Students' conceptions about the formula for a rectangle's area and some similarities to its historical context

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In this paper, we focus on a debate between grade 6 students about the formula for a rectangle's area, emerging during a 2-hours teaching, and raising questions about the possibility of using history in order to design a hypothetical learning/teaching trajectory of rectangle's area, and we analyse students' conceptions/misconceptions in relation to the historical context of area measurement.

The research reported in this article, is a part of a wider design research concerning the way 8 practising primary teachers, collaborated as members of an inquiry community of practice, in order to design a unit on the concept of the area of a rectangle. The questions guiding the teachers' inquiry were: "How can we teach area in a way that our instructional choices support children's conceptual understanding?" and "What kind of tasks would be both mathematical challenging and significant for students?". During their meetings, the teachers were supported by two teacher educators (the author of the article is one of them) having the role of a facilitator and an expert, and acting as "witness researchers" (Krainer 2008, p. 254): they provided them with theoretical tools concerning mainly recent research about area, the different types of tasks they could use and the essential role of questioning for classroom management. The teachers' educators, co-generated follow-up tasks or questions for the students, facilitated interactions between teachers, and between teachers and students and provided feedback jointly analysing the videotapes.

The findings presented here, concern a debate between grade 6 students about the rectangle area formula, emerging during teaching, and raising questions about the possibility of using history in the teaching of Mathematics. More precisely, an argument that has puzzled the

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historians' scientific community, the recapitulation argument, emerged as the object of our own reflection too.

In our case, we are not looking for answers to the question "How can we use history in order to teach the *concept* of rectangle' s area?", but "How can we present the *formula* for the area of a rectangle?" making use of the history of Mathematics.

Our analysis focused on identifying critical incidents that affected the learning of the area formula for a rectangle by grade 6 students. More specifically, the research question guiding the analysis of the instructional events presented in this paper was:

– Do students' difficulties with area reflect historical obstacles of treating magnitudes?

The social context of the study

The changes introduced in the new mathematics curriculum in Greece (Ministry of Education, 2011) – from kindergarten to grade 9 – demand important changes in mathematics classroom instruction and teachers – especially elementary and primary teachers – face many new challenges. The new standards not only add some new content, but also advocate innovative teaching including teachers' initiatives in mathematical tasks construction, the use of investigations and open-ended problems, students' self exploration and collaboration. In fact, the change requested by the new mathematics curriculum, is not so much a change in content, but rather a "change of culture" about the very nature of mathematics and mathematics teaching.

Until now, the teaching in Greece has been highly constrained by traditional approaches to exposition (Liakopoulou, 2011). The situation is similar to the one described by Castle & Watts (as cited in Tirosh & Graeber, 2003, p. 675): "[...] teachers are primarily deliverers of content, that curricular planning and decision making rest at higher levels of authority, and that professional development is unrelated to improving instruction". Furthermore, in Greece there is no organised lifelong training for teachers, and initiatives of professional development are sporadic and left to private initiatives for cooperation between researchers and teachers (Paschos & Vlachos, 2010).

The mathematical experiences of the teachers in our study were generally focused on procedural solutions for mathematics problems. As indeed the majority of primary teachers, they spent the majority of the lesson on routine activities, while questioning in the classroom and listening to students was not an essential activity.

The mathematics context of the study

The study of area measurement is, at least in Greece, an important strand in the mathematics curriculum, firstly, because of the wide variety of everyday applications of area concepts and secondly, because area concepts are often used in textbooks and by teachers to introduce many other mathematical ideas. The importance of area measurement, in combination with the fact that area is among the concepts studied towards the end of school year, made teachers of our community of inquiry to select "area" as the object of their inquiry process.

What we know about area measurement (in a nutshell)

The basis of area measurement lies in understanding how a specified unit can be iterated until it completely covers a flat surface, without leaving gaps or overlaps. "The concept of a unit is a central, unifying idea underlying all measurement" (Hiebert 1981, p. 38). To understand area a child must construct and coordinate units (Reynolds and Wheatley, 1996). However, Sarama & Clements (2009, p. 297) found in a study that about "73 percent of primary-grade students did not display this understanding". Many children use boundedness – that is, deploying units in ways that would not violate the boundaries of closed figures – rather than space filling (Lehrer et al., 1998; Lehrer, 2003).

For many researchers children's understanding of the underlying array structures in a rectangle is the prerequisite for students to understand the formulas for area in a concrete way. According to Outhred and Mitchelmore (1992) unless the array of squares is seen as groups of rows and columns, students will not understand the significance of the lengths of the sides of the rectangle to find the area by formula, nor that of the multiplication principle. But the students' ability to "see" a grid of squares as groups of rows and columns is not obvious. Outhred and Mitchelmore (1992, p. 202) found that young children do not "[...] automatically interpret arrays of squares in terms of their rows and columns" and concluded, "This could hinder their learning about area measurement". Battista et al. (1998) with second graders reached the same conclusion. They found that many students could not see the row-by-column structure in rectangular arrays. Their interpretation was that students do not simply "read off" these structures from the objects, but instead, employ a process of "constructive structuralisation" that enriches objects with non-perceptual content.

