A comparison of three frameworks for measuring knowledge for teaching mathematics

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This paper presents a comparison of three different frameworks used in research projects aimed at measuring knowledge for teaching mathematics. As the included cases all build on Shulman's theoretical framework for teacher knowledge, in which the categories *subject matter content knowledge* (CK) and *pedagogical content knowledge* (PCK) are central, his framework was used as a reference. To enable comparison across the frameworks, each framework's categories were analysed and organized taxonomically. The results indicate agreement on a superordinate level. However, important differences were found in the operationalisation of the basic level categories *mathematics CK* and *mathematics PCK*. As the basic level normally represents clear communication of categories, this paper suggests that more attention to the operationalisation of basic level categories is needed.

Although Shulman did not deal explicitly with mathematics when characterising the "knowledge that grows in the minds of the teachers" (1986, p. 9), a number of researchers have adapted or extended his framework of knowledge in order to develop new frameworks for teaching mathematics. Following Shulman's framework, in mathematics the category *subject matter content knowledge* (CK) becomes *mathematics CK*, i.e. knowing the facts, structures, rules and procedures of mathematics. Consequently, the category *pedagogical content knowledge* (PCK) becomes *mathematics PCK*, i.e. the special and unique knowledge that links mathematics CK and the teaching and learning of mathematics. Shulman expresses this knowledge as "the ways of representing and formulating [mathematics in order to] make it comprehensible to others" (Shulman, 1986, p. 9). However, there is currently no consensus on how to implement this recategorisation, and there is an on-going debate on how to define and delimit the categories of teachers' mathematics CK and mathematics PCK (Blömeke et al., 2008).

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Luca states: "The square of a natural number is always 1 more than the product of its two closest neighboring numbers".

Is Luca's statement correct?

Please show as many different ways of solving this as possible.

Figure 1. *Test item from the COACTIV project (Krauss, Neubrand, et al., 2008, p. 235), hereafter referred to as Item*$_{COACTIV}$. *Translation from German to English approved and item reproduced with permission*

3. Imagine that you are working with your class on multiplying large numbers. Among your students' papers, you notice that some have displayed their work in the following ways:

Student A	Student B	Student C
35 x 25 125 +75 875	35 x25 175 +700 875	35 x 25 25 150 100 +600 875

Which of these students would you judge to be using a method that could be used to multiply any two whole numbers?

	Method would work for all whole numbers	Method would NOT work for all whole numbers	I'm not sure
a) Method A	1	2	3
b) Method B	1	2	3
c) Method C	1	2	3

Figure 2. *Test item from the LMT project (Ball & Hill, 2008, p. 5), hereafter referred to as Item*$_{LMT}$

One purpose of extending Shulman's framework is to measure teacher knowledge. To this end, a clear definition and delimitation of categories is important. Two of the general steps involved in the process of developing a paper-and-pencil test to measure knowledge can be summarised as follows: (1) defining a framework, describing and delimiting the different categories of said framework and (2) operationalising the categories to design reliable and valid test items that span the entire framework (Blalock, 1968, p. 257; Crocker & Algina, 2006; Downing, 2006; Schmeiser & Welch, 2006).

Some <lower secondary school> students were asked to prove the following statement:

When you multiply 3 consecutive natural numbers, the product is a multiple of 6.

Below are three responses.

[Kate's] answer

A multiple of 6 must have factors of 3 and 2.
If you have three consecutive numbers, one will be a multiple of 3.

Also, at least one number will be even and all even numbers are multiples of 2.

If you multiply the three consecutive numbers together the answer must have at least one factor of 3 and one factor of 2.

[Leon's] answer

$1 \times 2 \times 3 = 6$

$2 \times 3 \times 4 = 24 = 6 \times 4$

$4 \times 5 \times 6 = 120 = 6 \times 20$

$6 \times 7 \times 8 = 336 = 6 \times 56$

[Maria's] answer

n is any whole number

$n \times (n+1) \times (n+2) = (n^2+n) \times (n+2)$

$= n^3 + n^2 + 2n^2 + 2n$

Canceling the n's gives $1 + 1 + 2 + 2 = 6$

Determine whether each proof is valid.

Check one box in each row.

		Valid	Not valid
A.	[Kate's] proof	☐1	☐2
B.	[Leon's] proof	☐1	☐2
C.	[Maria's] proof	☐1	☐2

Figure 3. *Test item from the TEDS-M project hereafter referred to as Item*$_{TEDS\text{-}M}$ *(Brese & Tatto, 2012, p.50). Item reproduced with permission*

The test items displayed in figures 1, 2 and 3 are from test instruments used in three different research projects. These were the Professional Competence of Teachers, Cognitively Activating Instruction and the Development of Students' Mathematical Literacy (COACTIV), the Learning Mathematics for Teaching (LMT) and the Teacher Education and Development Study in Mathematics (TEDS-M). The projects aimed, amongst other things, to measure teacher knowledge related to teaching and learning mathematics. Each project created a framework for measuring knowledge for teaching mathematics. All three used Shulman's framework as a basis to develop their own frameworks. In accordance with Shulman, knowledge for teaching mathematics was subdivided into the following two categories: mathematics CK and mathematics PCK (Ball, Thame. & Phelps, 2008; Baumert et al., 2010; Tatto et al., 2008).

As test items are seen as realisations of categories (Wilson, 2005), before reading on, the reader is invited to categorise the items in figures 1, 2 and 3 using only the limited information provided in this introduction. Drawing on Shulman's two categories, the items should be categorised as either mathematics CK or mathematics PCK.

Now, turning to the categorisation of the items developed by the various projects: $\text{Item}_{\text{COACTIV}}$ (figure 1) was an operationalisation of mathematics PCK; Item_{LMT} (figure 2) was an operationalisation of mathematics CK and $\text{Item}_{\text{TEDS-M}}$ (figure 3) was an operationalisation of mathematics PCK. Are your categorisations in accordance with those of the projects?

To understand a project's categorisation, it is necessary to look at how the project group models the concept to be measured, particularly how they define and delimit the categories of the model or framework. This paper thus focuses on the definition and delimitation of categories of knowledge for teaching mathematics in projects where mathematics teachers' knowledge is measured by paper-and-pencil tests. The present investigation outlines and compares the framework categories of the projects. Hence, the research question is as follows:

> What differences and similarities can be found in frameworks modelling knowledge for teaching mathematics that build on Shulman's general framework for teacher content knowledge?

