Arithmetic textbooks in Estonia, Iceland and Norway

Similarities and differences during the nineteenth century

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This paper identifies similarities and eventual differences in the development of public mathematics education in the nineteenth century in three Northern-European countries: Estonia, Iceland and Norway. Special attention is paid to how these developments were reflected in the first arithmetic textbooks written in the vernacular in these countries. By the end of the century, all three countries had taken serious steps to develop public education, and arithmetic textbooks, meant for self-instruction or for use in elementary schools, had been published. The content and style of presentation of these textbooks in Estonia, Iceland and Norway are described and compared in the paper, revealing their roots in Northern-European culture: Lutheran Protestantism, Enlightenment and pedagogical currents initiated by Comenius, Pestalozzi and Spencer, emphasizing meaningful learning. Their educational aims were important driving forces in growing national movements in the respective countries by contributing to capability to manage own resources and use of own vernacular, resulting in increased self-esteem.

Three Northern-European countries – Estonia, Iceland and Norway – were all dependent of other neighbouring nations in the early modern era. Iceland belonged to the Danish realm, as did Norway until 1814 when it submitted to the Swedish king. Estonia belonged to the Swedish crown until 1712 when it was merged to Russia. The three countries had also Northern-European cultural currents in common; the Lutheran Protestantism, adopted in the sixteenth century, with its general approach to suppose the population to be literate in its local vernacular; and a German version of Enlightenment, in the late eighteenth and early nineteenth century. The need for general public education became commonly recognised.

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In none of those societies was there any schooling for ordinary people in the early 1800s. By the end of the century all three countries had taken serious steps to develop public education. The first mathematics textbooks, written in the vernacular and meant for self-instruction or for use in elementary schools, were published shortly before or during the nineteenth century. These very first textbooks may be taken as marking the beginning of general mathematics education. They defined the content of mathematics and methods of its teaching for the common people at elementary school level.

The aim of this article is to compare the history of mathematics education in the three countries: with special reference to arithmetic textbooks. By doing a comparison between Estonia, Iceland and Norway, new knowledge is expressed in terms of the similarities and eventual differences found.

The paper is drawn upon several different sources on the history of arithmetic teaching. Concerning the global history, Keitel (2006) has written about perceptions of mathematics and mathematics education in the course of history, Howson (1995) has written on comparative studies on textbooks, Griffiths and Howson (1974) about mathematics society and curricula and Bjarnadóttir (2013) has written the history of arithmetic teaching. Andersen (1914), Christensen (1895), Piene (1937) and Solvang (2001) have contributed to the history of arithmetic teaching in Norway and Christiansen (2009, 2010) has written extensively about the contributions of Holmboe to mathematics education in Norwegian upper secondary school level. Hansen (2002) has written about the history of mathematics education in Denmark and Pestalozzian influence in the Danish school system. Andresen (1985) and Prinits (1992) have contributed to the history of mathematics education in Estonia and Bjarnadóttir (2006, 2009, 2012) to the history of eighteenth and nineteenth century arithmetic textbooks in Iceland.

Arithmetic textbooks fell early into a certain pattern. However, authors have had some freedom on pedagogical ordering of the material; to present mathematics in context and to develop natural curiosity. Early printed textbooks were not written for use in institutionalized school systems in which students were working towards examination qualifications. The relation of the text to the reader differed from that of a present day school text. There were usually not sets of exercises; the works were not presented in a readily identifiable "lessons" and the presence of a teacher was not assumed (Howson, 1995). This changed as school became more widely distributed, initially for the elite. Gradually, schools were also established for the general public, who, however, had to acquire their knowledge by self-study far into modern era in most European countries. We shall trace that development in the three countries during the nineteenth century. We begin by exploring educational trends in Europe with respect to the three countries and look into societal circumstances in each of them.

The origin and social environment of arithmetic textbooks

Libri d'abbaco reckoning books

During the fifteenth century, growth of towns and cities meant new occupational opportunities, including participation in activities of manufacture and commerce. This development, originating in Italian cities, such as Florence and Venice, gradually spread along trading routes towards west and north to the German area. Reckoning masters created their own manuscripts of typical problems, *libri d'abbaco*, reckoning books, written in their own vernacular, the language of the marketplace. A characteristic feature of these books is that the solution and detailed working out of the problem is given immediately after the enunciation. The modern habit of printing only the problem and leaving the solution to the reader, or at most giving only the answers at the back of the book, is not found. The books contained typically an explanation of numeration and the numerical place value, followed by the four arithmetic operations in whole numbers and fractions: addition, subtraction, multiplication and division; and extracting square roots. The remaining content concerned mathematical techniques for business use: use of the *Rule of three*, monetary exchange, problems of partnership, barter, percentages and interests (Bjarnadóttir, 2013; Van Egmond, 1980).

