A modelling approach to probability – analysing students' conceptual structures

THEODOSIA PRODROMOU

This research study investigates how middle school students use probability to model random behaviour in real-world contexts and how they articulated fundamental probabilistic concepts to show aspects of the mental models that they generated. This article is concerned with the conceptual structures that the students develop when exploring computer-based simulations. The results suggest that the students relied on their experience to provide a context reality from which to construct their mental model of the situation, from which they then defined the probability model. While the students attempted to build mental models, they checked the adequacy of the mapping between their probability models and reality by interrogating the context of their personal experiences. The results also suggest that the way students express this relationship between signal and noise seems to have a particular importance in building comprehensive models that link observed data to modelling distributions.

Statistics and probability are connected, since statistical inference is based on probability to draw conclusions about uncertain situations. Nonetheless, probability, as an idea, is difficult to pin down and is still subject to ongoing controversies about its different conceptions.

This paper is concerned with the modelling perspective of probability (Chaput, Girard & Henry, 2011), in which probability is considered "as a theoretical value of the degree of confidence that one can give to a random outcome" (p. 86). This modelling approach is based on the synthesis of the two primary conceptions of probability, the *classical* conception and the *frequentist* conception. In the classical conception, which is based on combinatorial calculus, probability is a fraction of which the "numerator is the number of chances whereby an event may happen and the denominator is the number of all chances whereby it may either happen or fail" (deMoivre, 1718/1967, p. 1). This conception introduces an

Theodosia Prodromou,
University of New England, Australia

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a priori approach to probability in which probabilities can be calculated before any physical experiment is performed.

The classical approach offers powerful theoretical results when using correct and accurately constructed mathematical models. These results are challenged by the empirical results obtained from an experimental reality.

The frequentist conception is based on observations of relative frequencies of an event associated with a random experiment that is repeated a sufficiently large number of times under the same conditions. In this view, experimental probability is estimated as a limit towards which the relative frequencies tend when stabilizing (von Mises, 1928). The idea of stabilization is based on the empirical laws of large numbers. When modelling (i.e., associating a model with experimental data), the Law of large numbers is very important. The use of computers allows students to generate large statistical series, and hence helps them to understand this law.

Chaput, Girard and Henry (2008) proposed three steps of modelling in teaching probability and statistics. The first step is the observation of the concrete situation and the description of it in usual terms. Any concrete situation is usually described using scientific knowledge based on pre-designed general models. Students are expected to describe the pre-designed general models in a simplified system presented in relation to everyday life and described using everyday language. Then one may proceed by choosing the appropriate characteristic properties of the real objects in order to design the related pseudo-concrete model. Then, the work hypotheses are set out to describe the situation that is modelled. For example, in an urn model, balls are supposed to have the same probability of being drawn. The experimental process also requires acting on the reality in order to study the invariants of the situation.

According to Chaput et al. (2008), the second step involves the mathematisation process, during which the students represent the model in a suitable mathematical symbolic system. The mathematisation process leads to the formalisation of the model and the selection of the right tools to solve the abstract mathematical problem.

Chaput et al. (2008) describe the validation of a model as the third step of modelling in probability and statistics. The validation involves the translation of the mathematical results in regard to the appropriate pseudo-concrete model, and juxtaposition of the model hypotheses to the answers to the initial problem derived from mathematical results. At the end, according to the authors, the validity of the answers has to be estimated.

The work of of Chaput et al. (2008, 2011) is interesting and provocative, but it only investigated the modelling process of a random situation for teachers' training. They did not provide any evidence of students making models.

Probabilistic models can be built into the computer software and used to generate simulations to be investigated by students, who might view the real world situation through a probabilistic model, instead of seeing the probabilistic model through the data (Pratt, Davies & Connor, 2011). Presenting the modelling approach using powerful technologies changes the way teachers teach probability and students learn stochastics (Chance, Ben-Zvi, Garfield & Medina, 2007). Eichler and Vogel (2012) researched mental models in students of grade 4 and grade 6 to explain the students' reasoning in situations that involve data and necessitate modelling with frequentist probability. Eichler and Vogel showed that, when students dealt with elementary situations of uncertainty they expressed a huge variability in actions and responses depending on the situation's complexity and representation. The distinction of the aspects' data, objects (or data generation) and mental simulation helps to facilitate the analysis of students' rationales and identify students' difficulties in mental modelling.

However, we have limited empirical knowledge about how students develop mental models when using the modelling process to model everyday situations. For example, Nilsson (2009) provided some empirical data about how Swedish students (12–13 years old) developed mental models about compound phenomena and how they developed conceptual structures of variation and coordination among theoretical and experimental interpretations of probability. Nonetheless, we still need to research further the conceptual models and the conceptual structures that students develop about stochastic concepts. On account of that, the aim of the current paper is to investigate what conceptual structures students may (or can) develop when modelling computer-simulations of real-world phenomenon. This overall aim is addressed through the following two research questions: (1) How do middle school students use probability to model random behaviour in real-world contexts? (2) What connections do they build among fundamental probabilistic concepts when treating probability as a modelling tool?