The frameworks are expected to differ even though the focus of this paper is on frameworks based on the categories found in Shulman's general framework for teacher content knowledge (1986). Hence, as the first part of the theoretical background section will show, categorisation is a cognitive process which depends on the categorisers' theoretical and practical background. However, the comparison of frameworks provides knowledge about similarities and differences. This contributes to raising awareness about possible disagreements about category content, i.e. framework constructs, which could be vital if the aim of the research community is to reach a consensus on *one* model of knowledge for teaching mathematics. Even if the objective is not to find one model, knowledge about agreed-upon categories of teacher knowledge offers several advantages. According to Ball et al. (2008), such knowledge carries the following three advantages: 1) it allows researchers to examine whether any of the sub-categories of teacher knowledge are better predictors of student outcomes than others; 2) it provides a clearer distinction in order to try out different approaches to teacher development and 3) it makes it easier to design teaching and support materials for both teacher and student education in schools. On the other hand, the lack of a common theoretical basis and disagreement about the categories to be used "prevents a convincing development of instruments" (Blömeke, Felbrich, Müller, Kaise. & Lehmann, 2008, p. 719), makes comparisons across projects difficult and could represent a problem when designing teacher education programmes.

In order to compare the three frameworks in this report, i.e. to discuss possible similarities and differences in the frameworks' categories, the background section starts by providing theories for categorisation, followed by an introduction on how one might organise categories taxonomically. Further on, before explaining the methods and presenting the chosen frameworks, a more detailed description of Shulman's content knowledge categories for teaching is provided. As previously mentioned, these are used as reference categories in the comparison that follows in the analysis and discussion section.

Theoretical background

Categorisation

Categorisation is a cognitive process, and as such it varies according to an individual's background and previous experiences (Harnad, 2005; Jacob, 2004). Consequently, proximity to the field of practice and/or theory influences the way an individual categorises, meaning that the categorisation of the test items in figures 1–3 most likely differs from person to person. For example, $\text{Item}_{\text{COACTIV}}$ (figure 1) could have been categorised as mathematics CK, while Item_{LMT} and $\text{Item}_{\text{TEDS-M}}$ (figures 2 and 3) could have been categorised as mathematics PCK. In the latter two test items, three student responses are proposed which require the respondent to react as a teacher would; in the first item, however, the respondent is asked to solve the task in multiple ways like a student working with textbook tasks would. Or all items could have been categorised as mathematics PCK because they all show situations or knowledge demands that, although typical for a mathematics teacher, are beyond the realm of mathematics CK. $\text{Item}_{\text{COACTIV}}$ requires mathematics PCK because a mathematics teacher needs to know that a mathematical problem can be solved in multiple ways. Knowledge of the different solution paths enables the teacher to choose the most effective way to guide his/her students. This is tested by asking the respondents to use different methods to solve the problem.

Not only do categorisations vary depending on a person's background and previous experiences, they also vary according to how the concept *category* is understood. In addition, categorisation depends on the criteria for categorisation that is to be utilised. There are different theories of categorisation, including the classical view, the prototype view and the exemplar view (Murphy, 2002). The present paper concentrates on the classical theory of categorisation (Smith & Medin, 1981) and the prototype theory (Rosch, 1978). These theories are of specific interest because of the substantially different impact they have on the validity of the inferences made on the basis of test results, i.e. that a test actually measures what it is designed to measure (Kane, 2006; Shadish, Coo. & Campbell, 2002).

In the classical theory of categorisation, a category is described as a collection of objects that match the category definition as determined by a checklist of object characteristics (Murphy, 2002; Smith & Medin, 1981). Thus, the act of categorisation entails confirming that an object has all the characteristics on the checklist, in which case the object is a member of the category. If the object does not have all of the characteristics, it must belong to another category. There are no in-between cases; the boundaries of the categories are clear-cut and the categories are mutually exclusive.

On the other hand, in the prototype theory, categories are more like diffuse areas around one or more prototypical examples, and the act of categorisation comprises a personal judgment of how similar an object is to the prototype(s) of a given category. If an object is judged sufficiently similar to the prototype of a particular category and sufficiently different from those of other categories, then it belongs to the particular category (Ellis & Hunt, 1993; Hahn & Chater, 1997). The boundaries of the categories in the prototype theory are not clear-cut and there might be some overlap, with objects in the overlap belonging to more than one category.

Even though the prototype theory is assumed to adhere more closely to the way people actually understand and structure the world (Aase, 1997), this theory is not as compatible with effective test development as the classical theory of categorisation. If valid decisions are to be made based on test scores, the categories in the construct to be measured need to be clearly defined and delineated (Downing, 2006). Furthermore, to ensure that the entire construct is measured, it seems essential to base categorisation on equality within categories and discontinuity across categories; hence, the items included in the tests used to measure this knowledge should be categorised into only one category, either as mathematics CK or mathematics PCK. If test developers adopt the prototype theory, which allows an item to be categorised as an operationalisation of CK *and/or* PCK, then how can one understand, interpret and report correct/incorrect responses to the test items?

Taxonomies of categories

Although there are many different taxonomies of categories, the hierarchical taxonomy, in which the categories are placed according to their level of specificity, inclusion or abstraction, is particularly important (Murphy, 2002; Murphy & Lassaline, 1997; Rosch, 1978). Murphy defines the hierarchical taxonomy as ”a sequence of progressively larger categories in which each category includes all the previous ones” (Murphy, 2002, p. 199), which makes it possible to identify the level at which agreement or disagreement regarding categorisation can be found and differences in categories can be discussed. A hierarchical taxonomy can serve as a framework for thinking and communicating.

In the present paper, a taxonomy with the following three levels of category abstraction is used: superordinate, basic and subordinate (e.g., Murphy, 2002; Rosch, 1978). Superordinate-level categories are more abstract and inclusive than lower-level categories. Lower-level categories are subordinate to higher-level ones, they are more refined and are more similar to each other than categories at superordinate levels (Murphy & Lassaline, 1997; Rosch, 1978). According to Rosch (1978) and Murphy (2002), the basic level of taxonomy is considered to be the most natural, useful and preferred level of specificity, with just the right level of identification and abstraction to allow for clear communication.

The taxonomy in figure 4 shows two subordinate-level categories of the superordinate level, one of which is referred to as basic, i.e. linear algebra and non-linear algebra are subordinate-level categories to the basic-level category of algebra, whilst algebra is superordinate to linear and non-linear algebra and subordinate to mathematics. Therefore, algebra is a first-order subordinate category of mathematics, and linear and non-linear algebra are second-order subordinate categories of mathematics.