In a later recent study, Outhred and Mitchelmore (2000) concluded that linking area measurement to both linear measurement and multiplicative concepts must occur before the learning of the area formula. "[...]

Children will at some stage be able to interpret covering as a means of measuring area and that their understanding of rectangular covering will then play a vital role in their understanding of area measurement" (p.147). Emphasis on formula memorisation rather than conceptual understanding is behind children's difficulties with area (Huang &Witz, 2011). In fact, in many countries, the incomplete understanding of the concept of the area due to restricted experimentation and the quick passage to the formula is the cause of many misunderstandings (Kordaki & Balomenou, 2006; Bonotto, 2003).

Recent research on area have focused on conceptual foundations of measure related to unit concepts trying to delineate a hypothetical learning trajectory, to formulate an informal theory of measure (Lehrer et al., 2003). According to Sarama and Clements (2009, p. 297) this learning trajectory develops through a series of levels: "area pre-recognizer", "incomplete coverer", "primitive coverer", "primitive coverer and counter", "partial row coverer", "row and column structure", "array structurer". During the last level, the "array structure" level, children "understand that the rectangle's dimensions provide the number of squares in rows and columns and thus can meaningfully calculate the area from these dimensions without perceptual support" (p. 299). This means that during this upper level, children associate the product of a rectangle's dimensions with the number of square units arranged in rows and columns.

The data of our research presented here, question the smooth transition to the "array structure" level, revealing many misconceptions behind this apparent association, concerning mainly the homogeneity in the product of the units of measurement. Furthermore, counting squares by lines or by columns, results in one of the two formulas (l for length and w for width):

$A = l$ (number of colums) x w (number of squares in each column) or

$A = w$ (number of lines) x l (number of squares in each line),

but not in the formula that usually appears in the textbooks, namely A (cm^2) = l cm x w cm. In the formula there is a numerical equivalence, but not a conceptual one.

Theoretically, the structure of a grid as a tessellation of rectangular units provides the link between multiplication, side length and the covering of the region, which is encapsulated in the formula *area = length* x *width*, but the instructional method for area is confusing to children because a square array cannot simply be understood by multiplying length and width (Stephan & Clements, 2003).

In the next paragraph we attempt to analyse and explain difficulties concerning the linking of length sides of a rectangle and its area, making reference to the historical evolution of area's formula, as product of magnitudes (and not simply numbers).

The task

The "filling rectangle" task was borrowed from the research of Battista et al. (1998, p. 507). Given the rectangle below (figure 1), students knowing the size of one square were asked to make a prediction of the number of squares it would take to cover the inside of a rectangle.

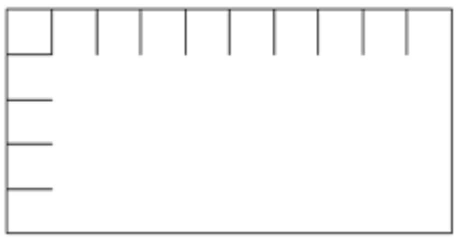

Figure 1. *Rectangle given to students.*

In the study of Battista et al. the task was given to second graders. In our case a similar rectangle (5 x 10) was given to sixth graders. The aim was to test whether students could justify the area formula through rows and columns.

Initially, teachers had not included this task in their instructional design, because it was considered as a "low cognitive load" task, given the students' level (5th grade/approx. eleven years old students). After difficulties encountered by the class teacher with the "little square" task (see appendix I), the group decided to include the "filling rectangle" task in Cycle 2 (instruction in a 6th grade class/approx. twelve years old students).

The "hidden" misconceptions

After the teacher had given the task, students immediately responded by using the area's formula. They were also able to justify the area formula through rows and columns.

Walking into the classroom, the teacher observes that a student (Bill) has written in his notebook: 5cm x 10cm = 50cm.

Teacher:	Bill can you read this out loudly?
Bill:	10 cm times 5 cm equals 50 cm.
Teacher:	Do you all agree?
Olga:	You are writing 10 cm times 5 cm equals 50. 50 what?
Bill:	Cm.
George:	10 cm times 5 cm equals 50 square cm.
Teacher:	Did you hear that? George says that the result, the area is not 50 cm, it is 50 square cm. What do you say about that? They can't both be correct and you can't agree with both.
Myrenia:	*All operations must be made with the same measure and the result will also be in the same measure.* Since we multiply cm times cm, the result can't be square cm.
Nick:	If we have 10 cm times 5 cm ... we can't multiply potatoes with potatoes and have oranges as a result.