Model-building

A modelling approach in the teaching of probability and statistics

The modelling approach reinforces use of probabilistic models that are formalised in a symbolic system and developed to represent concrete situations or problems arising from reality. The probabilistic models incorporate uncertainty, or random error in a formalized way (Borovcnik, Bentz & Kapadia, 1991). These probabilistic models, according to their inherent rules, are expected to simulate the behaviour of random phenomena and also predict specific outcomes of random phenomena (Borovcnik et al., 1991; Pratt, 2011).

From a classical, Laplace-oriented, conception of probability, the construction of a probability model begins with the investigation of the sample space that is the set of all possible outcomes of the experiment under study. When the sample space is constructed a probability is assigned to each of the possible outcomes. This structural approach can embrace these probabilities as numerical values or in the form of density functions such as the normal distribution. In this framework, a probability distribution of some discernible characteristics has the status of a model of the data that describes what one might see if many samples were collected from a population, enabling us to compare data from a real observation of this population with a theoretical distribution. The probabilities assigned to the set of all possible outcomes can be validated later.

In statistics and probability teaching, modelling is performed by the creation of sample simulations based on probabilistic models of populations that can be built into the computer software and used to generate simulations that can be investigated by students (Chaput et al., 2011; Pratt et al., 2011). These sample simulations are "on the determination of various parameters through a frequentist approach, or the testing of theoretical models by comparing their behavior with the real observed data" (Chaput et al., 2011, p. 92). As Chaput and colleagues pointed out, the use of computer simulations requires minimal knowledge of probability or any knowledge of a theoretical model to achieve its comparison with the real observed data.

The importance placed on formal symbolic systems in this description of the modelling approach coincides with the modelling used in various simulation activities. Such modelling is present in the two experiments of this study included essentially no formal symbolic work at all.

A modelling approach and statistics

In statistics teaching, the modelling perspective is in accordance with the *modelling* process of *contemporary* statistical thinking (Wild & Pfannkuch, 1999) that allows the application of theoretical results when making statistical inferences regarding particular observed sample statistics or data analysis.

Statistical practice often involves the use of chance models that describe the variability in observed data, either for comparison with experimental data or to simulate data to make an estimate about a particular population from data provided by a simulated sample (Garfield & Ben-Zvi, 2008). Statistical thinking involves constructing models and using them to study, model, and predict the behaviour of particular aspects of the world. Part of developing ideas of statistical modelling is to select appropriate models (Wild & Pfannkuch, 1999) and use these models when considering drawing statistical inferences from data. Inferences are made when using a model to compare a model with the experimental results, producing a *p*-value.

Lightner (1991, p. 628) argues that "statisticians must consider probabilistic models to infer properties from observed data". Statistical inference is based on probability and "whereas variability in data can be perceived directly, chance models can be perceived only after we have constructed them in our minds" (Cobb & Moore, 1997, p. 820). From this point of view, the development of understanding, builds up on mental models (Bartholomew, 1995) informed by "expert knowledge".

The acknowledgment that probabilistic models can be used to describe and predict behaviour adds to the existing difficulty of teaching inferential reasoning. The integration of statistical data analysis with theoretical probabilistic distributions and the assumptions underlying those models present a real conundrum in teaching. Research on students' informal and formal inferential reasoning would suggest that there are huge gaps in current knowledge about how best to enable learners to make the connection between probability and statistical inference.

Pratt (2011) advocates that a modelling approach to probability would fit more comfortably with the use of statistics in disciplines other than mathematics and may enable to students to connect probability to statistics. In particular, Pratt (p. 9) states that:

> presenting probability in the curriculum as a modelling tool will inevitably bring with it certain new challenges in how children learn but these difficulties can be embraced as essential steps to overcome in the development of students who will engage fully in modern society.

Pratt provides Prodromou's *Basketball* simulation (2008, 2012a) as an example of using the probability as a modelling tool. In this simulation the shots of a basketball are modelled by a normal distribution and the variations that are coming from measurement errors or random causes, are spread around a central value. The Basketball simulation used a simple model of a basketball player shooting baskets. Two mechanisms were used by the Basketball simulation to generate the trajectories of the balls following Newton's Laws of motion; one was fully determinist, the other was probabilistic, incorporating variation in variables that determined the trajectories of the balls. The software provided students with the opportunity to assess the data-centric perspective on distribution that was graphically presented as a set of data about the trajectories and success of shots at the basket. Students also had access to the graphical representation of the modelling distribution, which showed a distribution of values from which the computer would randomly choose. The students could set the modelling distribution by either adjusting interface controls (the arrows or the handle on the slider, see left in figure 1), or by directly entering their own value for each outcome interval (right in figure 1). In both cases, the simulation allowed students to transform the modelling distribution (which generates the output data) directly and thus they have indirect control over the data-centric distribution.