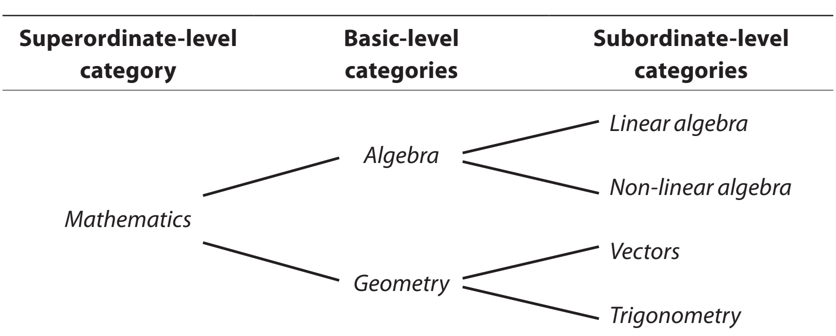

Figure 4. *A small, non-exhaustive example-taxonomy of mathematics*

Although algebra and geometry work quite well as basic-level categories in figure 4, they may not always be the most appropriate categories to use as basic-level categories. For example, if the topic were a course in algebra then the category algebra would be too general as a basic-level category, and the subordinate-level categories of algebra listed in figure 4 might be appropriate as basic-level categories. On the other hand, if the discussion were centred on school subjects, algebra would be too specific to be a basic-level category; instead, mathematics, history and language arts might be the categories most naturally placed at the basic level of taxonomy. The decision regarding which categories are most useful as basic-level categories always depends on the context. Furthermore, it also depends on the education, experience, background and training of those performing the categorisation (Murphy, 2002).

Shulman's framework for teacher knowledge

According to Shulman (1986), teacher knowledge can be divided into different domains, such as knowledge of individual differences among students, knowledge of generic methods of classroom organisation and management, and CK for teaching. The latter domain (CK for teaching) constitutes the theoretical background for knowledge in this paper. Shulman further suggested that teacher CK consists of the following three categories: subject matter CK, PCK and curricular knowledge.

Schulman defined subject matter CK as "the amount and organisation of [subject] knowledge per se in the mind of the teacher"; therefore, subject matter CK requires "going beyond knowledge of the facts and concepts of a domain" (Shulman, 1986, p. 9). The subject matter CK concerns not only what teachers know, but also how well-founded this knowledge is. PCK is subject matter knowledge for teaching and includes knowledge about "the most useful forms of representation [...], the most powerful analogies, illustrations, examples, explanations, and demonstrations" in the topic taught, as well as an "understanding of what makes the learning of specific topics easy or difficult: the conceptions and the preconceptions that students of different ages and backgrounds bring with them" (Ibid., p. 9). This domain thus includes knowledge of how to represent, explain and teach the subject matter as well as an understanding of how children learn the subject and common obstacles to this learning.

Curricular knowledge "is represented by the full range of programmes designed for the teaching of particular subjects and topics at a given level" (Shulman, 1986, p. 10). This adds the dimension of knowledge of plans for instruction and courses of study, including textbooks. The category of curricular knowledge also has the subordinate-level categories of vertical and lateral knowledge. Vertical curricular knowledge is about knowing the "topics and issues that have been and will be taught in the same subject area during the preceding and later years in school". Lateral curricular knowledge refers to the "teacher's ability to relate the content of a given course or lesson to topics or issues being discussed simultaneously in other classes" (Ibid., p.10).

The work by Shulman and his colleagues in the mid-1980s (and thereafter) has had a great impact on teacher knowledge research as it made an important distinction between a) the conventional subject knowledge and competence that teachers and those who study the advanced subject successfully acquire and b) the unique PCK that successful teachers develop (Shulman, 1986, 1987).

Methodology

The research strategy used in the present study resembles a collective case study (Cresswell & Maietta, 2002) in which representative cases are selected purposefully. Choosing a case study design has distinct advantages when a "question is being asked about a contemporary set of events, over which the investigator has little or no control" (Yin, 2003, p. 9).

Using the definition of a case as a system bound in time and place (Merriam, 1998; Stake, 1995; Yin, 2003), the framework for measuring knowledge for teaching mathematics of each included project is regarded as a case in this paper. The cases, i.e. the frameworks, included in this study were purposefully selected (Robson, 2002) based on the following three inclusion criteria: (1) the frameworks used Shulman's (1986) categorisation of CK for teaching as a part of the theoretical foundation; (2) the categories of the frameworks were operationalised in order to measure knowledge for teaching mathematics using a paper-and-pencil test and (3) information about the projects (reports, articles) was easily accessible. The following frameworks were included: *Professional knowledge of secondary school mathematics teachers* from the COACTIV project; *Mathematical knowledge for teaching* from the LMT project and *Knowledge for teaching mathematics* from the TEDS-M project (table 1).

The status of each of the included frameworks differs from one another. The framework of the LMT project is presented as a working hypothesis that the researchers acknowledge still requires improvements, whereas the frameworks of the TEDS-M and COACTIV projects are used to ensure that items included in the measurement instruments used in the respective projects span the entire framework. Other differences include the target populations of the projects (in- and pre-service teachers) and,

Table 1. *Frameworks of the projects included in the analysis*

Project	Framework	Origin	Measuring knowledge of	Teaching school level
COACTIV	Professional knowledge of secondary school mathematics teachers	Germany	In-service teachers	Lower secondary
LMT	Mathematical knowledge for teaching	USA	In-service teachers	K–8
TEDS-M	Knowledge for teaching mathematics	International *	Pre-service teachers	Primary & lower secondary

Note. * 17 countries participated: Botswana, Canada, Chile, Chinese Taipei, Georgia, Germany, Malaysia, Norway, Oman, the Philippines, Poland, the Russian Federation, Singapore, Spain, Switzerland, Thailand, and USA (Tatto et al., 2012).

to a certain degree, the school level at which the teachers are teaching or are expected to teach. However, the purpose of this paper is to compare proposed categories of knowledge for teaching mathematics in order to raise awareness about similarities and differences found in the research literature. To this end, the states, target population and school levels of the frameworks are not considered important. Moreover, the framework categories are not understood to be final.

For the purposes of comparison, Shulman's (1986) categories for teacher CK were placed taxonomically (see table 2). This hierarchical ordering of Shulman's categories facilitates the structuring of the categories of the included frameworks and serves as an analytical tool to help identify the levels at which comparisons can be made (table 2). Shulman's category *teacher CK* is regarded the most general category and was therefore placed at the superordinate level; subject matter CK, PCK and curricular knowledge are placed as first-order subordinate categories. As for second-order subordinate categories, Shulman seems to define only two such categories: lateral and vertical knowledge (subordinate to curricular knowledge). However, other subordinate categories could be constructed from his descriptions of subject matter CK and PCK, as exemplified in table 2.