The protestant reform and subsequent educational theories

The Protestant Reform in northern Europe in the early 1500s, according to its general approach to suppose the population to be literate, set out in the following decades to establish *Gymnasia* in greater towns to prepare its students for university studies. For Martin Luther's collaborator, Philipp Melanchthon, knowledge existed primarily for the service of moral and religious education and he praised mathematics for its ethical role. Most arithmetic textbooks of this time were written in the vernacular with self-instruction in mind. The mercantile community was the main target group and thus the books belonged to the reckoning book tradition (Grosse, 1901).

The conception of a school for the whole population was formed by Jan Amos Komensky (1592–1670), or Comenius by his Latinized name, in the seventeenth century. Comenius and Johann Heinrich Pestalozzi

(1746–1827) were prominent pedagogical philosophers who founded the basis of the development of the pedagogy for young children (Jänicke, 1888; Grosse, 1901).

Comenius, one of the leading educational thinkers of the seventeenth century, emphasized in his *Didactica magna* [Great didactic[1]] of 1628–32 that arithmetic and geometry must be taught partly for the various needs for life and partly as the scholarly topics awakened and sharpened the mind. In his *schola universalis sapientiae*, school of universal knowledge, mathematics was to be taught in all classes. Elementary schools for the common population, henceforth called primary schools, were already established in German states in the eighteenth century, and in the following period, didactical questions were discussed by educators (Grosse, 1901).

The achievements of early modern age entailed an explosion of trades, crafts, manufacturing and industrial activities in most European countries. This way the ability of a greater part of the population to appropriately deal with reading, writing and calculating became a condition for the functioning of the societies. So, in many European countries developing of school education for ordinary people became an important goal of their governments (Keitel, 2006).

The Enlightenment was an eighteenth century philosophical movement of intellectuals, characterized by rationalism, and an impetus toward learning. Its purpose was to reform society, using reason rather than tradition or faith, and advance knowledge through science. It promoted science and intellectual interchange (Sigurðsson, 2010). Promoting mathematics education therefore suited the Enlightenment philosophy well in its rationalism. The central idea of Enlightenment that knowledge is power and access to knowledge will empower people, forced developments in public education in many European countries in the beginning of the nineteenth century (Keitel, 2006).

The beliefs in the environment's influence on the individual and the value of knowledge from the Age of Enlightenment increased the significance of the childhood in the development of the humans. Early nineteenth century German speaking educators, the successors of Comenius, such as Pestalozzi, proposed teaching methods based on concrete experience, and educational aims concerned with the development of mental abilities that influenced the teaching of mathematics from kindergarten through the secondary school. Pestalozzi strove to combat the tyranny of method and "correctness". He set out to find ways of helping children to understand number and not merely to develop speed and accuracy in the mechanical working of examples. Pestalozzi's educational aims were based on his idea of "Anschauung" which concerns the reception by the mind of a sense-impression, observation, perception or intuition. The children were not given the products of learning but were guided to find them for themselves. Exercises in arithmetic were in the first instance related to circumstances in their environment and objects, such as beans and pebbles or whatever was at hand. The principle was the use of units, at first solid and later as marks on paper (Jänicke, 1888; Silber, 1976). Adolph Diesterweg (1790–1866) developed a specific brand of Pestalozzian reckoning. In his versions of school reckoning, the reckoning books returned, after a vigorous introduction into mental arithmetic along the lines of Pestalozzi.

Herbert Spencer (1820–1903) was an influential English philosopher. In his theories on pedagogy, which were based on those of Pestalozzi (Spencer, 1884), the fostering of students' initiative and ability to draw own conclusions was emphasized.

The global perspective of public education

Lack of schools and a tradition of self-study were a common pattern in many countries. For example, in the early 1800s, many school-age children in the United States rarely attended school. Of those who did attend, many, especially boys in the New England colonies or states, attended during the winter months only and did not study any mathematics beyond elementary arithmetic. Mathematical textbooks were published for several reasons but not for the pupils. Pupils rarely had a textbook. They copied in their own notebooks rules for mathematics and exercises with which to practice the rules. The textbook was mainly for the teachers or for self-taught individuals (Michalowicz and Howard 2003; Bjarnadóttir, 2013).

In 1821, Warren Colburn (1793–1833) published an innovative arithmetic textbook, *First lessons in arithmetic on the plan of Pestalozzi*, based on Pestalozzi's pedagogy, that gave students the opportunity to discover rules by induction from examples (Kilpatrick, 1992). Colburn's texts were an instant sensation among educators in the 1820s. Inevitably, a backlash set in against his inductive method. Merchants complained that students came to work with them with only a chaotic jumble of ideas about numbers and no practical knowledge. However, Colburn's methods built on a seedbed of dissatisfaction with the old methods and electrified educators. The vast diffusion of numerical skills in the United States from the 1820s to 1900 owed much to his influence (Cohen, 2003).

In England, from the 1830s onward, elementary education increased, owing to partial financial support of the government. The parliament passed the 1870 Elementary education act (The Forster education act), implementing compulsory elementary education for all children in

Britain. Elementary schools provided a curriculum that emphasized reading, writing, and arithmetic (Griffith & Howson, 1974). Pupils carefully inscribed copied examples of standard problems, thus creating for themselves a collection of worked examples. Many of them were taken from popular textbooks of the time (Stedall, 2012).

