

# Prolifing Swedish teachers' knowledge base in probability

PER NILSSON AND TORSTEN LINDSTRÖM

This paper aims at profiling Swedish teachers' knowledge base in probability. 43 teachers in compulsory school answered a questionnaire on probability estimation tasks and concept tasks. In the concept tasks, they were challenged to explain their solutions and the content involved in the probability estimation tasks. We distinguish five patterns in the teachers' knowledge profile: 1) a basic understanding of the theoretical interpretation of probability, 2) problems with structuring compound events, 3) difficulty with conjunction and conditional probability, 4) a higher degree of common content knowledge than of specialized content knowledge and 5) limited understanding of random variation and principles of experimental probability.

This paper is part of a larger, on-going project aiming at investigating the teaching of probability in the Swedish school-system. A long-term goal of the project is to provide valuable information on how we should design pre-service and in-service programmes to prepare teachers to teach probability. In this paper, we present the first step in this enterprise by describing and analysing the nature of Swedish school-teachers' knowledge base in probability.

In recent decades, there has been an increased interest in studying students' understanding of random experiments and how the students develop an understanding of probability in order to make predictions about the different outcomes of a random situation (Jones, Langrall & Mooney, 2007). This trend is also evident in the contemporary school curricula of many countries, including the Swedish curriculum for compulsory schools.

In the new Swedish curriculum for elementary schools, the combination of probability and statistics constitutes one of six central content

---

**Per Nilsson**, Örebro University and Linnaeus University

**Torsten Lindström**, Linnaeus University

areas (Swedish National Agency for Education, 2012). There is no doubt that this increased interest in chance and probability in the curriculum is important. We increasingly meet chance variation and random phenomena, not only in games, sports, and individual decision-making, but we also encounter them in the social security system, insurance, and elections that form the basis of our modern society (Jones, 2005). However, international studies show that teachers find it difficult to teach probability (Haller, 1997), and the question that motivates the stance of the present project is whether Swedish teachers are prepared for this increased emphasis in the school curriculum on teaching probability.

While we recognize that there are many factors affecting teaching performance – e.g. how teachers are affected by the school-system, how they perceive knowledge and teaching, how they assess students' understanding, how they relate to their professional community, and how they respond to curriculum change (Watson, 2001) – the approach of the present study builds on the stance taken by Shulman (1987). He concluded that (p. 41)

teaching typically occurs with reference to specific bodies of content or specific skills and that modes of teaching are distinctly different for different subject areas [...] the particular kinds of learners and the character of the setting also influence the kind of teaching [...] [and] most assessments must examine the applications of pedagogy to specific subject areas.

In relation to the teaching of the specifics of probability, Stohl (2005) observed that "the success of any probability curriculum for developing students' probabilistic reasoning depends greatly on teachers' understanding of probability as well as a much deeper understanding of issues such as students' misconceptions" (p. 351). In the present paper, the focus is on teachers' knowledge base in probability. Investigating teacher knowledge builds on the assumption that teachers' understanding of significant mathematical ideas has a profound influence on their capacity to teach mathematics effectively. Above all, their understanding of mathematical ideas influences what they intend students to learn and how they, the teachers, can develop the learning trajectories of these ideas. Moreover, the way teachers understand the content they teach has proven to be critical for their pedagogical orientation and their ability to make instructional and assessment decisions (Liu, 2005; Van Dooren, Verschaffel & Onghena, 2002).

The present study offers a Swedish perspective as we are investigating teachers who are educated in Swedish teacher programs and who teach in the Swedish school-system. However, we claim that the study is of

general interest, beyond the Swedish system, in terms of broadening our knowledge of teachers' understanding of probability and how they merge their understanding with their education and teaching experiences.

### *Research question*

Based on the introduction and the background that follows, the present paper addresses the specific research question: *What is the profile of Swedish teachers' knowledge base in probability?*

## Background

In this section, we discuss three themes which, between them, point to why and how probability is a particularly difficult subject to teach. The themes are: *the irreversibility of randomness*, *different meanings of probability*, and *peoples' informal ideas of probability*.

### *The irreversibility of randomness*

In contrast to arithmetic or geometry, probability deals with situations that are irreversible. An elementary operation like multiplication can be reversed, and this reversibility can be represented with concrete material. This is very important for young children who are still very linked to concrete situations in learning mathematics (Batanero & Diaz, 2012). Take, for example, the situation where four children have six marbles each and the question is how many marbles they have got altogether. The situation can be used to illustrate the multiplication "four times six equals 24". We can represent the multiplication of four times six in several ways with concrete materials and we experience that we always arrive at 24. That the process is reversible means that we always return to the home position if we start from the end and do the operation backwards. In our example, this means that we can concretely illustrate how the distribution of 24 marbles among *four* children always results in the situation where each child gets six marbles; we always return to the initial state. Such experiences are very important in helping children experimentally abstract mathematical structures (Batanero & Diaz, 2012). However, in the case of random experiments and probability thinking, we obtain different concrete experiences each time the experiment is carried out. The children cannot experience reversibility in a random situation. We do not get the first result again and, consequently, we cannot derive the process backwards, returning to the initial state. Say, for instance, you throw 100 thumbtacks. It is impossible to reconstruct the situation from how the thumbtacks landed on the table

to their initial state in your hand, just before you threw them. This makes the teaching of probability comparatively harder for teachers.