In order to decide which subordinate category level to label as basic, a previously published list of empirical advantages of basic-level categories was utilised. The list shows that basic-level categories are more informative than superordinate-level categories and that the names of basic-level categories are more frequently used in texts (Murphy, 2002,

Table 2. *Taxonomy of Shulman's (1986) teacher CK categories, showing superordinate, first- and second-order subordinate-level categories*

Project	Level of taxonomy		
	Superordinate	First-order subordinate	Second-order subordinate
Knowledge Growth in Teaching (Shulman, 1986)	Teacher CK	Subject matter CK	knowledge of facts*
			knowledge of concepts*
			knowledge to understand the structure of the subject matter*
		PCK	representing subject matter*
			explaining subject matter*
			teaching subject matter*
		Curricular knowledge	lateral knowledge
			vertical knowledge

Note. * Shulman (1986) did not propose these as categories; they are suggested by the author as examples of possible second-order subordinate categories based on the descriptions of subject matter CK and PCK presented in the previous section about Shulman's framework.

p. 214). Considering the vast amount of literature that used subject matter CK and PCK as subordinate-level categories of teacher knowledge, Segall (2004) even claimed that PCK has "become 'common currency' in the literature in and on teacher education" (p. 490). Therefore, the first-order subordinate categories subject matter CK and PCK (in table 2) are regarded as the most natural categories to place at the basic level along with Shulman's curricular knowledge. Following the taxonomy of Shulman's categories (table 2), the knowledge categories of each case were placed in a corresponding or similar taxonomy for comparison (see table 3).

Reporting case studies includes providing a description of the context of each case. The context descriptions of the three projects included herein are mainly based on information found in documents and research literature listed on the projects' respective internet homepages. Only documents that describe the frameworks; explain and justify the categories of the frameworks; present short histories/descriptions of the projects and/or show exemplary or actual items included in their measurement instruments were selected for context descriptions. The documents for every case were divided into primary and secondary sources of information. Although secondary sources are not necessarily directly referred to in this paper, they were used as a backdrop. The primary source documents from the projects were written in English and gave a good overview of both the framework and the project. Although they were not always written in English, the documents displaying exemplary or actual test items were also included as primary source documents in the present paper. The primary and secondary source documents are listed in the appendices for the COACTIV (appendix A), LMT (appendix B) and TEDS-M (appendix C) projects.

The three test items presented in the introduction section of this paper serve two purposes. They are examples of test items that were designed to measure knowledge for teaching mathematics in the respective projects, and they facilitate illustrations of similarities and differences in categorisation. However, one should note that the items are purposefully selected and thus represent a biased selection. Furthermore, none of the projects have released or published complete instruments, and it is thus assumed that the items found in the research literature are representative of items used in the projects; more specifically, they are representative of items designed to measure the stated category.

$\text{Item}_{\text{TEDS-M}}$ and $\text{Item}_{\text{COACTIV}}$ were part of the pool of items included in authentic tests. Hence, the categories presented here are the categories the items were representing as reported by the projects. Item_{LMT} is not used in actual measures of teachers; rather, it is part of a small set of published items demonstrating the LMT project's effort to write and pilot survey items (Ball & Hill, 2008), though Item_{LMT} is repeatedly used as an item that exemplifies teachers' mathematics CK (Hill & Ball, 2004; Hill, Schillin. & Ball, 2004; Hill, Sleep, Lewi. & Ball, 2007). However, whether or not the items are used in tests is not the issue. The purpose of the inclusion of these items here is to aid the comparison of categories across the three frameworks. According to Wilson (2005), items are seen as realisations of the categories; hence, Item_{LMT} and $\text{Item}_{\text{TEDS-M}}$ were selected because while almost identical in design, they are operationalisations of mathematics CK and mathematics PCK, respectively. $\text{Item}_{\text{COACTIV}}$ was selected because of its dissimilarity to both Item_{LMT} and $\text{Item}_{\text{TEDS-M}}$, and because it is categorised as mathematics PCK. However, this item could, as argued earlier, be perceived to only require mathematics CK because it is asking the respondents to solve a mathematical problem in as many different ways as possible. Based on an analysis of the included documents, a description of each case and case context is given before the comparison of the frameworks commences.

Cases and case contexts

The context and framework of the COACTIV project

The framework *Professional knowledge of secondary school mathematics teachers* guided the work and empirical investigations of the COACTIV project, which was conceptually and technically embedded in the German extension of the Programme for International Student Assessment (PISA) 2003 cycle (Baumert et al., 2010; Krauss, Baumer. & Blum, 2008). The target population of the COACTIV project was ninth-grade students and their mathematics teachers.

One aim of the COACTIV project was to investigate the theoretical assumption that mathematics CK is empirically distinguishable from mathematics PCK. A second aim was to investigate the impact teachers' mathematics CK and mathematics PCK had on student outcomes. Their hypotheses were as follows: "CK and PCK represent distinct knowledge categories, [...] PCK is directly associated with the quality of instruction", and "[the] effect [of instruction] on student learning is mediated by the quality of instruction" (Baumert et al., 2010, p. 135).

The COACTIV team conceptualised teachers' mathematics CK to be "a profound mathematical understanding of the curricular content to be taught" (Ibid., p. 142), and PCK as "a distinct body of instruction- and student-related mathematical knowledge and skills – the knowledge that makes mathematics accessible to students" (Ibid., p. 142). Then, based on the suggestions of Shulman (1986), COACTIV researchers further divided mathematics PCK into the following three categories (which they referred to as dimensions): "Knowledge of mathematical tasks as instructional tools, knowledge of students' thinking and assessment of understanding, and knowledge of multiple representations and explanations of mathematical problems", which are referred to as the instruction dimension (Ibid., p. 142). Since the project emphasised the professional teacher knowledge needed to cognitively activate the students, the mathematics PCK dimensions were developed on the basis of the demands of mathematics instruction (Baumert et al., 2010).

The context and framework of the LMT project

The framework *Mathematical knowledge for teaching* is the result of the following two US projects started in the 1990s: *Mathematics teaching and learning to teach* (MTLT) and the LMT project (Ball et al., 2008). The intent of these two projects was to "develop a practice-based theory of mathematical knowledge as it is entailed by and used in teaching" (Ball, Thame. & Phelps, 2008, p. 396). In the first project, the MTLT, a third-grade public school classroom was observed for an entire school year (1989–1990) to analyse the work of teaching mathematics. The researchers investigated the kind of knowledge used in and for teaching as well as the kind of knowledge a mathematics teacher needs in order to teach effectively (Ball, 1999; Ball et al., 2008; Hill & Ball, 2004; Hill, Sleep, Lewi. & Ball, 2007).

As a complement to the first project, the researchers developed measures of CK for teaching mathematics in their second project, the LMT (Hill & Ball, 2004; Hill, Schillin. & Ball, 2004). The project group wanted a set of analytical tools that could be used to coordinate mathematical pedagogical perspectives. They used the measures provided as a way to investigate the nature, role and importance of the different categories of mathematical knowledge for teaching suggested in the first project (Ball et al., 2008). The intention was to design a test to measure mathematical knowledge for teaching, which was defined as "mathematical knowledge needed to perform the recurrent tasks of teaching mathematics to students" (Ibid., p. 399).