A little later, Myrenia changes her opinion:

Olga:	Since we count the area with little squares that are square cm, we must write square cm, because if I take a ruler, I will find the same, *but* the result won't be in square cm but in cm. So we counted this[1] in square cm.
Teacher:	So you mean that the 10 aren't cm but square cm?
Olga:	Yes.
[...]	
Myrenia:	(changes her opinion) I agree with Olga because a square cm is a little square, each side of which is 1 cm long. In a cm we can draw a line with the ruler and see that this is just a cm. So a square cm is different because it is a square.
Teacher:	So what will 10 be? cm or square cm?
Myrenia:	All should be either cm or square cm. The correct for me is square cm.
Teacher:	What do you think Nick?
Nick:	There are 10 here and 5 for each line. So we say 10 times 5 equals 50 and that doesn't need cm. These are the squares. So 10 squares times 5 squares equals 50 squares.

During an interesting debate (see appendix II), three possible answers emerged:

- Bill's answer: 10 cm x 5 cm = 50 cm is the area of the rectangle. His basic argument: "Since we multiply lengths in cm the result must be in cm"

- George's and Ioanna's answer: 10 cm x 5 cm = 50 square cm.

– Olga's, Myrenia's and Nick's answer: 10 square cm x 5 square cm = 50 square cm. Their basic argument: "Since we find area, in square cm, the other two (numbers) should be square cm".

Bill's answer was abandoned, while the other two were supported with quite convincing arguments.

Ioanna: In a rectangle length and width are always in cm. So we won't say square cm, we'll just say cm.

Myrenia: But we will. *We read cm if we take a ruler on our notebook and measure a line*. The result is cm. If we multiply 5 times 10 we'll have 50 cm, not square cm. *In the rectangle we count squares*[2]. *A square cm is a little square whose sides are 1 cm each, and this is why we call it square cm*.

For Myrenia, the result of measuring depends on what you measure: if we measure a distance the result is in cm, while the side of a rectangle "filled of squares" is in cm^2)

Ioanna: Yes but when we look for the area, we don't use cm, we use square cm

Myrenia: Then why do you multiply lengths in cm?

Myrenia's response contains two mutually supported misconceptions: The multiplicands must have the same measure as the product, and this is verified by the fact that indeed rectangle's sides must be measured in cm^2.

Ioanna: Because *width and length aren't in square cm*, so we'll call them just cm.

Myrenia: Why? We don't just have lines with cm. *We have squares*.

Ioanna: You can't multiply the squares.

Myrenia: We multiply the line around them.

[...]

Olga: *Since we've placed the squares*, this is *the cm of the square*. [...] Why call it cm and not call it square cm since it is the square's side?

Olga expresses exactly the same opinion as Myrenia: "square cm is the measure of the side of a square".

Ioanna: We don't care about those squares. We care about the length and the width.

Olga: Why don't we care about the squares?

Ioanna: Because *it is much easier* to multiply this instead of counting the squares one by one.

[...]

Myrenia: Ioanna, what you say is that we don't care about what's inside. We only care about the line around. The length, and the width. Okay, but couldn't we fill rectangle with other lines? In this case you have also square cm, or cm?

We find exactly the same conception in Lamy's, a 17th century French mathematician, description of a rectangle, as we analyse in the next paragraph.

Students' conceptions in a historical context

The area formula is directly linked with the conception of magnitudes. This conception has evolved for two thousands years, from the Babylonians scripts to 18th century with the appearance of modern Algebra.

In the Euclidean tradition (1) lines are not associated with numerical values given that the introduction of a unit length had been excluded (Unguru/Rowe, 1981, in Schubring 2005, p. 281). Furthermore, (2) multiplication of magnitudes has no sense, given the rigid principle of homogeneity for all operations.

> In his geometry, Euclid never multiplies a magnitude by a magnitude; for example, the line of length b is never multiplied by itself to produce the square b^2. [...] In other words, in Euclid's geometry the square on the side is not the square of the side, or the side squared; it is a planar region, which has this size.
>
> (Grattan-Guinness, 1996, p. 365)

Unguru and Rowe attack Neugebauer's interpretation of Greek mathematics thinking as "algebraic", arguing that:

> The consequence of having homogeneity as the fundamental property which any operation has to possess is that it destroys the likewise basic assumption of the supposed "geometric algebra" [...] Since the product would be two-dimensional, it would not be homogeneous with the one-dimensional factors.
>
> (Unguru/Rowe, 1981, in Schubring, 2005, p. 280–281)

In other words: *For the Euclidean system*, given that homogeneity between the magnitudes is the basic precondition for all operations, *the product of two linear magnitudes is not an area*. An area is not conceived as the product of its sides. As a consequence, symbolic expressions as L cm x W cm = A cm^2 are meaningless.