### *Different meanings of probability*

It is possible to think of probability in different ways depending on the random experiment in question. Different ways of thinking are suitable for different situations, depending on what information is available. In line with the teaching and research in probability education, the present study focuses on the knowledge base teachers have in relation to both the theoretical and the experimental interpretations of probability. The theoretical interpretation of probability allows the calculation of probabilities before any trial is made. It implies the need for sample space-orientated ratio-thinking (Hawkins & Kapadia, 1984) where the probability of an event is obtained through the fraction of outcomes favourable for an event out of all cases possible (Borovcnik, Bentz & Kapadia, 1991). The key ideas of theoretical probability are: to develop an ability to list *all possible outcomes* for an event, and an understanding of how the underlying *sample space* regulates the probabilities of a random phenomenon (Chernoff & Zazkis, 2011). Experimental probability offers a posterior model of a random experiment. Through experimental probability, the probability of an event is obtained from the observed relative frequency of that event in several trials (Borovcnik et al., 1991). Similar to theoretical probability, experimental probability also implies ratio-thinking, but this time in terms of determining the fraction of the number of times an event appears out of all trials done. The key ideas to understand in experimental probability are: *random variation, sampling, independency between trials in a sample and the role of sample size* (the law of large numbers) (Gal, 2005).

In our investigation and evaluation of the teachers' knowledge base in probability, we consider their ability to discern and piece together different aspects of the two interpretations.

### *Previous research on teachers' knowledge of probability*

Teachers' understanding of probability has not been studied to the same extent as students' understanding has (Stohl, 2005). Therefore, in order to form a basis for the design of our survey instrument and subsequent analysis, we also include pre-service teachers in our literature review.

Studies show that in-service and pre-service teachers do not have a strong understanding of probability and that they often hold the same misconceptions as students (Batanero & Diaz, 2012; Begg & Edwards, 1999;

Haller, 1997; Liu, 2005). Begg and Edwards (1999) mapped the content knowledge of 22 teachers and 12 student teachers in elementary schools in New Zealand. They found that the participants had a weak understanding of central probability concepts with only about two thirds of them understanding equally likely events and even fewer understanding issues of independent random events.

Batanero, Cañizares, and Godino (2005) conducted a survey of Spanish pre-service teachers' understanding of probability. The results of a sample of 132 pre-service teachers showed that the participants frequently had three probabilistic misconceptions: *representativeness* (Kahneman & Tversky, 1972), *equiprobability bias* (Lecoutre, 1992) and *the outcome approach* (Konold, 1989). Applying representativeness basically means that people judge small samples as being equally representative of a population (or the underlying sample space) as large samples. The equiprobability bias concerns how people overgeneralize the assumption of equiprobable outcomes to situations that are not uniformly distributed. When inclined toward the outcome approach, people often base their predictions on causal factors, and tend to assign numbers as "probabilities" on the basis of the strength of the perceived causal relationship. If the strength is sufficient for a certain outcome, the outcome-orientated person would expect it to happen (Pratt, 1998).

The law of large numbers is central to the experimental interpretation of probability and for being able to make connections between the theoretical and the experimental interpretation of probability (Nilsson, 2009; Prodromou, 2012). Understanding the law of large numbers implies "understanding the *unpredictability* of random phenomenon in the short-run but *predictability* in the long-run trends in data" (Stohl & Tarr, 2002, p. 321). In relation to this, Lee and Hollebrands (2008) show how teachers may have difficulties with understanding the role of sample size when examining distributions and variability and how this understanding makes it hard for the teachers to address the heart of the frequentist conception of probability.

Liu (2005) investigated a group of high school teachers as they engaged in seminar discussions that progressed over eight sessions in two weeks. During these seminar discussions, it became evident how difficult it was for the teachers to develop a combined and coherent understanding of probability. The teachers expressed what Liu calls a compartmentalized understanding. Their understanding of specific concepts was closely connected to specific situations and they conceptualized learning as "knowing to solve problems" (Liu, 2005, p. 348).

## Methodological considerations

### *Overall structure of teachers' knowledge base in probability*

Ball and her colleagues (Ball, Thames & Phelps, 2008) have developed the notion of "mathematical knowledge for teaching" (MKT) for structuring the nature of knowledge needed for teaching mathematics. The notion of MKT is distinguished by six main domains: *knowledge of contents and curriculum, knowledge of content and students, knowledge of content and teaching, common content knowledge, specialized content knowledge* and *horizon content knowledge*. Each domain highlights different issues of what is understood to be crucial in the teaching of mathematics<sup>1</sup>. In the current study, the object of study is teachers' knowledge base in probability. For the current study, we use the principles of common content knowledge (CCK) and specialized content knowledge (SCK) to specify the overall structure of teachers' knowledge base in probability.

**Common content knowledge (CCK).** Generally speaking, CCK refers to the mathematical "knowledge that is used in the work of teaching in ways *in common with* how it is used in many other professions or occupations that also use mathematics" (Hill, Ball & Schilling, 2008, p. 377). In several studies, it has become clear how teaching is affected negatively and valuable time is lost when a teacher pronounces or uses terms incorrectly, or miscalculates or gets stuck when trying to solve a task. A teacher needs to know when students have got the answer wrong, to realize when a textbook gives an incorrect definition, and to be able to use terms and symbols correctly. Hence, teachers need to know the content they teach and they are responsible for their students development (Contreras, Batanero, Díaz & Fernandes, 2011).

