the transcribed data from the classroom, we highlight some ”core questions”, CQ, which were questions and input the teacher Kim had prepared for the lesson in order to keep track. He had written the questions and inputs on a piece of paper before the lesson started. In the transcriptions presented below, we have written CQ in brackets to indicate a core question.

Analysis

The students were seated in a circle without pencils and paper available. Kim started the lesson by saying:

Excerpt 1

1 Kim: I will give you some time to think on your own before you answer. You are allowed to discuss with the student next to you. I am not very concerned whether everything becomes completely correct. However, the important issue is for you to explain how you are thinking. [...] we are going to place numbers on a number line, and I don't want you to place them right away. I want us do some talk about the numbers first. OK? So, the goal for this lesson is to be able to place fractions, percentages and decimal numbers on a number line. [CQ]

Adherence to the textbook is one of the codes in the Foundation dimension of the KQ. The task here was taken from the text book. However, in the textbook it was meant as a task for the students to do in their workbooks. Kim's use of the task as an oral exercise demonstrates that he did not adhere to the textbook in the intended way, but rather in a way he found to be purposeful for his lesson.

Overt display of mathematical knowledge is another code within the Foundation dimension which surfaced in this lesson. Mathematically this task was dealing with relations between different representations of numbers (fractions, decimal numbers and percentages) and also comparing numbers in order to position them on a number line. According to Anghileri (2000) "[r]esearch suggests that an approach to fractions which identifies each as numbers to be located on a number line, without emphasising the way of partitioning a whole, will help to establish the equivalence with decimals and percentages" (p. 115). This is also supported by Askew (2000) and Kleve (2010a). Thus Kim's knowledge of factors which have been shown to be significant in the teaching of mathematics was displayed.

The premises for the lesson, which were to talk about, discuss and explain, were clearly expressed by Kim in this introduction. He emphasised that he did not want an immediate solution. His expectations for the lesson, which was to talk about the numbers before positioning them on a number line, were also clearly presented. He focused on the process and rather than the product and thus on developing understanding which reflects theoretical underpinning of pedagogy (Foundation).

After Kim's introduction to this lesson, he asked how it was possible to show the ascending order (CQ). After a student (Toril) had suggested converting all numbers into percentages.

A discussion whether percentages, fractions or decimal numbers was the easiest to compare, took place.

Excerpt 2

2	Kim:	Ok, Yes. *Why* do you like converting to percentages?
3	Toril:	Because then I find it easier to see the difference ... rather than when they are all different
4	Kim:	So you think it is easier when all these are percentages rather than a mix between fractions, percentages and decimal numbers?
5	Toril:	Because there is a difference between 75 and 57, one can see ... if it is the same
6	Stud:	I can convert to fraction. I can convert into decimal numbers. I can convert into percentages. However, what I find the easiest is percentages, [...] So, I stick to percentages
7	Kim:	But *why* is it like that? It seems as if percentages and decimal numbers are the easiest [ ] but *why*?
8	Hanan:	(Inaudible)
9	Kim:	So you think, isn't 4/5 as obvious as 75 %? [Students confirm] *Why* isn't it?
10	Pio:	I think that many find it confusing with different denominators and nominators

11 Kim:	Petra and then Eirik
12 Petra:	I find it easier with percentage and decimal numbers, because then it is visible how much it is. But if you have a fraction you will have to imagine how much it is. You will have to think: Yes I have five pieces and I have four out of those. So it is hard to ...
13 Kim:	mmm mmm [consent]
14 Eirik:	I don't find it so different. Because when it says three out of five it is quite much, but if it says three out of sixteen it is quite little [...] Just look at the lowest number [denominator] and see how much it is
15 Kim:	But the way in which I am thinking about how you learned to read, [...] you learned visual images [ordbilder], didn't you? For example, one of the first words you learned to read was "sun" [sol]. And now you don't need to spell that word. You just know that the three letters together – that visual image – that is "sun". And I think that perhaps you have seen this sight number [tallbilde] [wrote 1/4 on the board] so often that you know it is a quarter, don't you?
16 Studs:	mmm mmm [confirming]
17 Kim:	However, this [pointing to 4/5] you haven't seen that often, so you don't know what it means or implies? See what I mean?
18 Studs:	mmm mmm [confirming]