Over the years, and under the influence of Arabic mathematics, the distinction between numbers and magnitudes weakened. For example, the brilliant Niccolo "Tartaglia" Fontana (1500–1557) complained that some mathematicians were confusing "multiplicare", the multiplication of numbers, with "ducere", the multiplication of magnitudes" (MacLennan, 2005, p. 154).

Till the16th century, the rigid restrictions of Euclidean tradition mentioned above were restricted: geometrical objects are now understood in explicit arithmetic terms. Lines can be characterised by their length, which may be represented as a variable. Thus, it is possible to speak of the multiplication of lines as if it were the multiplication of two numbers.

> One now employed numbers in practical geometry and units of lengths and areas so that, at least in practical contexts, multiplication of geometrical magnitudes was interpreted numerically. On the other hand, the dimensional interpretation of the operations in geometry remained an obstacle [...] the use of numbers continued to be judged as inappropriate in geometry, from the viewpoint of theoretical mathematics. (Bos, 2001, in Schubring, 2005, p. 281).

In other words $L \times W = A$ and $3 \times 4 = 12$ is accepted for area, but not 3 cm x 4 cm = 15 cm^2.

A revolutionary step towards uniting arithmetic and geometry was effectuated by Viète (1540–1603). Viète *compounded magnitudes to produce heterogeneous magnitudes*: a magnitude multiplied by a magnitude of the same kind produces a magnitude, which is heterogeneous but related as a unit and a square unit. "Viete's 'comparative' quantities resulted from multiplying lengths and widths. The product of magnitudes increased in dimension, its dimension being the sum of the dimensions of the two factors. In Viete's approach, multiplication thus did not constitute a closed operation" (Bos/Reich, 1990, in Schubring, 2005, p. 282).

On the other hand, Descartes (1596–1650) preserved dimensional homogeneity: he receives a segment through multiplication of segments, by introducing a unit length segment. Nevertheless, "It is important to stress that Descartes – while operating with line segments and introducing a unit line segment – did not identify line segments with their numerically expressed lengths [...]" (Bos, 2001, in Schubring, 2005, p. 282).

A few years later, Arnauld (1612–1694), in his *New elements of geometry* (1667),

> establishes a resemblance between a rectangle and multiplication. He explains in Book I: I suppose that multiplication can be applied to all magnitudes, and not only to numbers. Because, for example, we multiply length by width, when having a piece of ground of 4 perches for length and 3 for width, we say that this piece of ground has an area of 12 perches. (Barbin, 2010, p. 248)

In contemporary terms, we could write: "number of squares (area) = number of squares (in length) x number of squares (in width)" or "4 (squares) x 3 (squares) = 12 (squares)" or (if we symbolise the square as cm^2), we could write the formula as "4 cm^2 x 3 cm^2 = 12 cm^2". Similarity with Olga's, Ioanna's, Myrenia's and Nick's arguments is obvious.

Myrenia: We can't multiply with a measure and have a different measure as an outcome.

Approximately the same time period (in *Elements of geometry*, 1695), Lamy (1640–1715) defines the multiplication of two lines as the area produced by the motion of one of the lines over the other.

> When we mark two lines by two letters, for instance, *a b* marks the multiplication of two lines *AB* and *BC*, we mean that these two lines make the rectangular shape *ABC*. It is evident that this shape is made by the motion of line *AB* moved from *B* to C, repeated or taken as many times as there are parts in *BC*. (Barbin, 2010, p. 246)

We easily recognise Myrenia's argument: "Ioanna, what you say is that we don't care about what's inside. We only care about the line around. The length and the width. Okay, but couldn't we fill rectangle with other lines? In this case you have also square cm, or cm?"

A century later, the relation between multiplication of numbers and quantities, is still open. For example, Leonardo Salimbeni (1752–1823), in regarding the multiplication of quantities, tried to solve the problem, by a "theorem":

> When one magnitude multiplies a magnitude, the product will be homogeneous with the multiplied magnitude. The product should thus always have the same dimension as the multiplicand – or, as the first factor. Multiplication should maintain the dimension of the multiplicand, and should not be affected by the dimension of the multiplicator. (Schubring, 2005, p. 283)

In an effort of combining Lamy's and Salimbeni's perspective we could conceptualise a rectangle's area as the "traces" of a "segment" *AB* ("column" of 3 squares), "running" on *BC* in 4 "steps". So, Area (squares) = 4 x 3 (squares).

Discussion

It is impressive that the answer 5 x 10 cm^2 = 50 cm^2 (five lines of ten squares) or 10 x 5 cm^2 = 50 cm^2 (ten columns of five squares) for the rectangle's area, was not proposed by any child. Even for counting squares, students apply the memorised technique of Area = length x width, which leads them to the formula A (cm^2) = 4 cm x 3 cm, that pupils recite without really understanding it, as our research has indicated.