**Specialized content knowledge (SCK).** In contrast to CCK, teachers need mathematical knowledge that is specific to the teaching of mathematics (Hill et al., 2008). In line with Contreras et al. (2011), we particularly focus on how teachers explain what they are doing, how they express an ability to use technical terms (c.f. Mason, 1998; Ryve, Nilsson & Pettersson, 2013) of probability, and are able to discern and express the content of a probability task.

### *Participants and data collection*

To answer our research question, we e-mailed approximately 240 teachers, asking them to answer a questionnaire about probability. 43 teachers

chose to answer the survey. Of those 43 teachers, six were educated upper secondary school teachers (the Swedish gymnasium), 18 were educated secondary teachers (grade 7–9) and 16 were educated primary school teachers (grade 1–6). For the last three participants, it was difficult to interpret their education from the information they gave. The questionnaire was completed anonymously using Google docs.

The sample is not completely representative for two reasons. First, only about a fifth of the 240 teachers completed the questionnaire and this raises questions as to whether it was only those who were most comfortable with the subject who responded. The second reason is that the respondents were drawn from a list of "Mathematics developers"<sup>2</sup>. As a Mathematics developer, you have often shown a certain interest in the issues of mathematics education and you are supposed to implement local developmental work and serve as a guide for research and other inspirational material. Thus, when we evaluate our results, we must remember that we have probably received a picture of the knowledge base among the most qualified teachers and that we may have good reason to assume that Swedish teachers in general would perform below the average of the sample in the present study.

In the e-mail, it was explained that the survey would take approximately 90 minutes to complete. The aim of this was to signal to the teachers that they should give the survey enough time and take it seriously, in order for us to receive a valid profile of their knowledge base. However, this may be one reason for the low rate of participation

### *The survey*

Two pilot questionnaires formed the basis of the design of the current survey. One pilot focused on the experimental interpretation of probability, the other focused on the theoretical interpretation. The questions in the pilot questionnaires worked well, and we made only minor adjustments to the wording of some of the questions. The present survey combines questions from both of the pilots, i.e. tasks which call for reflection on both the theoretical and the experimental concepts of probability.

The survey began with some background questions about the teachers' education, how long they had taught mathematics, and their level of education in probability. These questions were followed by four questions where the teachers were asked to assess their familiarity with *the law of large numbers*, *sample space*, *conditional probability* and *uniform distribution* respectively. Teachers could choose from five options: a) Never heard of the concept, b) I have heard of the concept, c) I know what the concept means, d) I can relate the concept to some other concept of

probability theory, and e) I can explain to my students what the concept means. Focus on the present paper is on the probability tasks, which constituted the largest part of the survey. There were 11 probability tasks (appendix 1). Each task consisted of a *probability estimation task* and a *concept question* (explorative question). The probability estimation tasks were supposed to provide information on the teachers' CCK and the concept questions to provide information on the teachers' SCK. The concept questions are marked with an apostrophe (').

The research literature on probability education constituted a large source when constructing the probability questions for the survey. Another source of motivation when constructing the questions was textbooks in probability directed at children in grades 4–6 (10–12 years old). Questions 16 and 17 (appendix 1) are the best examples of this category. Some of the questions were motivated by die-games directed at 10–12 year old children. Children may encounter game-situations where they play in accordance with the total of two dice (question 14). They may encounter situations where they roll two dice, but proceed according to the one showing the lower number (question 19). Children may therefore ask their teachers questions regarding non-uniform and asymmetric distributions.

### *General principles in the coding of the teachers' responses*

The analysis of the answers to the content questions was done through the iterative process of *content analysis* (Krippendorff, 2004). This means that the entire material composed of the teachers' answers was analysed repeatedly. Meanings and phrases which contained information relevant for the teachers' SCK were marked. All marks were then ordered in levels, forming the profile of the teachers' SCK.

Coding of data is a matter of interpretation and this is particularly the case when it comes to the coding of conceptual questions. To strengthen the reliability of the coding, the coding process of the current study was made in two steps. The two authors first coded the data individually, in order to ensure that none of the researchers would dominate the coding too much during discussions. Then the coding was discussed collectively, with a particular focus on instances where the individual assessments differed.

In table 1 below, we present the result of the entire analysis of the teachers' responses to the concept questions. In this section, we describe the overall principles of our analysis and our way of classifying different responses. For all questions, rather general or imprecise references to probability theory or combinatorics were classified as Level 0. For



instance, Level 0 reasoning was displayed when teacher 29 at question 9' articulated that there is a need to have a presentation about probability and statistics to deal with the diagram in a teaching situation. Imprecise references to steering documents or school curricula were also treated as Level 0 responses. Answers containing remarks about required prerequisites (e.g. fractions, per cent, coordinate system in question 9') were considered to be on a higher level than Level 0. The exact code given depended on the prerequisites referred to. For instance, in question 9', teachers 15 and 16 mentioned that they would never use such a task for the grade in question; pupils aged 10–12 have not established the use of either decimal numbers or of coordinate systems in such a way that the task would be feasible for them. This was a response that was valued higher than a response of just remarking on, for instance, the need to understand fractions and percentages.