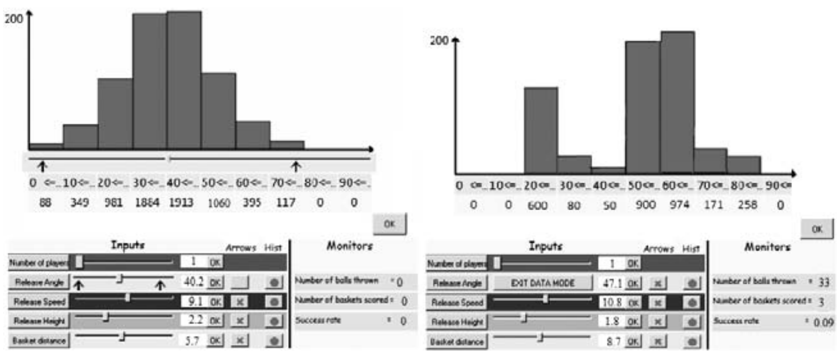

Figure 1. *The students altered the modelling perspective directly by moving either the arrows or the handle on the slider (left), or by setting numerical values associated with each possible outcome for a given variable (right)*

Prodromou's research investigated whether and how 15-year-old students built connections between the data-centric perspective on distribution and the modelling perspective on distribution. When Prodromou (2008, 2012a) and Prodromou and Pratt (2006) termed the "modelling perspective on distribution," she simply meant a set of values of basketball shootings modelled by a normal distribution, provided that the variations spread around a central value and are subject to random causes. Ultimately, in the design of the Basketball simulation, the normal curve is the model used to generate variation in the outcomes.

Prodromou's research (2008, 2012a, 2012b) showed that 15-year old students constructed three interpretations:

- *General Intention* (I_G): The modelling distribution (MD) was perceived as the intended outcome and the data distribution (DD) as the actual outcome, suggesting a connection being made in which the modelling distribution in some sense generates the data.
- *Stochastic intention* (I_{ST}): General intention becomes stochastic intention (I_{ST}) when randomness becomes a part of the interpretation of the student.
- *Target*: The modelling distribution (MD) was perceived as the target (T) to which the data distribution (DD) is directed.

These interpretations are mental models constructed by the students when trying to build connections between the data-centric and modelling perspectives, with the intentionality models dependent on appreciation of quasi-causal probability, and the target model dependent on recognition of quasi-causal emergence (figure 2).

When the students tried to build connections between the data-centric and modelling perspectives, they constructed these interpretations, which can be understood as mental models of the sort described by Johnson-Laird (1983).

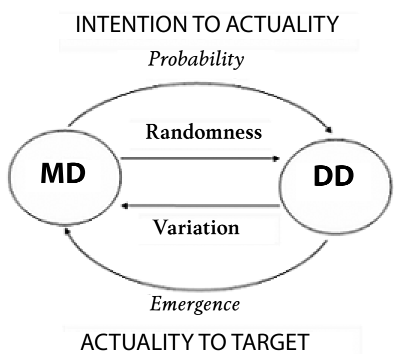

Figure 2. *A tentative model for the connection of the data-centric and modelling perspectives on distribution*

Conceptual structure: Johnson-Laird's mental models

To discuss the conceptual structures used by the students, we will use the theory of Mental Models first discussed by Johnson-Laird and then elaborated on by Schnotz (1994), Kintsch (1998), and Schnotz and Bannert (1999). Mental models are defined as representations of real situations (Kintsch, 1998). Johnson-Laird stated: "A mental model plays a direct representational role since it is analogous to the structure of the corresponding state of affairs in the world – as we perceive or conceive it" (1983, p. 156). Therefore, the mental modelling of a real situation acknowledges the physical objects of the real situation that is to be simulated, its constituent components, and the mechanisms governing the behaviour of these objects, the characteristics, and their relationships.

According to the information-processing model theory (Schnotz & Bannert, 1999), mental models are constructed in regard to a task and its requirements.

When dealing with requirements of a specific situation, the modeller builds a mental model that simulates certain aspects of a situation (Seel, 2001). The function of mental models provides us with answers via mental simulation processes (e.g., identifying possible outcomes of an experiment by constructing the event space of a coin flip). Mental simulations require sense-making of the simulation process.

Mental models could not be considered as fixed structures of memory that humans recall (Baguley & Payne, 1999). The mental models are considered situational constructions made to help process a situation. The elements of a mental model may be altered, improved, or modified while a modeller attempts to construct any representations of the real situation. Ultimately, mental models exhibit interpersonal difference in the way the models are structured and in their constituent functional elements (Schnotz, 1994). Since mental models are internal quasi-objects (Schnotz & Bannert, 1999), inferred from observable information that portrays the mental modelling involved in a particular situation or process, in this study the observable information involves students' articulations after students' use and direct manipulation of TinkerPlots (Konold & Miller, 2011) tools that conventionally would be regarded as abstract but which are represented as if concrete on the computer's screen.