For example, Ioanna, says that the result must be in square cm, because "it's the way we find it"!

Ioanna: When we find the area, the result is in square cm because we find it in squares and squares are square cm. This is why I say square cm.

Teacher: So you agree with 10 cm x 5 cm = 50 square cm. But what would you say to Nick who supported that we can't multiply potatoes with potatoes and find oranges?

Ioanna: This isn't relevant.

Teacher: Why do you say it isn't relevant?

Ioanna: Finding the area doesn't have to do with neither potatoes nor oranges.

Teacher: Doesn't it bother you to multiply two numbers that are in cm and find a product that is in square cm?

Ioanna: No.

Teacher: Why doesn't it bother you?

Ioanna: Because we write the area with square cm at the end, not cm, as we find it in squares. And we said that squares are square cm.

And a little later during a debate with Myrenia, she insists.

Ioanna: Yes but when we look for the area, we don't use cm, we use square cm.

Myrenia: Then why do you multiply it with cm?

Ioanna: Because width and length aren't in square cm, so we'll call them just cm.

Myrenia: Why? We don't just have lines with cm. We have squares.

Ioanna: You can't multiply the squares.

Myrenia: We multiply the line around them.

How could we, as teachers, persuade Myrenia? How could we strengthen the arguments of Ioanna? The area formula, as we teach it in middle grades, have similarities with vector multiplication, where the magnitude of the product equals the area of a parallelogram with the vectors for sides. But, teaching area in middle school, using unknown mathematics tools is a "conceptual inconsistency".

Can we use history, as a source of ideas, in order to help our students reinvent mathematical concepts and procedures?

Jankvist (2009a, 2009b) made an exhaustive categorisation of the "whys" and "hows" of using history in mathematics education, trying to give answers to three important questions:

- Why history may/should be used in the teaching and learning of mathematics.
- How history may/should be used in the teaching and learning of mathematics.

– In what ways these "whys" and "hows" are interrelated.

In this paper we read that "history as a tool" arguments, may have a motivation/affective, or cognitive nature. Referring to the role of history as a cognitive tool, in terms of our work, we will focus on the *evolutionary* arguments and more precisely on the *recapitulation* argument, that "ontogenesis recapitulates phylogenesis", initially formulated by Ernst Haeckel in 1874, according to which the ontogenetic development of a child – the psychogenetic stages – is but a brief repetition of the phylogenetic (that of mankind) evolution. To really learn mathematics a student must go through the same stages that mathematics has gone through during its evolution. According to the evolutionary arguments historical phenomenology may prepare the development of a hypothetical learning trajectory and history "can help us look through the eyes of the students" (Bakker, 2004, pp. 51, 87). These arguments are in close connection with the concept of epistemological obstacles, as introduced by Bachelard (1938). From these obstacles, according to Brousseau (1997, p. 87), "[...] one neither can nor should escape, because of their formative role in the knowledge being sought. They can be found in the history of the concepts themselves".

Many researchers, for various reasons, have contested this simplistic view. Jankvist, argues that "if (the "old fashioned" view of) recapitulation argument is taken for granted, then there is no other way to learn mathematics than through the use of history" (Jankvist, 2009b, p. 245). According to Piaget and Garcia (1989) the elements of knowledge acquired by the individual, as provided by the external world, can never be divorced from their social meaning, while Vygotsky and Luria (1994) argued that culture not only provides the specific forms of scientific concepts and methods of scientific enquiry, but overall modifies the activity of mental functions through the use of tools. For Mumford (2010) a concept can evolve differently in different cultures, so there is not always one historical evolution to study. Many more authors (Otte, 1994; Crombie, 1995; Radford 1997) have objected to the evolutionary arguments, revealing the important role of the socio-cultural context in the development of knowledge, and replacing the recapitulation argument with regards to acquisition of mathematical knowledge by arguments of historical parallelism. "Historical parallelism concerns the observation of difficulties and obstacles that appeared in history reappearing in the classroom. The idea of parallelism may also be used as a methodology or heuristic to generate hypotheses in mathematics education". (Jankvist, 2009b , p. 22)

In other words, regardless of whether the obstacles that appeared in the historical development of a concept appear to a greater or lesser extent in classroom, learning about the historical evolution of a concept or a formula, might help us better understand the way students think and learn. As phrased by Sfard (2008, p. 127):

> Being interested in learning, I focus in my analysis on the development of mathematical discourses of individuals, but I also refer to the historical development of mathematics whenever convinced that understanding this latter type of development may help in understanding the former.