Question 10 was, as was expected, the most difficult one. The example can actually be found in university level textbooks in probability theory (see e.g. Bertsekas & Tsitsiklis, 2008). It is a well-known case where asymmetry is caused by the conditional information that one side of the card is blue. Only two teachers answered this question correctly (teachers 7 and 23). However, not even these teachers reflected on the asymmetry or order embedded in the situation when they came to question 10'. As they managed to solve the probability estimation task, it is clearly visible that there are levels that should be expressed for students in the concept questions, but are not.

Calculating the rate of success in the probability estimation tasks was straightforward. The maximum score was defined prior to the participants' responses. This was not the case for the concept questions. Here, there were no maximum scores defined in advance. Instead, the rate of success in the concept questions was relative to the qualitative differences in the participants' answers and levels achieved. For example, in task 9', we distinguished five levels (scores from 0–4). The total possible score was  $4 \cdot 43 = 172$ . The rate of success achieved by the group was 47%, which is given by  $(0 \cdot 10 + 1 \cdot 3 + 2 \cdot 18 + 3 \cdot 7 + 4 \cdot 5) / 172$ .

## Results and analysis

In table 1, the coding of all the teachers' responses to the concept questions is presented and the absolute number of responses, classified in accordance with the specific levels is given in brackets. Table 2 presents the relative frequencies of the group's performance on the probability estimation tasks and the conceptual questions.

Table 1. *Classification of the teachers' responses to the conceptual questions*

Question	Level	Description
9'	0	References too general (3), listing of general teaching curricula (1), empty (6), or irrelevant (0)
	1	Events, relative frequency, fractions, per cent, what is actually done at 10-12 years of age? (3)
	2	Mean value, coordinate systems, decimal representation, number of trials essential (18)
	3	Reasoning why this is not suitable for ages 10-12, coordinate systems, fractions not yet covered, etc. (7)
	4	References to the law of large numbers or equivalent (5)
10'	0	References too general (6), listing of general teaching curricula (3), empty (8), or irrelevant (1)
	1	Fractions, per cent, sample space (19)
	2	Conditioning, one has to disregard the card that was green on both sides (6)
11'	0	References too general (3), listing of general teaching curricula (1), empty (8), or irrelevant (0)
	1	Fractions, per cent, sample space, discrete uniform probability law (13)
12'	0	References too general (4), listing of general teaching curricula (1), empty (13), or irrelevant (0)
	1	Fractions, division, per cent, etc. (6)
	2	References to the discrete uniform probability law and division if anything correct in the previous exercise (17)
13'	0	References too general (9), listing of general teaching curricula (1), empty (10), or irrelevant (3)
	1	The number of trials has an impact, several trials closer to expectation (14)
14'	0	References too general (11), listing of general teaching curricula (1), empty (8), or irrelevant (1)
	1	Combinations for the different sums, sample space or independent events are mentioned (9)
	2	There are a different number combinations for the different sums (7)
15'	0	References too general (0), listing of general teaching curricula (0), empty (7), or irrelevant (4)
	1	Only a few women are teachers, among teachers many are women, etc. (32).
16'	0	References too general (0), listing of general teaching curricula (0), empty (6), or irrelevant (7)
	1	Discrete uniform probability law gives the same probability (30)
17'	0	References too general (0), listing of general teaching curricula (0), empty (6), or irrelevant (6)
	1	There are more odd combinations than even ones (26)
	2	Conditioning implies some even sums should be removed from the sample space (2)
18'	0	References too general (0), listing of general teaching curricula (0), empty (7), or irrelevant (13)
	1	The set of men is larger than the set of young men, drive more, or equivalent (14)
19'	0	References too general (0), listing of general teaching curricula (0), empty (8), or irrelevant (12)
	1	There is a total number of 36 outcomes (13)
	2	Identification of the 10 or 11 (motivated) relevant outcomes (4)
	3	Notes order (6)

Table 2. *The rate of success on questions (see appendix)*

Question	Rate (%)	Question	Rate (%)
9'	47	14'	32
10	5	15	93
10'	22	15'	74
11	98	16	88
11'	57	16'	70
12a	95	17	77
12b	70	17'	30
12c	77	18	74
12'	36	18'	37
13	65	19a	26
13'	30	19b	53
14a	76	19'	30
14b	80		

Studying tables 1 and 2, we see that the teachers in general show limited SCK. Very often the teachers offer overall and imprecise explanations, or present suggestions that are particularly incorrect or irrelevant to the task at hand. We are aware of the difficulties of comparing the quantitative values of the results in the probability estimation tasks and the concept tasks. However, we see that the teachers are more capable of solving the probability tasks than of expressing answers in the concept tasks. Very often, they use theoretical ideas and concepts (implicitly) in calculations but find it hard to discern and express the content of the task or of what they have done. We understand this to mean that the teachers' knowledge profile is more developed in terms of CCK than in terms of SCK. Moreover, based on this difference, we claim that their knowledge profile is more computationally orientated than conceptually orientated. Next, we go into detail regarding the result of the survey. The presentation basically follows the patterns we identified in the responses to the probability estimation tasks. However, the analysis is done on the collective information we received from the survey.

In several probability estimation tasks, the teachers show evidence of relatively good CCK. We particularly note that the teachers are quite comfortable with simple random situations, which ask for a Laplace-based model of probability, based on proportional reasoning of favourable cases of an event out of all possible cases. In contrast, we see that they are less comfortable with the concepts of random variation and the stabilization of frequencies (question 13). The results also disclose that many teachers find it hard to deal with conditional probabilities and to structure the sample space of compound random events.