Ball and colleagues produced a "map of usable professional knowledge of subject matter" (Ball et al., 2008, p. 402) based on a synthesis of the results of their two projects which they called *Framework for mathematical knowledge for teaching*. Using Shulman's framework as a basis, the researchers suggested dividing teacher mathematics CK into the following six subordinate-level categories: common CK, specialised CK, horizon CK, knowledge of content and students, knowledge of content and teaching, and knowledge of content and curriculum.

Common CK was defined as "mathematical knowledge known in common with others who know and use mathematics" (Ibid., p. 403), while specialised CK was defined as "mathematical knowledge and skill unique to teaching, [...] knowledge not typically needed for purposes other than teaching" (p. 400). Horizon CK was described as an "awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (Ball et al., 2008, p. 403).

Knowledge of content and students, as well as knowledge of content and teaching, were defined to be at the intersection of subject matter CK and knowledge about students and teaching (Ball et al., 2008). Knowledge of content and curriculum was an adaptation of Shulman's (1986) subject independent category curricular knowledge.

Even though the researchers in the LMT project focused mainly on content knowledge and items were largely developed for common content knowledge and specialised content knowledge (Ball et al., 2008; Hill et al., 2004), they maintain the importance of carefully mapping teacher knowledge. Thus, their empirical findings suggest that CK for teaching is multidimensional and that the proposed categories are in need of refinement and revision (Ball et al., 2008).

The context and framework of the TEDS-M project

Unlike the COACTIV and LMT projects, the TEDS-M project was an international comparative study of teacher education. It focused on the preparation of mathematics teachers for the primary and lower secondary levels (Tatto et al., 2008; Tatto et al., 2012). The TEDS-M project paid particular attention to the links between teacher education policies, practices and outcomes.

The purpose of the TEDS-M project was to identify how the participating countries prepared mathematics teachers to teach in primary and lower secondary school, and to study the variation in the nature and impact of teacher education programmes on mathematics teaching and learning within and across the countries included. The researchers also aimed to measure the level and depth of mathematics and mathematics-related teaching knowledge that prospective primary and secondary teachers had attained by the end of their pre-service teacher education (Tatto et al., 2012).

To measure, the framework *Knowledge for teaching mathematics* was defined. It divided the concept of knowledge for teaching mathematics into two main sub-sets of knowledge: mathematics CK and mathematics PCK.

The framework for mathematics CK in the TEDS-M project was built on the content and cognitive domains of the TIMSS 2007 and TIMSS Advanced 2008 assessment frameworks (Tatto et al., 2008; Tatto et al., 2012). As the theoretical background for modelling teacher PCK in mathematics, the TEDS-M project group referred to research literature (Shulman, amongst others), the feasibility study of the TEDS-M project (MT21) and the frameworks of the LMT and COACTIV projects (Tatto et al., 2008; Tatto et al., 2012). Based on core tasks of professional teachers assessable in an international cognitive study (Schmidt et al., 2007), mathematics PCK was divided into the following three subordinate-level categories: mathematical curricular knowledge, knowledge of planning for mathematics teaching and learning, and enacted mathematics knowledge for teaching and learning.

Analysis and discussion

In order to discuss the differences and similarities between the framework categories in the present study, the categories of the three included projects were taxonomically organised. Using the analytic tool presented in table 2, the framework categories were placed as shown in table 3. The teacher knowledge concepts modelled in the frameworks of the LMT, TEDS-M and COACTIV projects are considered to be categories at the most abstract level. They are regarded as similar to Shulman's category *teacher CK*, and were hence placed at the superordinate level of taxonomy. Basic-level categories were easily identified in the TEDS-M and COACTIV frameworks in that their first-order divisions of teacher CK are labelled and described much in accordance with Shulman's categories subject matter CK and PCK.

In the frameworks of the TEDS-M and COACTIV projects, there were three levels of categories, whilst in the framework of the LMT project there were only two. However, since the six subordinate categories of mathematical knowledge for teaching were proposed by the researchers of the LMT project as a refinement to Shulman's categories, and the researchers themselves demonstrated the link to Shulman's subject matter knowledge and PCK (Ball et al., 2008, pp. 402–403), a three-level hierarchy of categories was applied to the framework of the LMT project as well. Hence, second-order category divisions of teacher CK in all three frameworks are placed at the subordinate category level in the taxonomy.

Based on the overt preference found in research literature for mathematics CK and PCK, the following comparison will focus on the basic-level categories and their subordinate-level categories. Moreover, compared to the subordinate-level categories of mathematics as a (school) discipline, the category mathematics PCK and its subordinate categories are relatively new and therefore of particular interest.

Although all three frameworks included in the present study build on or use Shulman's categories as part of their theoretical background, they all divide their superordinate-level categories into two subordinate-level categories, i.e. mathematics CK and mathematics PCK, instead of three as Shulman (1986) suggested (table 3). In the framework of the LMT project, Shulman's third basic-level category of curricular knowledge was provisionally placed as a subordinate-level category of PCK because LMT researchers were uncertain whether it would run across several categories or should be considered a category of its own (Ball et al., 2008). Since the framework of the TEDS-M project was based partly on the framework of the LMT project and partly on the framework of the TEDS-M feasibility study (in which mathematics PCK was subdivided into instructional planning), student learning and curricular knowledge (Schmidt et al., 2007) was apparently accepted as a subordinate category of mathematical PCK in the framework of the TEDS-M project. In the framework of the COACTIV project, curricular knowledge was not a category on its own. The COACTIV researchers claimed that mathematics CK for teaching is "defined by the curriculum and [it is] continuously developed on the basis of feedback from instructional practice" (Baumert et al., 2010, p. 142). Curricular knowledge seems to permeate all categories of the framework of the COACTIV project. For instance, the knowledge needed to select appropriate tasks as well as which examples and activities to assign at a given grade level both depend on the teacher's curricular knowledge. This

Table 3. *Taxonomy of framework categories by project, showing superordinate, basic and subordinate-level categories*

Project	Level of taxonomy		
	Superordinate	Basic	Subordinate
COACTIV	Professional knowledge of secondary school mathematics teachers	Content knowledge	Arithmetic
			Algebra
			Functions
			Geometry
			Probability
		Pedagogical content knowledge	Knowledge of mathematical tasks as instructional tools
			Knowledge of students' thinking and assessment of understanding
			Knowledge of multiple representations and explanations of mathematical problems
LMT	Mathematical knowledge for teaching	Subject matter knowledge	Common content knowledge
			Specialized content knowledge
			Horizon content knowledge
		Pedagogical content knowledge	Knowledge of content and curriculum
			Knowledge of content and teaching
			Knowledge of content and students
TEDS-M	Knowledge for teaching mathematics	Mathematical content knowledge	Algebra and Functions
			Number and Operations
			Geometry and Measurement
			Data and Chance
		Mathematics pedagogical content knowledge	Mathematical curricular knowledge
			Knowledge of planning for mathematics teaching and learning
			Enacting mathematics knowledge for teaching and learning

might be one of the reasons LMT researchers were uncertain about the placement of curricular knowledge.