In the way Sfard uses history of mathematics, one could recognise a kind of application of the indirect genetic method as it is described by Toeplitz (1927) and mentioned by Jankvist (2009b, p. 28–29):

> Either one could directly present the students with the discovery in all of its drama and in this way let the problems, the questions, and the facts rise in front of their eyes – and this I shall call the direct genetic method – or one could by oneself learn such an historical analysis, what the actual meaning and the real core in every concept is, and from there be able to draw conclusions for the teaching of this concept which as such is no longer related to history – the indirect genetic method.

Adopting such an attitude towards the historical development of mathematics, and furthermore, focusing mainly on the mathematising process of the mankind – instead of using only static historical snaps – may help us, as teachers, to guide our students to reinvent mathematics (Freudenthal, 1991). In other words, we may look for the source of the difficulty through the study of history – "sourcing" according to Ho (2009) – in order to turn the salient aspects of a historical point into actual lessons. This later process Ho (2009) called "implementation".

> Together, the backward sourcing and the forward implementation constitute the classroom realization of what Luis Radford (2000b) defines to be the articulation between the psychological domain and the historical domain, i.e., the articulation between students' learning of mathematics and conceptual development of mathematics in history. (Ho, 2009, p. 14)

The area formula (usually presented as: $E = h$ cm x b cm, h for height, b for basis) is taught very early in school – in fifth grade in Greece – and students of about 10 years old are not in a position to understand the historical trajectory of the formula described earlier in this paper. This is eventually the reason that, in spite of the fact that students naturally count the squares inside the rectangle per column or per raw, the formula is presented completely unattached of this process.

Can we proceed to the formula by "guided reinvention"? Yes, but the result will be "Area of rectangle = number (scalar) x number of (unit) squares" and not with the desired one "Area of rectangle = length x length" which shows that from two different magnitudes, a new one emerges by multiplication, i.e. as a new concept.

Therefore, would it be preferable to teach the area formula as an "arbitrary" concept/formula (in Hewitt's terminology), which will be accepted as a sign (in Radford's sense) that, through practice, will gradually acquire social meaning in students' mind?

Hewitt (1994) writes about the "principle of economy in the learning and teaching of mathematics", and distinguishes (1999) between "arbitrary" elements of the mathematical systems, i.e. those aspects of a concept used by a community of practice which can only be learned by being told and then memorising the "necessary" elements, i.e. those aspects of a concept which can be learned or understood through exploration and practice, elements of the mathematical system.

He acknowledges that as names, symbols and other aspects of mathematics representation system are culturally agreed upon conventions, students can feel them arbitrary, but "for students to become proficient at communicating with established members of the community of practice, they must both memorize the arbitrary elements and correctly associate them with appropriate understandings of the necessary elements" Hewitt (1994, p. 3).

According to Radford (2000a), viewing signs – in our case the area formula – as tools in order to accomplish an action, and given the complexity of the historical evolution, instead of seeing them

> as the reflecting mirrors of internal cognitive processes, we consider them as tools or prostheses of the mind to accomplish actions as required by the contextual activities in which the individuals engage. As a result, there is a theoretical shift from what signs represent to what they enable us to do. (Radford,2000a, p. 241)

Both approaches should be taken into account, trying to synthesize them in a harmonic way that will take care of the subject matter itself and the students (both as individuals and as members of a social environment, namely, the classroom). Besides, "learning should not only take us somewhere; it should allow us later to go further more easily" (Bruner, 1960, p.17)

If we accept that "knowledge is necessarily social knowledge" (Otte, 1994, p. 309), it is inevitably, certain social choices will appear as "arbitrary" for someone who ignores or is unable to understand the socio-historical context. In this case, and we think it is also the case of the rectangle's area formula, the best we can do as teachers is – paraphrasing

Hewitt (2001) – to use our awareness of pedagogy, history, the subject matter, and the student to make the best pedagogic decisions.

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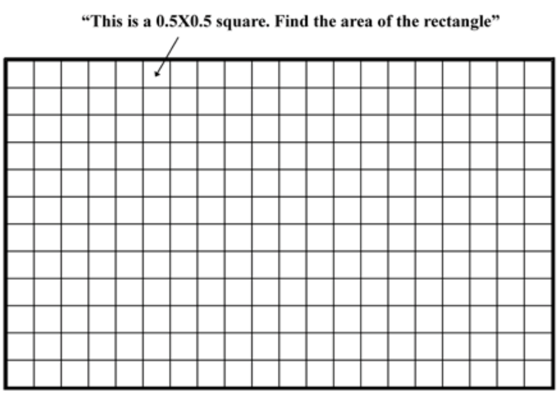
Notes