### *Familiarity with the basic principle of theoretical interpretation*

Among the four tasks with the highest rate of success, three of them (11, 12a, 16) concerned the calculation of single random events. These results imply that when the probability task is pretty straight forward, and the sample space is clear to the teachers, they have no problems with modelling probabilities in accordance with the theoretical, sample-based interpretation of probability.

Question 11 contained an extra difficulty. Here, we wanted to see how the teachers responded to a task without replacement. They would first answer regarding getting a black marble in a second draw from an urn, after first having picked a black marble that was not replaced. The same question was then asked regarding obtaining a red marble after picking a black marble that was not replaced. Studies show (e.g. Tarr, 1997) that students often fail in situations without replacement. In relation to the current task, this would show up as them noting that the probability of black has changed, as they have already drawn a black marble, but they should also consider that the probability of red has not changed. This problem is considered to be rooted in a way of thinking in which the student decides on the basis of part-part comparisons and not on the basis of part-whole comparisons (Tarr, 1997). However, based on the results of question 11, the teachers exhibited no difficulties in realizing that the probability of the outcome that was not obtained is the first draw (i.e., obtaining a red) had also changed. We interpret this as the basic ratio-principle of the theoretical interpretation being present in the teachers' CCK of probability.

In question 16, the teachers were asked to compare the probability of two situations, with the same proportions in sample space structure. If inclined toward part-part reasoning, it would be easy to decide that the probability of an event is highest in the situation with the largest number of favourable outcomes for the event in question. Of those who responded that the chance was different, most teachers also answered that the chance was largest for box B, i.e., for the box with most blue balls. This, exactly, reflects a comparison of the two parts involved and not of the proportions of favourable outcomes. However, 88% of the teachers answered that the chance was equal, and in their motivations they referred to the equal proportions of marbles in the two box situations.

### *The problem of random variation and stabilization of frequencies*

In question 13, the teachers were asked to consider the various chances of obtaining seven heads out of ten throws of a coin and 70 heads out of 100 throws. This was not an easy task for many of the teachers. Of

the 35% who gave incorrect answers, most of them answered that the chances were equal. These responses indicate that teachers have difficulties in understanding the law of large numbers and the stabilization of relative frequencies. It also indicates that the teachers' CCK is very much Laplace-orientated, involving proportions as a judging technique, without reflecting on the behaviour of random variation. The validity of this last interpretation is strengthened when we compare the teachers' responses to task 16, where they were asked to judge the probability of obtaining a blue marble when choosing from an urn containing six red and four blue marbles and an urn containing 60 red and 40 blue marbles. Question 16 proved to be easier for the teachers. In question 16, issues of frequencies or random variation were not part of the task as was the case in the question about the coin-flips. Connected to the previous paragraph, question 16 was more about sample space and, particularly, about fractions and equivalent fractions, which match a Laplace-based procedure for judging probabilities. Based on how responsive the teachers were to proportional reasoning in this task, we have good reason to believe that this was also very much affected by how they reasoned in the coin situation, valuing the likelihood of obtaining seven out of ten and 70 out of 100 as equal.

#### *The ability to structure the sample space of compound random events*

Above, we have argued that the teachers have an understanding of the basic principle of theoretical probability interpretation, that the probability of an event can be determined as the ratio between the number of favourable cases and the total number of possible cases. However, questions 14ab and 17 also follow this principle of probability modelling. But why then, is it that the teachers' rate of success is lower here than on the single random events discussed above? The interpretation we propose for understanding the low rate of success in 14ab and 17 is that these questions deal with compound random events and that compound events make greater demands on the ability to identify all favourable and possible outcomes. One specific issue with compound events is that you often have to take into consideration the order of the single outcomes to outline all possible outcomes. To illustrate, we look at question 13 regarding the total of two dice. It is possible here to form the eleven totals in 36 different ways. For example, there are two ways to arrive at the sum of three, either as  $1+2$  (1;2) or as  $2+1$  (2;1). Thus, we can form 36 unique pairs of the totals and six of these 36 pairs form the sum 7 and only one of them forms the sum 12.

It is notable that all teachers suggested certain totals as most likely (question 14a) or least likely (question 14b) respectively. Hence, none of the teachers suggested that all totals are equally likely, i.e. none of them demonstrated the equiprobability bias (Lecoutre, 1992), that everything is only a matter of chance, or the fairness resource (Pratt, 2000), that the totals are distributed uniformly since a single die is uniformly numbered.

### *The problem with conjunctive and conditional random events*

Basically, all questions, except for question 9 and question 13, build on the idea of favourable events out of total possible events. This is actually also the case for questions 10, 12bc 15, 18, and 19ab. However, these questions proved to be harder for the teachers. Questions 12b and 18 ask for the probability of *conjunctions*, and questions 10, 12c, 15, and 19 ask for the probability of *conditional random events*.

Concerning conjunctions, it is particularly interesting to compare the results of 12b to the results of 12a. Only one teacher failed to give the correct answer that the probability that a student likes tennis is  $\frac{6}{7}$ . But, only 70% gave the correct response,  $\frac{2}{7}$ , for the conjunction, that the student is a girl and likes tennis. That the rate of success was lower on 12b (and 12c) than on 12a follows the same pattern identified by Contreras et al. (2011) when confronting Spanish pre-service students with exactly the same question. Following the explanation proposed by Contreras et al. (2011), we have good reason to believe that the teachers find it difficult to identify what constitutes the "part" and the "whole" in the three different questions. It is difficult for them to discern the conjunction embedded in the question (to conclude on favourable cases) and to what sample space (total amount of possible outcomes) they should relate this conjunction.