In France, during Napoleon's Consulate and Empire, education was not a priority. Under the July Monarchy, however, the 1833 law of Guizot focused on curricula and the duties of local officials in primary instruction. The basic course of study in the public primary schools would include moral and religious instruction, reading, writing, and arithmetic. Each commune was required to support a public elementary school for boys. There was no comparative law for girls' public primary schooling and the church provided their popular education for a while (Quartararo, 1995).

The three societies

The three countries, Estonia, Iceland and Norway, are all situated in northern latitudes, enjoying oceanic contacts. Norway includes the latitudes for both the other countries, while Estonia is situated in the far east direction by the Baltic Sea and Iceland in the far north west of the Atlantic Ocean.

Estonia

From 1583 to 1712 Estonia belonged to the Swedish crown so the general education policy was similar in Estonia and in Sweden. In 1626, King Gustav II Adolf visited Estonia and, among other things, encouraged development of education and opening of schools for local people. A number of primary schools started to work in connection to churches in towns and villages. The language of studies there was mainly or half-Estonian. In 1684 a two-year seminar to train teachers for primary schools was established by Bengt Forselius who was a native Swede living in Estonia. He was strongly inspired from the ideas of Comenius and his book, *Orbis sensualium pictus* [The visible world in pictures]. The program of the seminar included reading, writing and also some arithmetic, and was very successful (Andresen, 1985).

The first two gymnasia were established in 1630 and 1631 which study languages were German, Swedish and Latin. In 1632 also the University of Tartu, *Academia Dorpatensis*, was opened and worked in the Latin language. The gymnasium and university level education was meant primarily for the children of upper classes (mainly Baltic Germans) and was not accessible for the common population (Kruze, Kestere, Sirk & Tijuneliene, 2009).

As the result of the Great Northern War (1700–21), Estonia was merged to Russia in 1712. The upper class and the higher middle class remained primarily Baltic German, mostly ethnically German inhabitants of the eastern shore of the Baltic Sea. Both ruling kingdoms, Sweden and Russia, guaranteed the continuation of Baltic Germans' special class privileges and administrative rights when they incorporated the provinces into their respective empires. This situation among other influences channelled German cultural currents to Estonia (Kasekamp, 2010).

As the consequences of the Great northern war and the merge to the Russian empire, Estonia had serious drawback in education. The whole system of elementary education was destroyed and the schools ceased to exist. Home education remained the major form of education for ordinary people during the next century. Only in around 1800, as a reflection to Enlightenment movement, organised schooling for ordinary people was started again. The abolishment of serfdom, adopted in 1816, claimed the beginning of a new era in Estonia. German landlords began to sell farms for freeholds to peasants. Also local courts and self-governments went slowly into the hands of Estonians. The population grew rapidly, the changes increased Estonians' self-assertion, and national awakening ideas reached Estonia (Kasekamp, 2010).

In the first decades of the nineteenth century, public education remained undeveloped and home schooling dominated. Still, there was the demand that all peasants' children, who were not taught at home, had to attend school. Children had to study two or three winters; they were taught reading, writing, arithmetic and choral singing. Mathematics was studied less than writing. Children had to memorize multiplication tables and master four basic functions of arithmetic. They also learned measurements and weights (Kruze et al., 2009).

The latter half of the nineteenth century was a period of rapid socio-economic transformation and modernisation; the first larger industrial enterprises emerged. Improved economic development stimulated urbanisation. The increased wealth and population spawned also cultural vitality. By the 1850s, Estonians had already experienced a *national awakening*, and a sense of national identity developed rapidly. Education was considered as important driving force in growing national movement (Kasekamp, 2010).

From 1860s, general 2–3 years' compulsory primary education was adopted. For the graduates of primary schools it was possible to continue their studies in lower secondary schools in parishes and towns but for economic reasons it was not the realistic option for many. Also access to upper-secondary education started to expand slowly as gymnasia were established in towns. As there was no instructional literature in Estonian until 1850, the need for local textbooks became obvious. Thus lots of textbooks were issued at the second half of the nineteenth century. Local schoolmasters and pastors were the authors of those textbooks (Andresen, 1985; Kruze et al., 2009).

Iceland

Iceland belonged to the Danish realm from the late fourteenth century. It lagged behind other European countries in many respects until the twentieth century, due to difficult climate and various calamities, such as a gigantic volcanic eruption in 1783–84. The population numbered 56,035 in 1835. Only 1.1 % lived in the capital Reykjavík which became a centre of administration at the beginning of the nineteenth century and acquired the position of a capital town (Jónsson and Magnússon, 1997). The Icelandic society was predominantly rural through the nineteenth century, and besides livestock farming, fishing played a large role in the economy.

The Lutheran Protestantism brought printing in the mid-sixteenth century and channelled German cultural currents to Iceland. The main emphasis was laid on religious publications and education. As a part of the Danish state, the Icelandic Enlightenment movement originated in Denmark, where in turn it was largely derived from Protestant Germany. The movement extended cultural and educational interests to secular books in the eighteenth century. The publishing activities of its champions can be seen as a major exercise in education of the general public (Sigurðsson, 2010).