1 She means the side of the rectangle.

2 Meaning that when we say for a rectangle's side "its 5 cm", we mean "5 squares with a side of 1 cm"

Appendix I

The little square task



Appendix II

Debate on the rectangle area formula

0.0 minutes

T:	Bill can you read this out loudly?
Bill:	10 cm times 5 cm equals 50 cm.
T:	Do you all agree?
Olga:	You are writing 10 cm times 5 cm equals 50. 50 what?
Bill:	cm.
George:	10 cm times 5 cm equals 50 square cm.
T:	Did you hear that? George says that the result, the area isn't 50 cm, it is 50 square cm. What do you say about that? They can't both be correct and you can't agree with both.
Myrenia:	All operations must be made with the same measure and the result will also be in the same measure. Since we multiply cm times cm, the result can't be square cm.
Students:	...
T:	Your opinion?
Nick:	Cm.
T:	Myrenia justified her opinion. You say cm. Why?
Nick:	If we have 10 cm times 5 cm ... we can't multiply potatoes with potatoes and have oranges as a result.
Olga:	I agree with Myrenia because we can't multiply cm with cm and have square cm as a result. All three numbers must have the same measure.
T:	But 50 represents the area.
Olga:	Yes.
T:	How do we deal with this problem?
Ioanna:	I think it is 50 square cm because when we look for the area, we talk about square cm, not cm. Since the product is in square cm, the other two numbers will be too.
Olga:	(changes her opinion) Since we count the area with little squares that are square cm, we must write square cm, because if I take a ruler, I will find the same, but the result won't be in square cm but in cm. So we counted this in square cm.
T:	So you mean that the 10 isn't cm but square cm?
Olga:	Yes.
T:	So Bill should write 10 square cm times 5 square cm equals 50 square cm?
Olga:	Yes.
T:	So here's another idea. We have three ideas. The first idea is 10 cm times 5 cm equals 50 cm. The second idea is 10 cm times 5 cm equals 50 square cm and the third idea is 10 square cm times 5 square cm equals 50 square cm. I can see three ideas.
Myrenia:	(changes her opinion) I agree with Olga because a square cm is a little square, each side of which is 1 cm long. In a cm we can draw a line with the ruler and see that this is just a cm. So a square cm is different because it is a square.
T:	So what will 10 be? cm or square cm?
Myrenia:	All should be either cm or square cm. The correct for me is square cm.
T:	What do you think Nick?
Nick:	There are 10 here and 5 for each line. So we say 10 times 5 equals 50 and that doesn't need cm. These are the squares. So 10 squares times 5 squares equals 50 squares.
T:	I write down the solutions we heard and the names of the ones who told them.

Bill's solution: 10 cm x 5 cm = 50 cm is the area of the rectangle. "Since we multiply lengths in cm the result must be in cm"

George's solution: 10 cm x 5 cm = 50 square cm.

Olga's, Ioanna's, Myrenia's and Nick's solution: 10 square cm x 5 square cm = 50 square cm. "Since we find area, in square cm. the other two should be square cm"

T: Children we need to conclude.

Constantina: Should we perhaps do length times width to find the area?

T: The length is in cm.

Constantina: Yes.

T: So you would write 10 cm time 5 cm equals what?

Constantina: 50 cm.

T: So who agrees with Bill's solution? 10 cm x 5 cm =50 cm? Those who agree with Bill's solution, raise your hands. 2 persons. Those who agree with Olga's solution, 10 cm x 5 cm =50 square cm, raise your hands.

T: Olga doesn't raise your hand? Two, three. Who agrees with George's solution? Most of you agree with that? And the rest of you who didn't raise your hands, with which solution do you agree? We have a serious issue. We must solve this problem.

17.18 min.

T: Bill's solution, 10 cm x 5 cm=50 cm. Who raised the hand?

Catherine: I did.

T: Tell me why you're in favour of this solution.

Catherine: Since the length is 5 and the width is 10, we'll multiply them and have 50.

T: Yes. Why we will have 50 cm?

Catherine: Why? Because the measure must be the same.

T: Who doesn't agree with this first opinion?

Ioanna: I don't.

T: Why? Why don't you agree?

Ioanna: When we find the area, the result is in square cm because we find it in squares and squares are square cm. This is why I say square cm.

T: So you agree with 10 cm x 5 cm=50 square cm. But what would you say to Nick who supported that we can't multiply potatoes with potatoes and find oranges?

Ioanna: This isn't relevant.

T: Why do you say it isn't relevant?

Ioanna: Finding the area doesn't have to do with neither potatoes nor oranges.

T: Doesn't it bother you to multiply two numbers that are in cm and find a product that is in square cm?

Ioanna: No.

T: Why doesn't it bother you?

Ioanna: Because we write the area with square cm at the end, not cm, as we find it in squares. And we said that squares are square cm.

T: So you're in favor of George's solution. Fine. Who is with Olga? Most. Why are you with Olga?

Constantina: (changed her opinion) Because square cm is a little square, a square cm. Ten square cm times 5 = 50 square cm. I agree with Olga's opinion.