Question 12c concerns the conditional probability of a student liking tennis when we know that the student is a girl. The problem here is also to understand the question in order to be able to identify the sample space and the number of favourable cases. From the condition given, that the student is a girl, we restrict the sample space to the 250 girls, and among them 200 like tennis. Hence, the probability of obtaining a student liking tennis, given that the student is a girl is  $\frac{4}{5}$ .

Dealing with conditional probability is known to be difficult (Falk, 1988). The two results that stand out in the survey are the results of questions 10 and 19. These two questions deal explicitly with conditional random events. Similar to the results on 12c, we propose that the difficulty may be grounded in a difficulty in understanding the questions. Understanding the logic involved in a conditional probability task is not as straightforward as understanding a task regarding the probability

of arriving at seven when throwing two dice. Moreover, in addition to understanding the logic of conditional random events, there is a need to be able to structure the entire sample space and figure out favourable outcomes. For instance, being able to list all 36 possible outcomes when throwing two dice will likely increase the possibility of the teachers solving tasks 19a and 19b. In question 10, you should understand that the information given (the condition) allows you to limit the sample space. So, when understanding the logic given by the condition that the card is blue on one side, you would be in a good position to understand that you need not worry about the card with two green sides. Using an index to mark the three blue sides of the two remaining cards, we can identify the three possible cases,  $(B_1, G)$ ,  $(B_2, B_3)$  and  $(B_3, B_2)$ , and conclude that there is twice the chance ( $2/3$ ) that the card is blue on both sides, compared to being green.

## Conclusion and implications

This paper is part of a larger project aiming at investigating the teaching of probability in the Swedish school-system. In the present analysis, we have tried to profile practicing teachers' knowledge base in probability by structuring our research instrument and analysis based on the principles of CCK and SCK (Ball et al., 2008). We conclude that five particular patterns can be discerned from our results and analysis.

First, it is clear that the teachers are quite comfortable with the basic principle of the classical interpretation of probability. In single random events, they have no problems thinking in terms of favourable outcomes out of the total number of possible outcomes, when modelling the chance of different events.

Second, however, to extend the application of the theoretical probability model, our results indicate that teachers need to develop an understanding of structuring tools that can help them to generate a complete set of possible outcomes when a random experiment consists of several steps or of several random variables.

The third pattern we note in our results is that the teachers find conjunctions and conditional probability particularly difficult. This is of course not at all surprising, these problem-types are known to be difficult, not only for children but also for adults (Shaughnessy, 1992). However, it is important to discover these difficulties, as we believe that a modern school, aiming to educate for a modern society, needs teachers who can think beyond simple random events and who can distinguish between dependent and independent random situations and the combination of events.

Fourth, we distinguish in the teachers' profile that they display a higher level of CCK than of SCK. There are many times when the teachers are able to solve probability tasks and implicitly use concepts and principles relevant to probability theory. However, they have major difficulties in explaining what they are doing, using the technical terms (c.f. Mason, 1998; Ryve et al., 2013) of probability, and discerning and expressing the content of a probability task. Against this background, it is worth reflecting on how good Swedish teachers are at interpreting the probability curriculum and evaluating the content of different tasks when planning instruction (cf. Contreras et al., 2011). We would point out that the survey gives a limited picture of the teachers' complete knowledge base. Above all, the survey provides limited insight into the teachers' ability to gain insight into and update their understanding of a new mathematical domain.

The present survey was biased towards sample space reasoning. Basically, only question 9' and question 13 challenged the participants to reflect on random variation and the experimental interpretation of probability. However, as the fifth pattern, we note that the teachers find these two questions rather problematic, in relation to both their CCK and their SCK. That the teachers find these questions harder confirms the view that the theoretical interpretation of probability is given priority in school and that this priority follows from the theoretical interpretation being in line with a deterministic mind-set (Stohl, 2005) and reversible logic (Batanero & Diaz, 2012). When it comes to experiments, students and teachers are concretely exposed to random variation and situations that are irreversible. Therefore, based on the result of the present study, we encourage future research to investigate in more detail the profile of teachers' knowledge base in relation to the experimental interpretation of probability. Such an investigation would also be in tune with the increased emphasis on experimental probability (Shaughnessy, 2003) and the connection between theoretical and experimental probability (Nilsson, 2009) in probability education.

Finally, we recall that the teachers in the current study are mathematics developers in the Swedish school system, and that the rate of response was only 18%. This means that the knowledge profile expressed by the teachers in the current study is probably located at the top of the population of teachers. Nevertheless, even if the level is slightly higher than in general, we have no reason to suspect that the five identified patterns are not representative of Swedish teachers' knowledge profile of probability, though more research is needed to support or confirm this claim.