Two cathedral schools existed since medieval times for education of the clergy. They developed into learned schools with the same aim after the Lutheran reformation and were united into one Learned School, a gymnasium, according to Danish regulations, in Reykjavík in 1801. The Learned School was attended by sons of the clergy and other official, in addition to sons of merchants, landowners and better-off farmers (Karlsson, 2000).

The families, supervised by the parish priests, were responsible for children's religious and reading knowledge by an ordinance of 1743 and a King's letter in 1790. Law no. 2/1880 prescribed children's knowledge of writing and arithmetic, including at least the four arithmetic operations in whole numbers and decimal fractions. The children were tested at their confirmation. Throughout the nineteenth century, home-education was seen by the Icelandic authorities as the only method available in the rural areas until legislation on compulsory school education for the age 10–14 was issued in 1907, even if Denmark had public school legislation in 1814. Until then, public education was mainly upon private initiative. A primary school was established in Reykjavík in 1862. Around 1880, primary schools had been established in several towns, and ambulatory schools in many rural areas. A state-run lower secondary school was established in 1880 (Bjarnadóttir, 2006).

Changes that had taken place in Iceland by the turn of the twentieth century were far less extensive than e.g. in Norway and Denmark; the technological changes that had taken place elsewhere only reached Iceland to a limited extent; and there were relatively fewer towns than in most other European countries (Sigurðsson, 2010). Iceland had its Theological Seminary from 1847 without mathematics education at tertiary level but from where mathematics educators came for the following decades. University of Iceland was established in 1911 but mathematics was only taught there from the 1940s (Bjarnadóttir, 2006).

Norway

In 1380, Norway together with Iceland was absorbed into a union with Denmark that lasted more than four centuries. In 1814, Norwegians resisted the cession of their country to Sweden and adopted a new constitution. Sweden then invaded Norway but agreed to let Norway keep its constitution in return for accepting the union under a Swedish king. Rising nationalism throughout the nineteenth century led to a 1905 referendum granting Norway independence (Kunnskapsforlaget, 2006).

The Lutheran reformation was introduced by a royal decree in Norway in 1536, and church services were conducted in Danish. Independence from Denmark in 1814 was pivotal in the development of Norway's educational policy. Several textbooks in arithmetic were written for the common population before 1739, but none in mathematics, as the first book of the *Elements* was used as a school textbook. After a school reform in 1739 by the new king Christian VI, there was a need for suitable schoolbooks, and much work was done in the last decades of the eighteenth century to reintroduce mathematics as a subject in learned schools. The kingdom Denmark/Norway introduced a new school reform in 1800 which in many ways strengthened the position of the discipline of mathematics, and from then on may we talk about proper teaching in mathematics in the higher education (Piene, 1937). By a governmental decree in 1809, the pupils started at the learned schools at the age of 9–10 years.

Throughout the eighteenth, and into the nineteenth century, the number of illiterates was considerable, and the demand for better schools for common people grew. In 1827, Norway introduced the *folkeskole*, a primary school which became mandatory from the age of 7 in 1889, and which was meant for all strata of the population. The number of permanent schools was limited because of dispersed settlement, so the law required ambulatory schools in every parish. These schools were normally in people's living rooms, where the household were occupied with their daily chores, and the teaching could for that reason be poor. The education of the teachers of the ambulatory schools was normally less than mediocre, and the pupils of these schools only received a few weeks of education every year. The law of 1827 also initiated improvement in teacher education, and more up-to-date textbooks. In 1848 there was a law of primary schools in towns with an extended curriculum, classes for 18–24 hours per week, and a 45 weeks scholastic year. In 1860, ambulatory schools were exceptional, and in 1889 there was a common school system for all social classes, but with different rules for rural areas and towns. The written language in Norway at this time was Danish, a language that is closely related to Norwegian, and henceforth easily understood by Norwegians (Andersen, 1914; Christensen, 1895; Christiansen, 2009, 2010; Solvang 2001).

Arithmetic textbooks in the nineteenth century

By the early nineteenth century, secondary education had already a well-established tradition in the three countries. In Iceland and Norway, they were under the Royal Danish Directorate of the University and the Learned Schools [Den kongelige direction for universitetet og de lærde skoler], and in Estonia according to the German/Prussian tradition. Public education and mathematics education in particular, was, however, still weak, although literacy was assumed according to the protestant philosophy that the general public was to be able to read religious texts (Myhre, 2001).

Estonian textbooks

The first Estonian language mathematics textbook, *The art of reckoning and calculation*, was published by Peter Hinrik Frey in 1806. Educated in the University of Halle, Frey worked as pastor in Estonia and so has learned the local language. This very early book includes the description of the four arithmetic operations and deals with whole numbers and fractions. To explain the content, the form of dialogue between the schoolmaster and the students is used in the book (Prinits, 1992). It is difficult to understand the author's motivation to compile that textbook at such an early time, when even not all primary school teachers were able to calculate. Most probably it was primarily meant as the guide for the teachers and parents to teach reckoning (Lepmann, 2006).