T: So, it seems that most of you agree with one of these two solutions: The first is 10 square cm x 5 square cm = 50 square cm (Constantina, Olga, Myrenia, Nick) and the other is 10 cm x 5 cm = 50 square cm (Ioanna, George). I guess we rejected the third one because the area isn't in cm? We have these two opinions. I want two representatives of these opinions to come up. Ioanna, Myrenia ... Ready for a debate? We'll listen to your arguments. Be careful. You have the responsibility of persuading the audience with your arguments.

21.31 min.

Ioanna: In a rectangle length and width are always in cm. So we won't say square cm, we'll just say cm.

Myrenia: But we will ...We read cm if we take a ruler on our notebook and measure a line. The result is cm. If we multiply 5 times 10 we'll have 50 cm, not square cm. In the rectangle we count squares [Meaning that when we say for a rectangle's side "its 5 cm", we mean "5 squares with a side of 1 cm"]. A square cm is a little square whose sides are 1 cm each, and this is why we call it square cm.

Ioanna: Yes but when we look for the area, we don't use cm, we use square cm.

Myrenia: Then why do you multiply it with cm?

Ioanna: Because width and length aren't in square cm, so we'll call them just cm.

Myrenia: Why? We don't just have lines with cm. We have squares.

Ioanna: You can't multiply the squares.

Myrenia: We multiply the line around them.

T: When the audience needs to decide in favor of the one or the other they should have listened to all the arguments. Does anyone want to help? Because I imagine that some of you are on the one advocate's side and some on the other's. On the court we also have assistants. Whoever wants to, can help Myrenia with her arguments or Olga. Constantina who are you helping?

Constantina: I help Myrenia. When we learned to count the area, we didn't count it in cm but in square cm.

Ioanna: Only in the outcome.

Constantina: No, not only in the outcome.

T: George who do you want to help?

George: I help Ioanna. Listen. We have a shape. How do we measure a shape? We can't count the width with square cm but with cm.

Myrenia: Yes, but in a shape, its side isn't in squares, it's in a line.

George: The line is counted in cm.

Myrenia: Yes, but these are squares, they are square cm.

George: We multiply length times width which is multiplied in cm. And when we want to find the area, we have square cm as a result.

Myrenia: It's as if we want to find the perimeter, in your way with cm. But I'm talking about the inside, the area, they are squares, it's shaped in squares, not plain lines.

George: When we have a field for example, we draw squares in the field in order to measure it?

[Students are laughing]

George: Yes, but we measure it, meters times meters and we find square meters.

Myrenia: But we can't multiply potatoes with potatoes and find tomatoes.

[Students all talk together on the subject and create noise]

George: This is not relevant.

T: Please, students, be quiet.

George: This isn't relevant to what we're talking about. In square cm each side is one cm, which is relevant to the cm, whereas potatoes and tomatoes are not relevant.

T: Fine. Which is your disagreement Myrenia?

Myrenia: We can't multiply with a measure and have a different measure as an outcome.

George: But it is the same measure.

Ioanna: Yes, but we've learned that in the area we count in square cm. You can't call that square cm.

T: What?

Ioanna: The length. You can't call it square cm. You just call it cm or m or mm.

T: Olga what do you want to help on?

Olga: George says that we need to count it in cm. Since we divided rectangle in squares, why not call it square cm? Since we've placed the squares, this is the cm of the square. One cm of the square, as we've already divided the squares. Why call it cm and not call it square cm since it is the square's side?

Ioanna:	We don't care about those squares. This is what we care about. The length, and the width.
Olga:	Why don't we care about the squares?
Ioanna:	Because it is much easier to multiply this instead of counting the squares one by one.
Olga:	But I won't count the squares one by one.
Ioanna:	What will you do?
George:	You'll just make the squares.
T:	So, Ioanna? I think that your main problem is when I have a multiplication, I can't multiply cm and find square cm. So if Ioanna finds an example where you multiply two things and have a different outcome you'd be persuaded?
Myrenia:	Yes.
T:	Girls is there anything to add, otherwise the class will decide.
[Girls aren't talking]	
T:	Then let's count the votes. Who is in favor of Myrenia? Square cm x square cm = square cm? I'm counting 7. Who's in favor of Ioanna? 7. Well in this case ...
Myrenia:	Can I say one more thing?
T:	Let's go to a second round.
Myrenia:	Ioanna, what you say is that we don't care about what's inside. We only care about the line around. The length, and width. Okay, but couldn't we fill rectangle with other lines ...? In this case you have also square cm, or cm?
Ioanna:	All these years when we try to find the area the result is in square cm. See?
Myrenia:	But we have also learned that we can't multiply two of the same things and have a different outcome. It can't happen. How do you support that?
Ioanna:	Yes, but in that, you can't measure the width and length in square cm.

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