TinkerPlots

Recent developments in pioneering software used in statistics education, such as TinkerPlots, are ideally suited for supporting the modelling approach for students by providing them with modelling tools based on probability (i.e., the user can set the probability of some event) that can be used to construct models used by computer-based simulations. TinkerPlots software is the most recent version of TinkerPlots software that has been designed for use by students in grades 4–8. TinkerPlots provides students with tools to develop understandings of data, statistical concepts, and probability by designing and running probability simulations, using "the sampler" (figure 3) to create modelling distributions. For example, modelling distributions are created by defining the sizes of the sectors of a spinner, or by determining the heights of the bars of a histogram that describes the population in a mixer, or by drawing a curve to define a probability density function. All these options of the sampler engine inevitably bring with them new challenges in how students combine these tools when use probability as a modelling tool to build a model that simulates of a real-world phenomenon. Additionally, this will consequently bring with it new challenges in how children learn and give rise to research questions about the conceptual development of students who will engage in constructing chance models.

It is of paramount importance to understand how students construct mental models while designing, running probability simulations, interpreting outcomes, and drawing connections amongst them when making informal statistical inferences, as well as understanding the challenges students might encounter in a modelling-approach pedagogy.

The next two sections of this paper will describe how middle school students use probability to model random behaviour in two contexts, namely, the Virtual school and the Facebook task. The studies of students' engagement with designing the Virtual school simulation and the Facebook task simulation show how students construct mental models and the connections they build among fundamental probabilistic concepts when treating probability as a modelling tool.

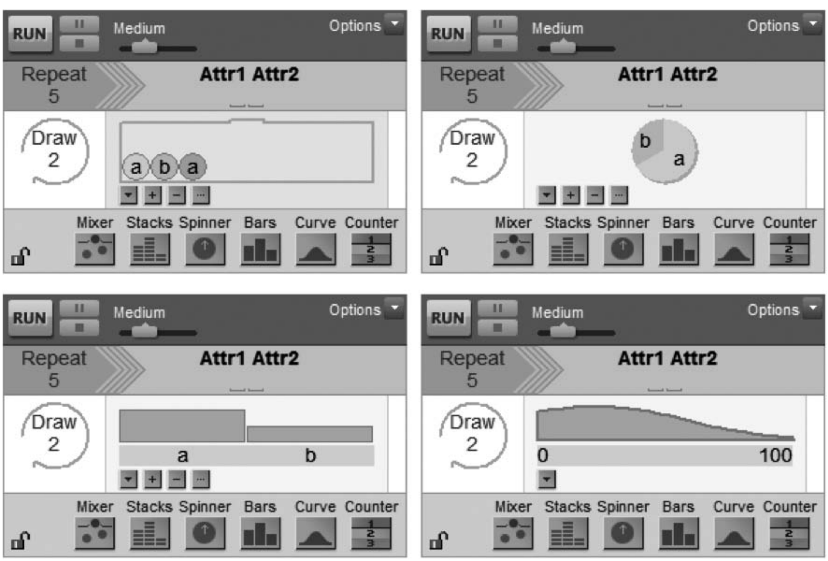

Figure 3. *Examples of samplers in TinkerPlots™ 2*

First study: Virtual school simulation

The Virtual school simulation study investigated how 15 year-old students built connections between probability, statistics, and simulated phenomena within a TinkerPlots computer-based environment.[1]

Students were asked to use TinkerPlots features to build a "Data factory" (see figure 4) to generate a number of individual "virtual students" to populate a "virtual secondary school" with each virtual student created using student-defined probability distributions for each of the different variables (e.g. gender, name and height).

TinkerPlots provided students with tools that enable students to use probability as a modelling tool using the sampler, which is essentially a non-conventional form of probability distribution. The students' articulations show variability in their actions depending on how they used a combination of tools to model the situation under study. The analysis of the students' utterances and actions shows that their construction of the sample was based on their personal understanding of the situation when activating experiences from daily life.

The students began by setting up the sampler as a spinner, which allowed them to visually assign different angles for the sectors that correspond to the probability of selecting a gender for each virtual student (figure 4).

We start by discussing how the students, George and Rafael, chose unequal angles for the sectors, thus giving unequal probabilities of getting male (m) students and female (f) students; in this case there was a much larger probability of getting male students than females. The students explained their choice:

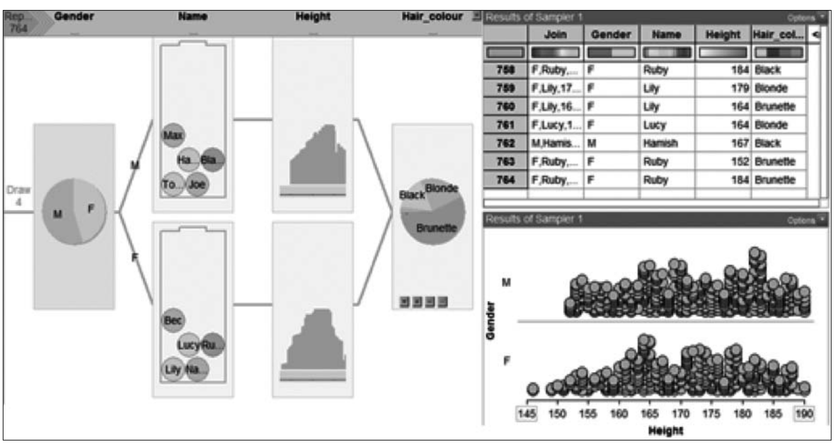

Figure 4. *Data factory that simulates a "virtual school"*

Ge: in schools there are generally more males.