In this excerpt we notice how Kim used a number of pedagogical practices (Boaler & Humphrey, 2005) in order to make the discussion flow. The pedagogical practices he used, can be identified through the use of KQ; he responded to students' ideas (Contingency) by acknowledging and reformulating (4) and further probing by asking "why" (2, 7 and 9).Toril had suggested percentages, without any further justification which made Kim challenge her to do so. Also in (6) the student did not justify his preference (percentages) and Kim also asked him to justify with a "why" (7). After Hanan's contribution he reformulated and probed further with yet another "why" (9). Pio's contribution in turn (10) initiated a discussion about whether fractions as representations of the numbers were easier than decimal numbers or percentages. Here we see interactions between Kim and his students and effective questioning which may develop understanding. These aspects of a teacher's mathematical knowledge are considered within the Transformation dimension of KQ (Rowland et al., 2009). In (15) he introduced a visual image, "sun", as an analogy to 1/4 (Transformation). Thus he acknowledged students' difficulties in seeing values of unfamiliar representations of numbers.

Drawing a line on the board, he asked: "We have a number line here. When we talk about percentages, fractions and decimal numbers, how should we draw it or refine it?" (CQ). Through this CQ he initiated yet another discussion in which they agreed to refine the number line to one between 0 and 1. In order to refine the number line further he went on with a new core question.

Excerpt 3

19 Kim:	Let us take a look at the numbers we have, and I want to know: are there areas of the number line which we will not touch at all? Are we going to have numbers spread all over, or are there areas we will not touch? [CQ]
20 Simen:	Perhaps not in the very beginning, I don't think we will touch that many there
21 Kim:	*Why* not?
22 Simen:	Because I don't think the numbers are small enough to get there
23 Kim:	But *how can you decide* that the numbers are not that small?
24 Simen:	I will convert into decimal numbers to see where to put them
25 Kim:	Ok, Xia?
26 Xia:	I don't think we will touch anything below o point five
27 Kim:	You don't think we will touch anything below o point five? That *you have to explain* to me!
28 Xia:	I have converted them into percentages
29 Kim:	You have converted them into percentages. So you mean that I can divide it into two pieces? Am I approximately in the middle now? [pupils: yes, just about] You say that here in this area [he shaded between zero and o point five on the number line] we will not have any numbers?
30 Xia:	No [consent]
31 Kim:	Petra?
32 Petra:	Because looking at the numbers you can see that they all are above one half way point, so therefore they cannot be below the half-way point
33 Kim:	Ok, *can you give an example?*
34 Petra:	That three out of five, half of five is two point five, however that this not possible in a fraction, but it is. And three is more – so therefore we must have it above
35 Kim:	Ok. That is completely right folks, very good. Anybody else who can see what Petra says, and what Simen says and what Xia says? That there is no, that we don't have any below the half? O point seven is bigger than o point five, isn't it?

Kim did not present a ready refined number line, but started off with just a line, and refined it further through interactions with the students. By choosing an empty number line, on which the number order is preserved but the intervals are not marked (Anghileri, 2000), Kim focused on the relations between the numbers rather than placing them exactly. This is also emphasized above where he asked if he was "approximately" (29) in the middle when marking 0.5 on the number line. His choice of the empty number line as a representation exemplifies an aspect of the Transformation dimension of the KQ, and emphasizes a consciousness with regard to students' conceptual development and an awareness of students' contribution (Mason, 1998; Mason & Davis, 2013). In asking for examples and justifications Kim was making the mathematics accessible for all students in the class, also for students that so far had not participated in the discussion.