## References

- Ball, D. L., Thames, M. H. & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, 59(5), 389–407.
- Batanero, C., Cañizares, M. J. & Godino, J. (2005). Simulation as a tool to train pre-service school teachers. *Proceedings of the first African regional conference of ICMI*. Ciudad del Cabo: ICMI. CD ROM.
- Batanero, C. & Diaz, C. (2012). Training school teachers to teach probability: reflections and challenges. *Chilean Journal of Statistics*, 3(1), 3–13.
- Begg, A. & Edwards, R. (1999). *Teachers' ideas about teaching statistics*. Paper presented at the combined annual meeting of the Australian association for research in education and the New Zealand association for research in education, Melbourne, Australia.
- Bertsekas, D. P. & Tsitsiklis, J. N. (2008). *Introduction to probability* (2 ed.). Belmont: Athena Scientific.
- Borovcnik, M., Bentz, H. J. & Kapadia, R. (1991). A probabilistic perspective. In R. Kapadia & M. Borovcnik (Eds.), *Chance encounters: probability in education* (pp. 27–73). Dordrecht: Kluwer.
- Chernoff, E. & Zazkis, R. (2011). From personal to conventional probabilities: from sample set to sample space. *Educational Studies in Mathematics*, 77(1), 15–33.
- Contreras, M., Batanero, C., Diaz, C. & Fernandes, J. (2011). Prospective teachers' common and specialized knowledge in a probability task. In M. Pytlak, T. Rowland & E. Swoboda (Eds.), *Proceeding of the seventh congress of the European society for research in mathematics education*. Rzeszów: University of Rzeszów. Retrieved from <http://www.erne.unito.it>
- Falk, R. (1988). Conditional probabilities: insights and difficulties. In R. Davidson & J. Swift (Eds.), *Proceedings of the second international conference on teaching statistics*. Victoria: University of Victoria. Retrieved from <http://www.stat.auckland.ac.nz/~iase/publications.php?show=icots2>
- Gal, I. (2005). Towards "probability literacy" for all citizens: building blocks and instructional dilemmas. In G. A. Jones (Ed.), *Exploring probability in schools: challenges for teaching and learning* (pp. 39–64). New York: Springer.
- Haller, S. K. (1997). *Adopting probability curricula: the content and pedagogical content knowledge of middle grades teachers* (Unpublished doctoral dissertation). University of Minnesota.
- Hawkins, A. S. & Kapadia, R. (1984). Children's conceptions of probability: a psychological and pedagogical review. *Educational Studies in Mathematics*, 15(4), 349–377.
- Hill, H. C., Ball, D. L. & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education*, 39(4), 372–400.
- Jones, G. A. (Ed.) (2005). *Exploring probability in school: challenges for teaching and learning*. New York: Springer.

- Jones, G. A., Langrall, C. W. & Mooney, E. S. (2007). Research in probability: responding to classroom realities. In F. K. Lester (Ed.), *The second handbook of research on mathematics teaching and learning* (pp. 909–956). Reston: NCTM.
- Kahneman, D. & Tversky, A. (1972). Subjective probability: a judgment of representativeness. *Cognitive Psychology*, 3(3), 430–454.
- Konold, C. (1989). Informal conceptions of probability. *Cognition and Instruction*, 6(1), 59.
- Krippendorff, K. (2004). *Content analysis: an introduction to its methodology* (2 ed.). Thousand Oaks: Sage Publications.
- Lecoutre, M. P. (1992). Cognitive models and problem spaces in "purely random" situations. *Educational Studies in Mathematics*, 23(6), 557–568.
- Lee, H. & Hollebrands, K. (2008). Preparing to teach mathematics with technology: an integrated approach to developing technological pedagogical content knowledge. *Contemporary Issues in Technology and Teacher Education*, 8(4). doi: <http://www.citejournal.org/vol8/iss4/mathematics/article1.cfm>
- Liu, Y. (2005). *Teachers' understandings of probability and statistical inference and their implications for professional development* (Unpublished doctoral dissertation). Vanderbilt University.
- Mason, J. (1998). Enabling teachers to be real teachers: necessary levels of awareness and structure of attention. *Journal of Mathematics Teacher Education*, 1(3), 243–267.
- Nilsson, P. (2009). Conceptual variation and coordination in probability reasoning. *The Journal of Mathematical Behavior*, 29(4), 247–261.
- Pratt, D. (1998). The construction of meanings in and for a stochastic domain of abstraction (Doctoral thesis). University of London. Retrieved from <http://proxy.lnu.se/login?url=http://search.ebscohost.com/login.aspx?direct=true&db=edsoai&AN=edsoai.730439535&lang=sv&site=eds-live&scope=site>
- Pratt, D. (2000). Making sense of the total of two dice. *Journal for Research in Mathematics Education*, 31(5), 602–625.
- Prodromou, T. (2012). Connecting experimental probability and theoretical probability. *ZDM – The International Journal on Mathematics Education*, 44(7), 855–868.
- Ryve, A., Nilsson, P. & Pettersson, K. (2013). Analyzing effective communication in mathematics group work: the role of visual mediators and technical terms. *Educational Studies in Mathematics*, 82(3), 497–514.
- Shaughnessy, M. (1992). Research in probability and statistics. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 465–494). New York: Macmillan.