Ra: Umm, generally the schools that I've been in there have been more males. This school for example has a lot more males than females.

They then created two different "mixers", one for the names of each gender, and placed names in each mixer, from which one name was chosen at random for each virtual student of the appropriate gender.

The students decided to introduce another attribute for the virtual students, height, so they set up two samplers, one each for boys' and girls' heights, as probability density functions by drawing two curves, as shown in figure 4. Rafael stated, "This is getting complex. For a student, we have a choice of gender, name, and height. We are going to have many data for each student". George agreed that although "each student is one, varied information is provided for a student".

Rafael and George chose the characteristic aspects of the "virtual school" in order to develop a pseudo-concrete model that was supposed to represent the essential features of a school. Hypotheses were set out to describe the situation: For instance, a virtual student assumed to consist of a cluster of pieces of data. Twenty minutes into the activity, Raphael and George decided to have the data factory generate 675 virtual students. The simulation generated sample data for the virtual school. The students compared the distributions of heights of the virtual students with the curves drawn in the sampler (see figure 5).

Ra: [pointing to the distribution on the bottom right of figure 5a] We had it rising there, then we had a drop, then we had a big rise there which is why I'm guessing all these came from before it dropped down there.

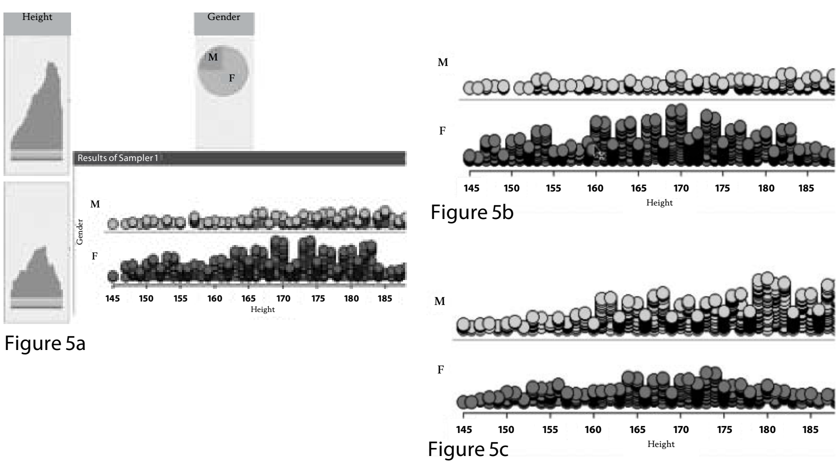

Figure 5. *Distributions created in the sampler and distributions of height*

Ge: Well the females theory of the rise [pointing to the graph on the bottom left of figure 5a], which is here [pointing to 150–155 of the bottom left graph], and that goes down a bit before continuing, so it would be going, down, continuing up [170–175], around here [180–183], before decreasing again. With the males, we thought the height was a lot higher that was there [pointing to the graph on top left].

Ra: There, there's a lot less males though, so even, the end results aren't as packed there, the more spread out, or they seem to be spread out as much as the females but there's no piles. Like you can't see them as high as well as you can see the females' height increases but you can actually see the increase there [175–185 right bottom].

Ge: There is an increase but it's not as prominent as the female increase.

Rafael and George decided to make 1000 virtual students, because they believed that the graphs would show the distribution of the height of males clearer. They observed:

Ra: But now you can see the increase in the males a lot better (pointing to the top figure 5b). Like you can see the towers higher, stacked higher than they were before. It's clear see that there is increase, but with the females is you can still see the increase, the huge increase the females have had as well.

Rafael paid attention to slices of prominent features of the distributions such as higher areas of accumulated data of the distribution. George attempted to equalise the number of male students with female students. After generating a new set of virtual students with a 50:50 ratio of male to female students, the boys observed the new distribution (figure 5c):

Ra: Yeah, it's a lot clearer to see the increase in the males [pointing to the distribution of male height], now because it is a probability it's not going to look exactly like that [pointing to the distribution of male height they created in the sampler]. There's going to be exemptions. But you can see, you can see the overall that it's increasing. Getting higher here before dropping down again [pointing to the distribution of male height], which is what our graph showed [pointing to the distribution of male height they created in the sampler]. The females you could tell fairly compact in the middle ... and there's not as much any to the side which is show here [pointing to the distribution of female height].

Ge: Exemptions? [2]

Ra: I suppose there is always gonna be exceptions to the graph. They're not always gonna be exactly as we plan out. Because this is based on probability and probability, just because we see that [pointing to the distributions they created in the sampler], like it doesn't mean that it will follow that (100 %).

The above excerpt shows how Rafael's attention was, at this point, focused on the way the shape of the distributions of heights was changing compared to the distributions they created in the sampler. The extended discussion of the boys showed that Rafael could not recognize the absolute resemblance of the data distribution of heights with the distribution they created in the sampler due to uncertainty caused by probability.