After the introductory question to this excerpt, Kim asked "why not" (21), "how can you decide" (23), "you have to explain" (27) and "can you give me an example" (33). This we see both as aspects of Contingency (responding to students' ideas) informed by Transformation (using questioning effectively in order to develop students' knowledge) and Connection (asking questions to elicit students' understanding of connections between mathematical ideas). Considering Petra's reasoning (34), Kim legitimated it, but did not discuss decimal number as a nominator any further. This was beyond the scope of what he had planned for this lesson, and Kim chose not to deviate from the planned agenda (Contingency). Instead Kim summed up the discussion and a consolidation took place (35). Students confirming "yes" could be heard, and Kim seemed to be pleased as he went on with a new core question: What is the easiest number to position [on the number line]? Through a discussion orchestrated by Kim, an agreement that 75 % is the easiest number to position took place. Kim commented that 75 % could be something familiar, and he related to the earlier introduced visual image, "sun". Use of analogies and linking to something well-known and familiar are captured both in the Transformation and Connection dimensions in the KQ.

The last excerpt from this lesson was initiated through yet another core question from the teacher: "What is the most difficult to place?" Again a discussion took place in which Kim probed with "why"-questions, and he asked explicitly for justifications. Here we noticed that the students took part even more actively than earlier in the lesson and that they explicitly connected to each other's ideas, which showed that they were actively listening to each other's input.

Excerpt 4

36 Dan: I think 57 is the most difficult

37 Kim: Why?

38 Dan: I am quite used to o point sixty five, it is quite easy, but fifty seven becomes like between something, a bit – it is a bit difficult to explain, but it is approximately sixty or a little more than fifty. More than fifty and less than sixty.

39 Kim: Yes, so you may think that on a number line which is not divided in advance, then it is hard to do it precisely, is that what you mean?

40 Dan: Yes.

41 Kim: Even

42 Even: O point sixty five is the same because it is just one half up to the next number – six, seven

43 Kim: Yes, very good! Oda?

44 Oda: I think three of five!

45 Kim: Three of five, *why*?

46 Oda: Because I don't have any clue what it is

47 Kim: You don't have any clue?

48 Oda: No!

49 Kim: Does it have anything to do with the visual image and the "sun"? Petra and then Toril

50 Petra: But actually you have to think you have to multiply the whole fraction until you get a ten. That I would have done. That I did already

51 Kim: That's good. We will have a look at that later. Toril?

52 Toril: I didn't find the fractions very difficult because it is just for you to double and then you get six of ten. And ten in a way is what we've got now, so then we get o point six.

53 Kim: Yes!

54 Toril: And the other one [pointing to 4/5] becomes eight of ten

An interesting feature in this excerpt is how the students were connecting ideas. Even supported Dan that 0.65 was easy. When Oda expressed not to have any clue about 3/5, two other girls immediately supported her by suggesting methods to make the fractions more manageable. This indicates that discussing was established as a classroom norm. They communicated ideas and discussed mathematical problems, solutions and strategies with each other. Here, Kim's voice was hardly heard; he had left the arena to the students. We suggest that this demonstrates an awareness of students' mathematical thinking and expressions, and thus his insight during instructions (Contingency) which involves not to intervene when a mathematical discussion between students is taking place. However, he supported the students with a clarifying question in (39), and in (49) he again linked to the "sun", which also mirrors his insight during instruction.

Summing up and Discussion

In his orchestration of the classroom discussion, Kim was doing relatively little in the moment (he restated and asked some whys). This is in accordance with findings of Boaler (Boaler & Humphreys, 2005). Observing Kim we noticed that classroom norms which supported the discussion, were already established. However, our use of the KQ in carrying out a deeper analysis, revealed aspects of Kim's mathematical knowledge which we suggest were crucial for a successful discussion to take place. We have pointed out how aspects of codes and dimensions of the KQ surfaced in the lesson, and also how the important interplay between the dimensions was unfolded in the pedagogical moves the teacher made.

Kim's foundational knowledge formed the ground in this lesson. He used the task from the textbook consciously in carrying out an oral discussion rather than an individual exercise in the students' work books. He displayed overt mathematical knowledge in the introduction to the lesson and his focus on the process rather than the product reflected theoretical underpinning of pedagogy.