On the basic level, the three frameworks are strikingly similar in regards to the number of categories and their labels (table 3). Nevertheless, to determine how similar the categories actually are, we need to investigate how the frameworks of the projects delimited and defined the basic-level categories through their subordinate categories.

It is noteworthy that the subordinate-level categories of subject matter knowledge in the framework of the LMT project differed from those in the frameworks of the TEDS-M and COACTIV projects, as presented in table 3. The effort made by Ball and colleagues in the LMT project in "developing in more detail the fundamentals of *subject matter knowledge for teaching* by [...] elaborating subdomains, and by measuring and validating knowledge of those domains" (Ball et al., 2008, p. 402) might provide researchers and teacher educators with an opportunity to learn more about what is special about the mathematical (discipline) knowledge demands of a teacher.

Table 4 provides a structured overview of the division of mathematics PCK in the three frameworks alongside Shulman's original notion of PCK from 1986. It contains a thorough, yet shortened description of the subordinate categories of the frameworks. For the framework of the TEDS-M project, all the topic samples (examples of core situations) given in the conceptual framework (Tatto et al., 2008) are shown. For the frameworks of the LMT and COACTIV projects, short paraphrases or quotations of the published category descriptions are used.

The frameworks of the LMT, COACTIV and TEDS-M projects all divided mathematics PCK into three subordinate-level categories, and the names of these categories could offer hints as to how the researchers chose them. Questions like when or in relation to what/whom do teachers use their knowledge seem to be key features of the subordinate-level category definitions.

Recalling that categorisation is a cognitive activity, it should not come as a surprise that the subordinate-level categories of the three frameworks are different. The categories originate from three projects with different contexts, different expert members and different aims and goals. Nevertheless, all three frameworks included (i) knowledge about content and students; (ii) knowledge about content and teaching/instruction; (iii) knowledge about planning for teaching the content and, as previously described, (iv) curricular knowledge. Furthermore, elements from Shulman's notion of PCK can be found in all subordinate-level categories of the three frameworks. For example, knowledge of the most useful representations is found in the subordinate-level categories concerning teaching or enactment of mathematics, and knowledge of students' conceptions and preconceptions is either a category of its own (in the frameworks of the LMT and COACTIV projects) or is incorporated in the knowledge needed to plan and enact mathematics for teaching and learning (in the TEDS-M framework). However, knowledge of students (and curriculum) necessarily informs the teacher when choosing appropriate examples or representations as described in the subordinate teaching/instruction categories of the frameworks of the LMT and COACTIV projects, in which the subordinate categories combine general knowledge of teaching and instruction with mathematics in order to ensure the students achieve

Table 4. *An overview of the subordinate categories of mathematics PCK, including Shulman's notion of PCK from 1986*

Shulman (1986, p. 9)	Subordinate-level categories of mathematics PCK		
	COACTIV (Baumert et al., 2010)	LMT (Ball et al., 2008)	TEDS-M (Tatto et al., 2008, p. 39)
Pedagogical content knowledge is subject matter content knowledge for teaching and comprises – knowledge of the most useful forms of representations of mathematical ideas – knowledge of the most powerful analogies illustrations examples explanations demonstrations – an understanding of what makes the learning of specific topics easy or difficult – knowing/ understanding that students bring different conceptions and preconceptions to class – knowing strategies most likely to be fruitful in reorganising the students' understanding if the preconceptions are misconceptions ... in a word, [knowing] the ways of representing and formulating the subject that make it comprehensible to others	**Knowledge of mathematical tasks as instructional tools (TASKS)** "Knowledge of the potential of mathematical tasks to facilitate learning" (p. 142) – includes the "ability to identify multiple solution paths" (p.149)	**Knowledge of content and curriculum** "We have provisionally placed Shulman's third category, curricular knowledge, within pedagogical content knowledge" (p. 402)	**Mathematical curricular knowledge (CURRICULUM)** – establishing appropriate learning goals – knowing different assessment formats – selecting possible pathways and seeing connections within the curriculum – identifying the key ideas in learning programs – knowledge of mathematics curriculum
	Knowledge of students' thinking and assessment of understanding (STUDENTS) "Knowledge of student beliefs (misconceptions, typical errors, frequently used strategies) and the ability to diagnose students' abilities, prior knowledge, knowledge gaps, and strategies" (p. 142-143)	**Knowledge of content and students** Knowledge combining knowing about students and about mathematics, teachers must (paraphrased from page 401); – anticipate what students think and find confusing – know what students will find interesting and motivating, easy or hard – anticipate student responses – be able to hear and interpret students emerging and incomplete mathematical thinking	**Knowledge of planning for mathematics teaching and learning (PLANNING)** – planning or selecting appropriate activities – choosing assessment format – predicting typical students' responses including misconceptions – planning appropriate methods for representing mathematical ideas – linking the didactical methods and the instructional design – identifying different approaches for solving mathematical problems – planning mathematical lessons
	Knowledge of multiple representations and explanations of standard mathematical problems (INSTRUCTION) Knowledge of how to guide and support the students' "achievement of a deep understanding of mathematical content (...) by offering multiple representations and explanations "(p. 143)	**Knowledge of content and teaching** Knowledge combining knowing about teaching and about mathematics, teachers must (paraphrased from page 401); – know how to sequence particular content for instruction – choose appropriate examples, representations, method(s), procedures – know how to evaluate the instructional advantages and disadvantages of representations used	**Enacting mathematics for teaching and learning (ENACTING)** – analyzing or evaluating students' mathematical solutions or arguments – analyzing the content of students question – diagnosing typical students' responses, including misconceptions – explaining or representing mathematical concepts or procedures – generating fruitful questions – responding to unexpected mathematical issues – providing appropriate feedback

a deep understanding of the subject. The frameworks of the LMT and COACTIV projects thus included both the pre-active and interactive part of teaching in the subordinate teaching/instruction categories, as they define knowledge needed before, during and after interaction with students (i.e. teachers plan and choose appropriate examples or cognitively challenging opportunities to present to the students; teachers interact, guide and support the students and are continually evaluating the instructional choices made). The choice of verbs used to exemplify the categories of the framework of the TEDS-M project, on the other hand, may help distinguish between the pre-active part of teaching (PLANNING) and the interactive part of teaching (ENACTING) (Tatto et al., 2008).