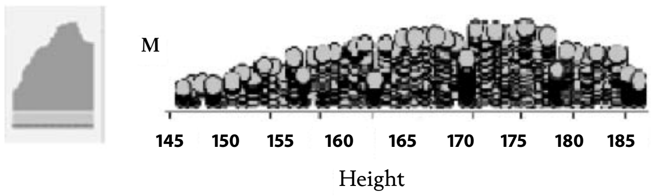

Figure 6. *The distribution of male height they created in the sampler and the distribution of male height*

George suggested excluding female students from their model and creating a virtual school with 500 male students. After creating their all-male school, they observed:

Ra: Well, it's steadily increasing with slight jumps. Like it lows there and jumps up a bit, which guessing that 140, 155 region there [pointing to the left graph in figure 8].

Ra: It's a 163 [left graph9. And that 163 just there [right graph].

Ge: It's going up and just going down a little bit 8right graph].

Ra: That's probably where that is come from 160. 164 here has just a low. But then it continues. Then it jumps back up and keeps going before dropping down a bit again here [pointing to both graphs at the same time].

Ge: Just there that one, for reason it's just steadily dropping [pointing between 180–185 of right graph]

Ra: ... except there [180–185]. That seems to have a sharp drop which I'm guessing is just off one of these areas here. Where it just seems to drop down [both graphs].

Re: Do you believe that the final graph resembles of what you created in the sampler?

Ra: Yeah, fairly accurately.

Rafael was able to conclude that the two distributions eventually resembled each other.

When engaged in a modelling-approach activity, the students, in an intermediate step in their reasoning, developed a mental model in which a holistic entity is perceived of as a cluster of independent pieces of data.

After building the all-male school, they wanted to add another feature, and they chose to set a sampler as a probability density function by drawing a curve based on the users' personal experiences, they drew connections between probability, statistics, and the simulated phenomenon. When the boys engaged in drawing a curve that illustrated a probability density function of heights, they first acknowledged a central area where the most common heights were gathered and then talked about the variations of heights spread around that central area. These students appeared to recognise that the description of the notion of probabilistic models and the nature of a reasonable approximation of real or simulated phenomena lies in the relationship between a central area of values and variations spread around that central value that are coming from random causes.

The students seemed to appreciate that the co-ordination of a central area of values and the variations spread around that central value requires a fairly good knowledge of how objects and physical processes work, so that the students could design the TinkerPlots Data Factory in such a way that it would generate sample data that would resemble the real-world situations being modelled. A good co-ordination of a central area of values and variations spread around that central value seem to be essential for mental modelling of statistical knowledge and experience.

The students actively constructed mental models in the service of understanding and explaining their experience. They seemed to realise the importance of the choices they made as modellers, thus they focused their attention on creating in the sampler modelling distributions of the attributes, such as height, gender and hair colours.

Results from the students' activity showed that the positioning of probability as a modelling tool used to build models in computer-based simulations brings, as Pratt (2011) suggested, distribution of data into the foreground, not as a pre-determined entity, but as a non-fixed entity, open to debate.

Raphael and George compared the data distribution of the simulated heights to the density curve they drew in the sampler and articulated situated heuristics, for example

> there is always gonna be exceptions to the graph. They're not always gonna be exactly as we plan out. Because this is based on probability and probability … just because we see that [pointing to the distributions they created in the sampler] … like it doesn't mean that it will follow that 100 %.

This situated abstraction can be interpreted as a construction of a relatively naïve conception of the use of probability as a modelling tool. It portrays one of the general properties of mental models: Mental models are often constructed spontaneously to understand a situation and to make predictions via mental simulation of these models. One of the students in the study could not recognize or accept the absolute resemblance of the distributions of heights to the curves students created in the sampler due to the existence of many variables that caused sampling fluctuations in the repetition of running the simulation. When they excluded one variable from the attributes of the model, the students were better able to identify the resemblance of the distribution of the generated data with what they designed in the sampler and progressively construct the notions of sampling fluctuations and model.

This empirical evidence shows that students seemed to be checking the adequacy of the mapping between the model and reality when comparing the data distribution of the simulated heights to the distribution they drew in the sample. This mapping between the model and reality feed back into the mental model and to actively contributing to the development of or aspects of the mental models.

Second study: Facebook task simulation

This was another task in which the students relied on their everyday experience in the world in order to build their models. In this case they were interested in the hours they and/or their friends spent on Facebook and the impact of hours spent on Facebook on simulated students' school performance/grades. One pair of students – Chris (C) and John (J) – decided that all students spent an equal amount of time, 5 hours, and they drew curves in the interface to define the probability density functions that would be used to generate the simulated data.

The Facebook task focuses on the interaction between at least two variables: hours they and/or their friends spent on Facebook and the impact of hours spent on Facebook on simulated students' school performance/grades.

They had generated 1000 virtual students and they compared the distributions created in the sampler to the distributions of the generated data:

C: Well, this is a lot more accurate [pointing to the graphs on the right, see figure 7]. Because it shows like, out of the thousand like samples, rather than that's just like a rough drawing [pointing to the graph on the left].