Systematic compilation of Estonian language textbooks started only at the second half of the century. Two comprehensive series of school textbooks were published in 1850s, both containing also arithmetic textbooks: published by F.F. Meyer and F. Gebhardt. Both authors, having German background and working as pastors, most probably used German arithmetic textbooks as their models (Andresen, 1985). The first Estonian geometry textbook was issued in 1878 by J. Kapp, and the first textbook in algebra by J. Kurrik in 1879 (Prinits, 1992). No Estonian language textbooks in advanced mathematics were issued till the end of the nineteenth century. The languages at the University of Tartu were still Russian and German.

One of the most influential Estonian arithmetic books of this period, *A sensible reckoner*, was compiled by Rudolf Gottfried Kallas (1851–1913) in 1874 (Prinits, 1992). It was written mainly for primary school teachers and is the combination of a textbook and a methods book. The book is structured into two parts. The first part (44 pages) is devoted to general problems of arithmetic teaching. In the second part (290 pages), mathematical content together with its teaching tips is presented.

In the first part of the book, the author provides the reader with suggestions on how teaching is to be done. In this part he refers to the work of Komensky, Pestalozzi, Diesterweg and Hentschel. It is interesting how the author promotes meaningful learning, visualisation and using of challenging problems by teaching mathematics. The author of the book contrasts a rational reckoner ”who always knows why he does what he does the way he does” with a machine-like reckoner ”who never knows why he does what he does” (Kallas, 1874, p. 22). In the following some of his ideas are presented.

– Difficulty level of the tasks should be carefully designed: ”teaching goes step by step from the easier to the more complex” (ibid., p. 22). For this purposes he had developed a gradation of arithmetic tasks.

For example, while teaching addition of integers the following order was suggested:

Adding tens (20 + 30, 20 + 40, ...)

Adding tens and ones (20 + 1, 20 + 2, ...)

Adding tens and two-digit numbers (20 + 13, 20 + 14, ...)

Adding one- and two-digit numbers without carrying over first (1 + 11, 1 + 12, ...) and then with carrying over (7 + 14, 8 + 14, ...)

Adding two-digit numbers (11 + 12, 23 + 34, ...) first without carrying over and then with carrying over

– Repetition and training are important parts of mathematics learning and should be systematic.

Kallas' understanding was that repetition is to develop deeper understanding, to show the already known from a new perspective: "Repetition is like climbing the mountain: reaching the top, students see the whole landscape" (ibid., p. 19). Learning should be meaningful: "reckoning should be thinking, keep from the old dead machine reckoning" (ibid., p. 22). The author advised that pure mechanical memorization of calculation algorithms ("machine-like reckoning") should be avoided. Teacher's role should be to explain why the rules hold.

– Motivating students is important.

For this purpose, Kallas had several suggestions, to use practice related word problems among others. "Children's willingness will grow if they notice that reckoning is connected to their lives" (ibid., p. 25). The book contains lots of word problems with the content of daily life in the farms – well known to the potential readers. He suggested teachers also to compose such tasks themselves on the bases of data from the newspapers and calendars.

– Usage of visualisation helps to learn and understand arithmetic.

This seems to be one of the favourite ideas of the author. He argues that "senses are the gates through which the knowledge enters" (ibid., p. 7) and presents lots of examples in the book how the visualisation of arithmetic concepts could be done. Kallas warns teachers against teaching which is based on the teachers talk only. He promotes teaching by visual representations and emphasizes the importance of developing students' thinking. As one can see, many of the methodological suggestions set out by Kallas in his *A sensible reckoner* have its clear roots in the ideas of Comenius and Pestalozzi.

In the second part of the book the mathematical content is presented. This part is divided into three chapters: integers, fractions and proportional calculation. Quite often the material in this part is presented in the form of a table: in the first column the mathematical content and in the second column author's comments, suggestions and also the solutions of the tasks are given. Suggestions with teaching hints are often illustrated in the form of intended dialogues between the teacher and the student. With these suggestions the author skilfully demonstrates how to implement the pedagogical ideas presented in the first part of the book.

In the first chapter (160 pages), addition, subtraction, multiplication and division of integers are dealt with. Differently from his many predecessors, Kallas suggests concentric approach to teaching arithmetic: firstly all four operations on one-digit numbers (1 to 10), then on two-digit numbers (from 1 to 100) and only after that generally. And this is how the material in his book is structured into paragraphs. The fourth paragraph in this chapter is about "numbers with names" – numbers with monetary and measurement units and calculations with them.

The second chapter (50 pages) is devoted to common and decimal fractions and four arithmetic operations with them.