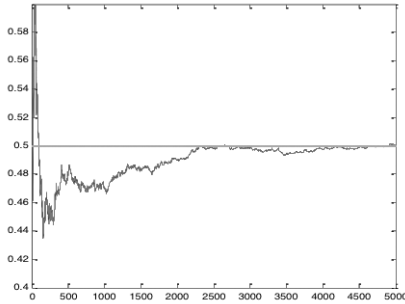
- Shaughnessy, M. (2003). Research on students' understandings of probability. In J. Kilpatrick, W. G. Martin & D. Schifter (Eds.), *A Research companion to Principles and standards for school mathematics* (pp.216–226). Reston: National Council of Teachers of Mathematics.
- Shulman, L. S. (1987). Assessing for teaching: an initiative for the profession. *Phi Delta Kappan*, 69 (1), 38–44.
- Stohl, H. (2005). Probability in teacher education and development. In G. A. Jones (Ed.), *Exploring probability in school: challenges for teaching and learning* (pp.345–366). New York: Springer.
- Stohl, H. & Tarr, J. E. (2002). Developing notions of inference using probability simulation tools. *The Journal of Mathematical Behavior*, 21 (3), 319–337.
- Swedish National Agency for Education. (2012). *Curriculum for the compulsory school system, the pre-school class and the leisure-time centre 2011*. Stockholm: Author.
- Tarr, J. E. (1997). *Using research-based knowledge of students' thinking in conditional probability and independence to inform instruction* (Unpublished doctoral thesis). Illinois State University.
- Van Dooren, W., Verschaffel, L. & Onghena, P. (2002). The impact of preservice teachers' content knowledge on their evaluation of students' strategies for solving arithmetic and algebra word problems. *Journal for Research in Mathematics Education* (5), 319. doi: 10.2307/4149957
- Watson, J. M. (2001). Profiling teachers' competence and confidence to teach particular mathematics topics: the case of chance and data. *Journal of Mathematics Teacher Education*, 4 (4), 305–337.

## Notes

- 1 Please consult (Ball et al., 2008) for a comprehensive description of each of the six domains.
- 2 The Swedish term is *Matematikutvecklare*.

## Appendix: The probability estimations tasks and the conceptual questions of the survey.

- 9' The graph below describes the proportion of Heads when the number of coin-flips are varying from 0–5000. Specify the mathematics you need to explain the graph to a group of students in grade five.



- 10 There are three cards in a box. One card is green on both sides, one card is blue on both sides and one card is green on one side and blue on the other side. You pick one card and look on only one of the sides. You see that the side is blue. What colour is it most likely that it is on the card's other side?
- It is most likely that the side is blue.  
 It is most likely that the side is green.  
 The chance is equal for both colours.
- 10' Describe the mathematical content of the task.
- 11 A box contains three red marbles and four black marbles. You pick a marble at random from the box and see that it is black. Without replacing the marble you pick one more marble.
- a) What is the probability that the marble is black?  
 b) What is the probability that the marble is red?
- 11' Describe the mathematical content of the task.
- 12 A survey of a random sample of boys and girls in a school gave the results:
- |                  | Boys | Girls | Total |
|------------------|------|-------|-------|
| Liking tennis    | 400  | 200   | 600   |
| Disliking tennis | 50   | 50    | 100   |
| Total            | 450  | 250   | 700   |
- a) What is the probability that the student likes tennis?  
 b) What is the probability that the student is a girl and likes tennis?  
 c) The student selected is a girl. What is the probability that she does like tennis?
- 12' Describe the mathematical content of the task.

- 13 You toss a fair coin. Which of the following alternative do you think is most likely?
- To obtain Tail 7 times within the 10 first throws.
  - To obtain Tail 70 times within the 100 first throws.
  - Both of the alternatives are equally likely.
- 13' Describe the mathematical content of the task.
- 14 You throw two regular dice.
- a) Which total/totals are most likely?
  - b) Which total/totals are least likely?
- 14' Describe the mathematical content of the task.
- 15 Which of the following statements do you think is most likely?
- That a woman is a teacher.
  - That a teacher is a woman.
  - Both of the alternatives are equally likely.
- 15' Explain your solution.
- 16 Box A and box B are filled with red and blue marbles and then shaken. You want a blue ball, but only allowed to draw one ball without looking. Which box should you choose?
- Box A, with 6 red and 4 blue marbles.
  - Box B, with 60 red and 40 blue marbles.
  - It does not matter.
- 16' Explain your solution.
- 17 Kalle and Lisa draw one card each with values 1, 2, 3, 4 from an urn. Kalle wins if the sum of the two cards is even and Lisa wins if the sum is odd. Which of the following statements do you think is correct?
- Kalle has the greatest chance to win.
  - Lisa has the greatest chance to win.
  - Both have an equal chance to win.
- 17' Explain your solution.
- 18 During one day everyone who takes the car to work are recorded. What is most likely?
- That a man has an accident.
  - That a man who is younger then 21 years old has an accident.
- 18' Explain your solution.
- 19 You are rolling two fair six-sided dice.
- a) What is the probability that the die that displays the lowest value displays one?
  - b) What is the probability that the die that displays the lowest value displays six?
- 19' Explain your solution.

### Per Nilsson

Per Nilsson is professor in mathematics education at the School of Science and Technology, Örebro University and associated professor at Linnaeus University. His research interests concern classrooms communication in mathematics, the nature of mathematical reasoning and knowledge for teaching probability.

per.g.nilsson@lnu.se

### Torsten Lindström

Torsten Lindström is professor in mathematics at the Department of Mathematics, Linnaeus University. His research interests concern dynamical systems with applications to biology. He has a teacher's exam in mathematics, physics, and chemistry.

torsten.lindstrom@lnu.se