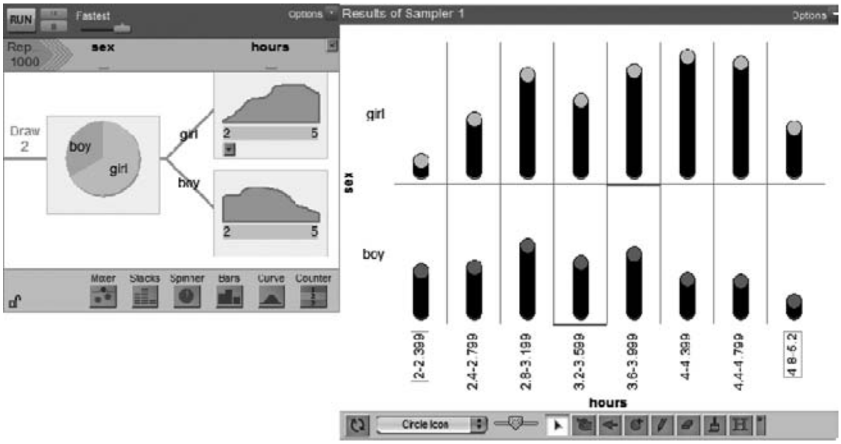

Figure 7. *Data factory that simulates the "Facebook task" (left) and the distribution of the generated hours for boys and girls*

J: Yeah, it's just a rough copy ... It's to show us, it's to get the real data. To get the more accurate data in order to put it into the plot form [pointing to the graphs on the right].

The students connected the density function and the distribution of the generated data, thus coping with aspects of connecting theoretical and empirically generated data. The density function and the generated data are the observable information from which the mental models that the students constructed can be inferred. The students used this observable information to perceive the data distribution as the actual outcome, suggesting a connection being made in which the modelling distribution in some sense generates the data. They suggested that the data, time spent on Facebook, were generated from the model that they built in TinkerPlots but they did not find it necessary to use information from the real data to feed in back into the model "interrogating" the context of real data. It seemed that the function of their mental models allowed for deriving answers via mental simulation by anticipating possible results.

While the students were working on observing representations of the situation, it appeared that there was no need to change, enrich or modify the elements of the mental models that they built.

The students spent time discussing how to incorporate in their model a representation or representations that would show the impact of Facebook on simulated students' school performance. The students relied on their experience to provide a context reality from which to construct their mental model of the situation, from which they then defined the

TinkerPlots sampler. They reflected on the impact of hours spent on FB on their marks and on the marks of their friends. For instance, Chris commented that he does not spend any time on Facebook, but he does spend time on sport activities that influence his school performance. George mentioned:

> Well I know that a lot of boys do use FB, but I think there's more girls than boys who use FB more hours. But girls' grades are better than boys, so the hours they spent on FB do not have such a negative impact on girls' grades.

Both students mentioned attributes of the given problem, which entail the results of possible mental simulations concerning the FB task. However, both students were not able to adequately simulate the given situation mentally. The identification of the appropriate attributes of a situation under study that essentially influence the data generations seems to be an integral component for mental modelling.

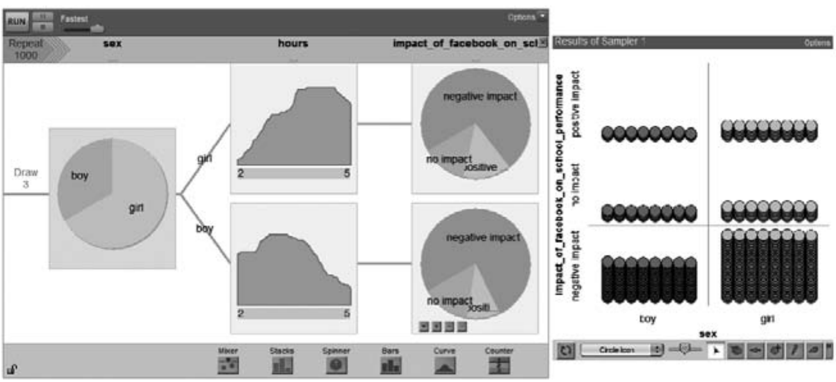

Figure 8. *Data factory that simulates the "Facebook task" (left) and graphs of the impact of Facebook on boys and girls*

The students added a spinner-type sampler by defining the sizes of its sectors: positive impact, no impact, and negative impact (figure 8). Although the properties of these on-screen objects, the spinners and mixers of TinkerPlots, are unknown to the students, it is possible to bring to mind the main influences of the data generation by activating experiences from daily life, especially other factors that have a causal impact on students' performance. They argued that in some cases, "Facebook would not have much of an impact, because if a student is not doing any study, s/he can't really go down because s/he haven't been really studying in the first place." They also articulated:

J: The boys spent less time on FB ... umm so they do have an extra 3 hours of time. 3, 4 hours, no 2 or 3 hours of time. They'd probably use that playing video games ... or you know training for their sport or something like that ... but it would have a positive impact on their school sporting if that counts too.

While the students attempted to build mental models, they checked the adequacy of the mapping between their probability models and reality by interrogating the context of their personal experiences. What the students attempted to do was to seek and get some statistical information from the context of their real experiences and compare the generated graphs to this context. When the students observed the empirically generated graphs (figure 8):

C: Yeah, It's having more of a negative impact on girls than it is having on boys ... I don't think it is very realistic actually because boys who spent time on FB wouldn't be getting study done either so they'd, they would have negative impact on school too.