Kim responded to the students' contributions in ways which challenged students in order for their understanding to develop. In the KQ, responding to students' ideas is a code within the Contingency dimension. As earlier pointed out, we encompass situations which occur in the classroom which were not necessarily unexpected, in the Contingency dimension of the KQ since we here are dealing with an experienced teacher (Ainley & Luntley, 2005; Kleve, 2010b; Rowland et al., 2005a). These ways of responding to and probing further, exemplify aspects of the Transformation dimension of Kim's mathematical knowledge which again informs the Contingency dimension and thus the interplay between them. We saw a teacher with a "Teacher insight" who worked within the ZPD of his students which is of crucial importance in order for the students' to learn. Kim acknowledged students' contributions and probed further thinking. He gave attention to students' voices, but also restated and sometimes reformulated students' contributions in order to clarify and to make mathematics accessible for all the students in class, which is an important aspect of the Transformation dimension (Rowland, 2009). He demonstrated an awareness that students were constructing mathematical ideas and that they were in a ZPD and he helped with scaffolding questions (Rowland, 2012). His questioning including "why" and "can you explain" challenged the students to reason and justify their contributions.

Kim had prepared some core questions as a help to keep track. These input demonstrate important aspects of his foundational knowledge which informed the Connection dimension in his anticipation of sequencing and in the way they were binding the lesson together.

Kim demonstrated an awareness of when to use students' contributions to make a mathematical point for the rest of the class and also when and how to pose a new task to facilitate further learning. Furthermore, he demonstrated ability to listen and interpret pupils' incomplete thinking which requires interaction between mathematical understanding and knowledge about pupils' mathematical thinking. Kim acted flexibly and productively with his students; he consolidated and ensured that students kept track as the lesson went on. Hence he directed students' attention to what really matters. This we see as the connective tissue between mathematical awareness and in-the-moment pedagogical choices of action and moves (Mason, 1998; Mason & Davis, 2013).

In this study we found the KQ appropriate as a tool in analysing an experienced teacher's orchestration of a whole class discussion in mathematics. We acknowledge that Ball's et al. framework, MKT, also could be used. We suggest that a further analysis of this lesson using Ball's et al. framework for MKT, the categories SCK, KCT and KCS can tell us *what* mathematical knowledge teachers need in order to carry out whole class discussions. As pointed out by Turner and Rowland (2009), we thus see that the two frameworks, MKT and KQ, have the potential to complement each other.

Orchestrating whole class discussions in mathematics requires a range of aspects of mathematical knowledge and the dynamics and interplay between the dimensions of the KQ appeared to be crucial. Many mathematics teachers steer away from such discussions, because they are difficult to carry out, and teachers have simply not had the opportunities to learn (Boaler & Humphreys, 2005). Through the use of the Knowledge quartet we saw how an experienced teacher's mathematical knowledge surfaced in his pedagogical moves. Taking this back to the original context of the Knowledge quartet, we can see how the quartet has the potential to be used in providing a tool for reflection on promoting mathematics discussion in the classroom. In mathematics teacher education, more emphasis should therefore be put on the aspects of a teacher's mathematical knowledge which surfaced in this lesson and which enabled the orchestration of the classroom discussion which in turn will develop oral skills for the students.

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Notes

1 They distinguished between *Common content knowledge* and *Specialised content knowledge for teaching* (SCK), and suggested dividing Shulman's *Pedagogical content knowledge* into two components: *Knowledge of content and students* (KCS) and *Knowledge of content and teaching* (KCT).

2 *Foundation*: Adherence to textbook, awareness of purpose, concentration on procedures, identifying errors, overt subject knowledge, theoretical underpinning, of pedagogy, use of mathematical terminology.

Transformation: choice of examples, choice of representation, use of instructional material, teacher demonstration.

Connection: Anticipation of sequencing, making connections between procedures, making connections between concepts, recognition of conceptual appropriateness.

Contingency: Deviation from agenda, responding to students' ideas, use of opportunities, teacher insight during instruction.

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