The third chapter of the book (70 pages) is about proportional calculation, percentage and methods for business use. First, solving of problems of computing the fourth proportional in equation of two ratios (The rule of three) is discussed, for example: "5 pounds of butter costs 1 rouble 30 kopecks. How much do 7 pounds cost?" (ibid., p. 258). Differently from authors of the earlier textbooks, Kallas suggests solution method based on transforming to units: "1 pound costs 5 times less, it is 26 kopecks. So, the cost of 7 pounds is 7 x 26 = 186 kopecks = 1 rouble 86 kopecks" (ibid., p. 258).

In the second part of the chapter, the concept of percentage is introduced in a financial context. Traditional types of percentage problems are discussed (to find interest rate, capital, interest, time) and unit-method is advised to solve these problems, as well. At the end of this chapter, even examples on solving problems with compound interest are given.

Throughout the entire book Kallas stresses the importance of mental calculations and in many places alternative solutions in addition to routine calculation approaches are given, for example, after presenting the solution to 648 : 72 using written algorithm, he also demonstrates the way to solve it mentally (648 : 8 : 9).

With his textbook, R. G. Kallas, himself being influenced by the ideas of Comenius and Pestalozzi, changed traditional understandings of arithmetic teaching in Estonia and set new standards for many years to come (Prinits, 1992).

Icelandic textbooks

During the heydays of the Icelandic Enlightenment in the 1780s, two remarkably substantial arithmetic textbooks, written in Icelandic, were published: *Greinilig vegleiðsla til talnalistarinnar* [A clear guide to the number art] (Olavius, 1780), and *Stutt undirvísun í talnalistinni og algebra* [A short teaching in the number art and algebra] (Stefánsson, 1785). At this time there were no primary schools and the books were the basis for the self-education of common people into the 1840s. The *Short teaching* was based on university lectures and introduced latest knowledge, such as decimal fractions, while the author of the *Clear guide* remarks in his foreword that he was guided by the Danish educator Christian Cramer (1699–1764) who wrote *Arithmetica tyronica* (Cramer, 1735), and by the German Christlieb von Clausberg (1746) whom he declared to be among the cleverest German reckoning masters. Clausberg wrote *Demonstrative Rechenkunst,* first published in 1732. He wrote in the spirit of Comenius as he emphasized comprehension and the great joy felt through understanding why a certain rule could be used for solving a problem (Jänicke, 1888). Both Clausberg and Olavius emphasized mental calculations and large parts of their books contain what Clausberg called "Vorteile", advantageous rephrasing of calculations, such as $425 \cdot 72 = 425 \cdot 8 \cdot 9$ (Olavius, 1780). Otherwise, the two textbooks introduced the traditional content; the four arithmetic operations in whole numbers and fractions, and conversions between measuring and monetary units, solved by the technique of the Rule of three. That the *Clear guide* was in circulation for about sixty years as one of two available arithmetic textbooks leads to the conclusion that Comenius had some influence on mathematical education in Iceland in the early 1800s even if no direct reference to Comenius has been detected. There were no teachers of the common people, and therefore no didactical discussion took place.

Low status of the learned schools and education in general due to the calamities of the 1780s and economic problems in the aftermath of Denmark's participation in the Napoleonic wars, reduced the effects of the two textbooks. Sources say that everyone who reached the upper class of one of schools was given a copy of the *Short teaching*, but that it was up to the pupils whether they ever opened the book or not (Helgason, 1907–1915). The University of Copenhagen was the natural continuation of the Icelandic learned schools. The university introduced more stringent requirements in mathematics under regulations of 1818 and Danish textbooks were introduced at the sole Learned School.

It was only by the mid-nineteenth century that a new Icelandic arithmetic textbook was written and published. Therefore no trace has been found of Pestalozzian influence during the early nineteenth century in Iceland, contrary to Denmark (Hansen, 2002). The three textbooks, published later in the century, were: an advanced arithmetic textbook by mathematician Björn Gunnlaugsson, and two elementary arithmetic textbooks, one by Jón Guðmundsson (1841), and the other by Eiríkur Briem (1869).

Björn Gunnlaugsson (1788–1876) was a prominent product of the Enlightenment movement without ever being admitted into a learned school. During 1808–17, Gunnlaugsson became acquainted with Danish and Norwegian land surveyors, who gave him books and guided him in geodesy and mathematics, enough for him to earn a gold medal for a prize winning mathematical problem at the University of Copenhagen before being admitted in early 1818 (Melsteð & Jónsson, 1883). He studied mathematics and geodesy at the university in 1818–22. He returned home to teach at the Learned School for forty years.

Gunnlaugsson's book on advanced arithmetic, *Tölvísi* [Number wisdom] (Gunnlaugsson, 1865) which he wrote in 1865 after his retirement, reflects the writings by Euler, Lagrange and Kästner, professor in Göttingen, as well as works by Fermat, Leibniz, Maclaurin, d'Alembert, Fourier and Gauss in the sense that the author refers to them in his text. All these works were developed before 1822, when Gunnlaugsson left Copenhagen for Iceland. No later nineteenth century mathematicians were mentioned but several contemporary Danish textbooks, such as those by L. S. Fallesen and C. Ramus, which Gunnlaugsson taught at the Learned School.