J: It is real yeah, realistic in a way that, it does have more of a negative effect than any other effect.

The students disagreed about whether the Facebook task model was realistic or not. On the one hand, Chris believed that the model was not realistic because the impact of FB on girls' school performance was more obvious than the impact that FB had on boys' performance. On the other hand, John observed the negative impact that FB on students' school

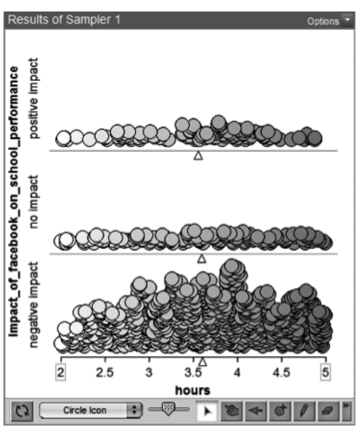

Figure 9. *The distribution of the impact of FB on school performance versus hours spent*

performance. It seems that the mental models that Chris and John constructed showed interpersonal differences in the way that the two boys interpreted the empirically generated graphs.

Chris checked the adequacy of the model by relying on his personal experience observing the graphs of the generated data. He compared the distributions of the empirically generated data (figure 9) from the model to the probability models in the sampler. He seemed to work with representations of the situation in order to construct his mental model that will simulate the situation under study.

G: ... it peaks up here and peaks in there [pointing to the distribution of data].

C: It doesn't, it kinda peaks in there.

Discussion

Inspired by Cobb and Moore (1997), who argued that "whereas variability in data can be perceived directly, chance models can be perceived only after we have constructed them in our own minds" (p. 820), in this paper we can see how the modelling approach leads to the development of cognitive/conceptual structures, the mental models. The mental model (abstract) is instantiated as an on-screen (Papert, 1996), quasi-concrete (Turkle & Papert, 1991) object, the sampler, that can be manipulated and experienced as concrete and tangible (Turkle & Papert, 1991).

In both studies, when students were introduced to probability through activities designed in accordance with the modelling approach, they built mental models by integrating statistical knowledge from data (real or generated) with information from the context of everyday life to continually check the adequacy of the model and its mappings to reality. Students interrogated the context of real life and sought information in the form of data as they observed the empirically generated distributions.

The generated data presented in graphical form, provided students with insights that fed into the modelling approach to probability, which generated the data. Moreover, insights gained from the context of real life and students' personal experiences (subjective probability) informed the designing of the modelling approach to probability in the Sampler (for example density curves drawn in the Sampler), and, in turn, generated statistical knowledge that provided users with feedback to improve the model. When students were engaged with interventions based on the modelling approach to probability, for example manipulating TinkerPlots tools that conventionally would be regarded as abstract constructs. Hence, they created mental models out of an amalgam of the context reality, expert knowledge, and their own subjective beliefs, from which they then designed TinkerPlots models that generated sample data that in turn resembled as much as possible the real-world phenomena it was intended to model.

The modellers constructed the on-screen models of the phenomena that in turn fed in to their mental models. We suggest that students' engagement in mentally modelling the given phenomena show that they were predominantly able to describe the factors that influenced the data generation and the properties of these factors, since they were able to reconstruct the phenomena.

From the first study, we found that a good co-ordination of signal and noise requires a fairly good knowledge of the phenomenon at hand, so that modellers can make "optimal" choices when designing a model that will generate sample data that resembles as much as possible the real-world phenomena it is intended to model. The way students express this relationship between signal and noise shows the adequacy of the mental model they hold when dealing with the physical structural situations represented in the tasks (Batanero, Henry & Parzysz, 2005). The first activity dealing with the relation of signal and noise involves elements of abstraction and simplification of reality with respect to the real situation studied. Such activities could enhance the comprehension of students' mental modelling and might help students move to the second modelling stage (Batanero et al., 2005) so they could develop chance models that are represented in a symbolic system suitable for probability calculus. Depending on the problem and the education and experience of the thinker, the notions of signal and noise can be part of the way we think about the world and thus be integral part of our mental models of the context reality.

Computer simulations such as TinkerPlots must lead researchers to consideration of students' construction of mental models when associating a probability distribution with a random experiment, with the important part played by building links between variation, theoretical models, simulations, and probability. These are the areas where sophisticated understanding and application of chance can be useful to students in decision making and modelling when modelling everyday phenomena.

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Notes

1 Aspects of the Basketball simulation and the Virtual school study reported here have also been discussed in Prodromou (in press).

2 When Rafael was asked about the word ”exemptions”, he shifted to using the word ”exceptions.”

Theodosia Prodromou

Dr Theodosia Prodromou is a mathematician, statistician and educator. She lectures mathematics education at the University of New England (UNE) in New South Wales, Australia. Her research interests are focused on exploring the relationship between technology and mathematical thinking – especially statistical thinking and probabilistic thinking. She is very interested in statistics education.

theodosia.prodromou@une.edu.au