The apparent danger of overlap in subordinate-level categories within a framework gives rise to the question of mutual exclusiveness, hence compromising the aim to span the whole construct and the ability to report test results by subordinate-level categories. A solution could be to report on the basic-level categories, given that the line between mathematics CK and mathematics PCK is easier to draw. To shed light on this issue, the items categorised in the introduction are reconsidered. The context of Item_{LMT} and $\text{Item}_{\text{TEDS-M}}$ (figures 2 and 3) presents a typical situation and task of mathematics teachers, with students being represented by their work. Upon reaching the questions posed in these items, test respondents are brought into a school context as they are presented with three students' work and asked to react to each of the students' responses as they would in the classroom. So far, the impression might be that these test items require either more or something other than mathematics CK; hence, it seems plausible at this point to presume that both items could be operationalisations of the basic-level category mathematics PCK, which is only the case for $\text{Item}_{\text{TEDS-M}}$.

In Item_{LMT}, respondents were asked to judge if the methods shown can be generalised for all whole numbers or not, and in $\text{Item}_{\text{TEDS-M}}$ the task was to determine whether or not each proof was valid. Item_{LMT} was designed to measure knowledge of mathematics, in particular the "[mathematical] knowledge not typically needed for purposes other than teaching" (Ball et al., 2008, p. 400); i.e. specialised mathematics CK. $\text{Item}_{\text{TEDS-M}}$ was an operationalisation of ENACTING, and the most likely corresponding sample topic is "analyzing or evaluating students' mathematical solutions or arguments". Analysing or evaluating students' mathematical solutions is indeed a core task of teaching. However, in order to generalise methods or validate proofs, as is the case in Item_{LMT} and $\text{Item}_{\text{TEDS-M}}$, a correct score requires CK of mathematics, as is the case for similar questions such as "Is this true for all numbers?", "Can this method always be used to solve this kind of question?" and "Is this true for all similar situations?" Considering the similarity between the test items, in the framework of the LMT project, $\text{Item}_{\text{TEDS-M}}$ would probably be categorised as mathematics CK, whereas in the framework of the TEDS-M project, Item_{LMT} would probably be categorised as mathematics PCK.

One thing worth noticing here is that the process of designing test items, operationalising the categories defined in a framework and making sure the whole construct is adequately covered is difficult and time consuming (Downing, 2006). Thus, it might be cost-effective to follow the example of the TEDS-M project team. They had the opportunity to build on the frameworks of both the LMT and COACTIV projects; even so, they obtained permission from the LMT project to use some test items developed in and for that framework (Tatto et al., 2012). The apparent similarity between Item_{LMT} and $\text{Item}_{\text{TEDS-M}}$ might be due to this sharing of items.

In the COACTIV framework, the three subordinate categories of teacher PCK were assessed in the paper-and-pencil tests as teachers' (i) "ability to identify multiple solutions paths" (TASKS); (ii) "ability to recognise students' misconceptions, difficulties, and solution strategies" (STUDENTS) and (iii) "knowledge of different representations and explanations of standard mathematics problems" (INSTRUCTION) (Baumert et al., 2010, p. 149). As the following observation illustrates, this emphasis seems to be a strong signal of the importance of mathematics CK: "PCK is inconceivable without CK [and] we assume that PCK is needed over and above CK to stimulate insightful learning" (Baumert et al., 2010, p. 145).

In $\text{Item}_{\text{COACTIV}}$, an operationalisation of the subordinate level TASKS category, the respondents are asked to provide as many solutions to the given problem as possible. One could argue that the knowledge needed to respond to this item is mathematics CK. However, one could also argue that it is particularly necessary in and for teaching mathematics, thus making it mathematics PCK (as is the case in the framework of the COACTIV project). The more versatile and flexible the teachers' mathematics CK is (regarding multiple solution paths and multiple representations), the easier it also becomes to give feedback and guidance to students (Baumert et al., 2010). However, this ambiguity makes basic-level categorisation all the more difficult. Thus, categorising $\text{Item}_{\text{COACTIV}}$ as both mathematics CK and PCK indicates that the prototype theory of categorisation might apply better than the classical theory where the object to be categorised is put in only one category (no in-between cases or overlap). If categorizing on the taxonomical basic level across frameworks, $\text{Item}_{\text{COACTIV}}$ would probably be categorised as mathematics CK in the framework of the LMT project in that it only requires mathematical knowledge (recognising the importance for teaching, it might further be placed in the subordinate category specialized mathematics CK). In TEDS-M it might be placed as mathematics PCK as it could be designed to elicit future teachers' knowledge about different approaches for solving mathematical problems (as in the subordinate category PLANNING), or it might be placed as mathematics CK as the item context does not explicitly show a core teaching situation.

The examination of the relationship between the framework categories and their operationalisation in test items might suggest that the ambiguity (as indicated when categorising $\text{Item}_{\text{TEDS-M}}$, Item_{LMT} and $\text{Item}_{\text{COACTIV}}$ in and across frameworks) could potentially reveal missing categories in the frameworks. Furthermore, it could simply be the outcome of poorly constructed categories or test items, i.e. some test items just do not fit as operationalisations of the frameworks' categories.

Concluding remarks

Compared to Shulman's original notion of PCK, the subordinate-level categories of mathematics PCK showed that the three frameworks all adequately covered his description even though they categorised mathematics PCK differently. For example, knowledge about content and students was a category on its own in the frameworks of the LMT and COACTIV projects, whereas in the TEDS-M project it was part of the two categories: planning for teaching mathematics and enacting mathematics for teaching and learning.

One subordinate mathematics PCK category stood out as an addition to Shulman's description: the TASKS category of the COACTIV project. By including this category in their framework, the COACTIV project has indicated the importance of knowing and recognising the potential of mathematics tasks to facilitate learning. Other categories worth mentioning are the subordinate categories of mathematics CK in the framework of the LMT project. The effort made by the LMT expert group to also present subordinate teacher categories of mathematics CK may benefit further development of teacher knowledge categories because it might support the understanding of (and agreement on) basic-level categories across projects. Furthermore, the subdivisions of Shulman's basic-level categories might have advantages when investigating what kind of teacher knowledge best predicts student outcomes. Indeed, this was indicated by Ball et al. (2008), but did not fall within the scope of this paper.