The Icelandic Literary Society published the book. Four hundred printed pages were published, stopping abruptly in the midst of continued fractions. What remained was never printed. The printed part of *Tölvísi* contained sections on numbers, number notation in base ten and various other bases, the four arithmetic operations in whole numbers and algebra, and common and decimal fractions; introduction to number theory – modular arithmetic, congruencies, prime numbers, Euclid's algorithm and Fermat's little theorem. Furthermore it contained emphasis on numerical methods and error bounds, exponents and roots, irrational and imaginary roots, imaginary numbers and binomial theorem. The manuscript contained continued fractions, ratio and proportions, equations, differential calculus, indeterminate diophantine equations solved by continued fractions, exponential equations, logarithms, convergent and divergent series, interests, annuity, permutations and combinations (Lbs. 2397, 4to; Bjarnadóttir, 2009).

The list of contents reflects Gunnlaugsson's view of life. He was nearly 30 years old when he could enter university studies in mathematics. The spirit of the Enlightenment and land surveying is expressed in his emphasis on numerical methods, accuracy in calculations and on error bounds. His life as a farmer is reflected in his concern for the agricultural environment and his choice of metaphors in problems chosen to demonstrate theories or techniques. The book reflects also Gunnlaugsson's attitude towards mathematics as a divine science and his philosophical and religious views on the nature of mathematics, especially in his treatment of zero and infinity, the imaginary numbers and the laws of exponents. All topics in *Tölvísi* were thoroughly explained from the first principles with several examples, but they developed, however, quite steeply into higher mathematics (Bjarnadóttir, 2009). The book is said to have turned out to be the "book that everyone praises but no one reads" (Melsteð & Jónsson, 1883).

The Icelandic Literary Society was a forum for the Icelanders' campaign for independence from Denmark in the nineteenth century and it enjoyed great support. In a population of less than 70,000, the Literary Society had over 700 subscribers in 1865 (Líndal, personal communication), who were obliged to buy *Tölvísi*. Gunnlaugsson's students during his 40 years at the Learned School, around 300 in total, were the most likely to understand the text, but there was also a tradition of attaining knowledge among farmers who could afford books. There were no schools except the Learned School, which was only attended by few. The book was never used there as a textbook (*Skólaskýrsla fyrir Reykjavíkur lærða skóla 1845–1904*). It may have been considered wiser to use the Danish textbooks authorized or recommended by the Royal Directorate of the University and the Learned Schools.

Few could follow Gunnlaugsson's thoughts. He worked alone on mathematics all his life after he left Copenhagen. He was far ahead of his Icelandic contemporaries in mathematics, but fell inevitably behind his European colleagues. Before Gunnlaugsson's time, there was no one to teach mathematics at the Learned School. After his time, in 1877, mathematics teaching at the school was cut back by a political decision, and it did not rise again until 1919. Therefore the history of Gunnlaugsson and his book, *Tölvísi*, and the works of his former pupils, Guðmundsson and Briem, based on the heritage of the two eighteenth century arithmetic textbooks, is nearly the complete history of creative mathematics education in nineteenth century Iceland (Bjarnadóttir, 2006).

The second half of the nineteenth century was characterized by a growing struggle for autonomy from Denmark, lead from Copenhagen by Jón Sigurðsson and guided by the spirit of Enlightenment and rationalism. Sigurðsson's prime supporter in Iceland, Jón Guðmundsson (1807–1875), wrote an arithmetic textbook in 1841, *Reikníngslist*, *einkum handa leikmönnum* [Arithmetic art, mainly for laymen] (Guðmundsson, 1841). The *Arithmetic art* was, like all textbooks published into the first quarter of the twentieth century, aimed at home- and self-education. The author defined his target group mainly for self-instruction of those who already could read.

The book includes the traditional content. It introduces the placement numerical system by a rhyme, known from one of the eighteenth century textbooks. A section on percentages is new vis-à-vis the eighteenth century textbooks. It contains topics such as the Euclidian algorithm and testing of operations, and examples on trade with fish as a barter unit, but no algebraic explanations. Guðmundsson (1841) is much concerned with teaching economical allocation of income and revenue of farming, including warning its readers against expenditure on the imported goods: coffee, tobacco and spirits. No evidence has been found as yet concerning use of the book except at the Learned School in 1842–43 by Gunnlaugsson. No other schools existed where the book might be used (Bjarnadóttir, 2012).