Overall, the analysis and discussion herein showed that each project provided a framework that, for the most part, had much in common both with Shulman's framework and with each other. However, differences in the operationalisation of the basic-level categories were observed. Moreover, this paper supports the claim that the disagreement between the

LMT and TEDS-M projects regarding the categorisation and/or operationalisation of categories by Item_{LMT} and $\text{Item}_{\text{TEDS-M}}$, and the question of whether or not knowing different solution paths to one problem (as in $\text{Item}_{\text{COACTIV}}$), represents a problem – the problem of how to distinguish genuine mathematics PCK from pure mathematics CK. These two categories should be mutually exclusive if they are categorised in the classical perspective in order to validly measure and report teachers' cognitive abilities.

As one aim of the current and forthcoming regulations in the Nordic countries is for teacher education to be research-based (Breiteig & Grevholm, 2010; Dahl, 2010; Grevholm, 2010; Gunnarsdóttir & Pálsdóttir, 2010; Niemi & Jakku-Sihvonen, 2011), the problem of distinguishing mathematics CK and PCK comes into play if the goal is to design courses and teacher education programmes based on research focusing on what kind of CK teachers have (or should have). As this paper demonstrates, categorising can be done in various ways for the same object; namely, knowledge for teaching mathematics. However, the multitude of frameworks and categories might imply a pragmatic stance within the teacher education research community towards different categories since categorisation is a cognitive activity that draws on personal background and experience. Thus, if the frameworks analysed here are to be used as a basis to design courses for mathematics teacher education, the similarities and differences between them must be kept in mind. Most importantly, the fact that they seemingly do not have the same idea regarding what constitutes mathematics CK and mathematics PCK when operationalising (and eventually reporting test results on) these basic-level categories must be acknowledged.

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Appendix A – The COACTIV project

Authors (year)	Document**	Title	Published
Baumert et al. (2010)*	Article	Teachers' Mathematical Knowledge, Cognitive Activation in the Classroom, and Student Progress	American Educational Research Journal, 47(1), 133-180
Krauss, Baumert, and Blum (2008)*	Article	Secondary mathematics teachers' pedagogical content knowledge and content knowledge: validation of the COACTIV constructs	ZDM, 40(5), 873-892
Krauss, Neubrand, et al. (2008)*	Article	Die Untersuchung des professionellen Wissens deutscher Mathematik-Lehrerinnen und-Lehrer im Rahmen der COACTIV-Studie.	Journal für Mathematik-Didaktik, 29(3/4), 223-258
Kunter et al. (2007)	Book section	Linking aspects of teacher competence to their instruction: Results from the COACTIV project	In M. Prenzel (Ed.), Studies on the educational quality of schools : The final report on the DFG Priority Programme (pp. 39-59)
Krauss, Brunner, et al. (2008)	Article	Pedagogical Content Knowledge and Content Knowledge of Secondary Mathematics Teachers	Journal of Educational Psychology, 100, 716-725
Jordan et al. (2008)	Article	Aufgaben im COACTIV-Projekt: Zeugnisse des kognitiven Aktivierungspotentials im Deutschen Mathematikunterricht	Journal für Mathematik-Didaktik, 29(H. 2), 83-107

Notes. * Primary source documents for the present study.
**All documents are listed on the COACTIV internet home page, http://www.mpib-berlin.mpg.de/coactiv/en/index.php (Eng. version).

Appendix B – The LMT project

Authors (year)	Document**	Title	Published
Ball, Thames, and Phelps (2008)*	Article	Content Knowledge for Teaching: What Makes It Special?	Journal of Teacher Education, 59(5), 389-407
Hill and Ball (2004)*	Article	Learning Mathematics for Teaching: Results from California's Mathematics Professional Development Institutes	Journal for Research in Mathematics Education, 35(5), 330-351
Ball and Hill (2008) *	Released items	Mathematical Knowledge for Teaching (MKT) Measures	www.sitemaker.umich.edu/lmt
Hill, Schilling, and Ball (2004)	Article	Developing Measures of Teachers' Mathematics Knowledge for Teaching.	The Elementary School Journal, 105(1), 11-30
Hill, Rowan, and Ball (2005)	Article	Effects of Teachers' Mathematical Knowledge for Teaching on Student Achievement	American Educational Research Journal, 42(2), 371-406
Hill, Sleep, Lewis, and Ball (2007)	Book section	Assessing Teachers' Mathematical Knowledge: What Knowledge Matters and What Evidence Counts?	In F. K. Lester (Ed.), Second Handbook of Research on Mathematics Teaching and Learning (Vol. 1, pp. 111-155)
Hill, Dean, and Goffney (2007)	Article	Assessing Elemental and Structural Validity: Data from Teachers, Non-teachers, and Mathematicians.	Measurement: Interdisciplinary Research & Perspective, 5(2), 81 - 92
Hill, Ball, and Schilling (2008)	Article	Unpacking Pedagogical Content Knowledge: Conceptualizing and Measuring Teachers' Topic-Specific Knowledge of Students	Journal for Research in Mathematics Education, 39(4), 372-400

Notes. * Primary source documents for the present study.
** All documents, except the first, are listed on the LMT internet home page, http://sitemaker.umich.edu/lmt

Appendix C – The TEDS-M project

Authors (year)	Document	Title	Published
Tatto et al. (2008)*	Conceptual Framework	Teacher Education and Development Study in Mathematics (TEDS-M): Policy, practice, and readiness to teach primary and secondary mathematics. Conceptual framework	http://teds.educ.msu.edu/reports/***
Tatto et al. (2012)*	International Report	Policy, Practice, and Readiness to Teach Primary and Secondary Mathematics in 17 Countries: Findings From The IEA Teacher Education and Development Study in Mathematics (TEDS-M)	http://teds.educ.msu.edu/wp-content/uploads/IEA_TEDS-M-International-Report1.pdf
Schmidt et al. (2007)*	MT21 Report**	The Preparation Gap: Teacher Education for Middle School Mathematics in Six Countries	http://usteds.msu.edu/MT21Report.pdf
Brese and Tatto (2012)*	Released items	TEDS-M 2008. User Guide for the International Database. Supplement 4. IEA Secretariat.	http://www.iea.nl/?id=20
Blömeke, Houang, and Suhl (2011)	Article	TEDS-M: Diagnosing teacher knowledge by applying multidimensional item response theory and multiple-group models	Vol. 4. IERI Monograph Series: Issues and Methodologies in Large-Scale Assessments (pp. 109-129)
Tatto, Senk, Bankov, Rodriguez, and Peck (2010)	Assessment Framework	TEDS-M 2008 Assessment Frameworks: Measuring future primary and secondary teachers mathematics and mathematics pedagogy knowledge	http://teds.educ.msu.edu/reports/

Notes. * Primary source documents for the present study.
** Reporting the feasibility study of TEDS-M.
*** http://teds.educ.msu.edu is the internet home page for the TEDS-M project.

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