Eiríkur Briem (1846–1929) published his *Reikningsbók* [Arithmetic] (Briem, 1869) when he was only 23 years old. Briem enjoyed home-education and stayed in Reykjavík for only three years during his studies at the Learned School and the Theological Seminary which normally took eight years. Briem became influenced by Spencer's educational theories when he stayed in Copenhagen in 1879–1880 (Bárðarson, 1931). Briem translated Spencer's book, *On education* (Spencer, 1884) and published his own book in an extended version in 1880 after the author's stay in Copenhagen. The book was widely used into the 1910s. His aim was to create a practical handbook for the self-instructing youth, devoid of theoretical explanations, but with the important additions, logarithms:

> [...] for the chapter about algebra, equations and logarithmic calculations I have, however, expected that people had some instruction; in that chapter I have, as elsewhere, avoided supporting the rules prescribed by reasoning; when I have made exceptions in several places, it is because the reasoning could as well be an exercise or it was so clear that it could be used to support the memorizing of the rule [...] in order to gain full use of what one learns in arithmetic, one must know it so well that one calculates *easily* and is not apt to make errors; this will not happen except with *considerable exercise* and by reviewing more than once what one has learned; especially those who have enjoyed teaching are much apt to forget if they do not regularly revise what they have learned. (Briem, 1880, p. iii)

One can spot Spencer's theories here, to foster the learner's initiative and not to reveal too much, while Briem did not mention Spencer explicitly.

The first two lower secondary schools for the general public were established in the early 1880s and the book was used at both of them and the Learned School for a while. Examination of the content of Briem's (1880) *Arithmetic*, as well as of Guðmundsson's (1841) *Arithmetic art*, not the least the examples they took, reveals that they followed the German ethical tradition in the spirit of Melanchthon. They both added decimal fractions to the traditional commercial textbook model, described earlier. Briem was not concerned with philosophical discussion on the number concept, but went directly to the placement notation, guided by the same rhyme as his predecessors. A study of the textbook reveals the author's roots in a self-sustaining society of sheep and cattle farming, supported by seasonal fishing, as well as his wish to provide people with financial education and advice concerning cautious allocation of their income.

When Guðmundsson and Briem defined their target groups, school use was secondary. The authors aim at self-education and expect the ordinary reader of books to be an educated man who is able to read the text on his own. In many of his story problems, Guðmundsson recounted informative matters, such as the time span since Columbus discovered America, the strength of the silk thread compared to the thread in a spider web, the furs of the Russian emperor, the distance of a lightning computed by the time delay of the thunderclap, etc. Guðmundsson was thus faithful to the Enlightenment vision, to enlighten the public. Gunnlaugsson only presented story problems to demonstrate some mathematical theories or methods, while he demonstrated his utilitarian aspects by thorough treatment of approximation theories, necessary for land surveying.

There is no noticeable difference between the societies that Guðmundsson and Briem described in their textbooks. Both arithmetic textbooks contain examples on the cost of supporting paupers, besides the usual examples on harvesting hay, feeding cows and sheep and buying or weaving material for clothes. Real money was scarce. Shopping at the merchant's was still barter trade; the farmer's credit was fish, wool, meat and fat, for which he bought rye, oatmeal, beans, coffee and linen. Briem warned farmers seriously against borrowing except to keep the production going. For example, borrowing wool was expensive but the farmers had to ensure enough wool for their family and servants to spin for knitting and weaving in the late-winter and early spring before the bustle of lamb-births, shearing and hay-making; the old farming cycle in a nutshell.

All three textbook authors became well-known in Iceland; Gunnlaugsson enjoyed respect and fame, mainly for his land-surveying; Guðmundsson as the editor of an influential journal supporting the movement for autonomy; and Briem as a professor of theology, member of parliament and the warden of the first National bank. They were all avid supporters of Jón Sigurðsson, the leader of the movement for Iceland's autonomy from Denmark (Ólason, 1948–52).

Guðmundsson and Briem, both theologically educated, were faithful to the protestant tradition in including ethic education in their arithmetic textbooks. They warned people of spending money on luxurious goods, such as coffee, sugar and spirits, for their own financial prosperity but also for the society's. The homes were expected to consume their own products as was the society as a whole. The overall goal of all these men; Gunnlaugsson, Sigurðsson, Guðmundsson and Briem, was Iceland's autonomy from Denmark, and for that the population had to be educated and capable of allocating their own personal resources in a wise manner. The nineteenth century textbooks were important channels for conveying ideas that the authors gathered would lead Iceland and Icelanders to prosperity and financial independence (Bjarnadóttir, 2012).

Norwegian textbooks

In the beginning of the nineteenth century the use of textbooks in the primary schools must have been very varying, and some Danish textbooks have probably been used. A number of textbooks were published in Norway, and Andreas M. Feragen (1818–1912) was an important teacher, teacher educator, textbook author and publicist, and his way of presenting the subject matter in his textbooks made it easier for the pupils to understand (Ribsskog, 1941). Among his textbooks are:

– *Regne-ABC, eller opgaver af de fire grundregningsarter med ens- og uensbenævnte tal tilligemed korte vink til praktisk tavleregning for de første begyndere* [Reckoning ABC, or exercises in basic arithmetical operations with abstract and concrete numbers ... for the novices], Kristiansand, 1856.

– O*pgaver og korte vink til praktisk tavleregning (*utvidet utg. av *Regne-ABC)* [Exercises and short hints to reckoning ...], Kristiansand, 1856.

– *Udmaalingsbog, indeholdende veiledning til praktisk udmaaling og beregning af flader og rumstørrelser med 110 oplysende tegninger og 112 øvelsesopgaver* [Book of measuring, containing guidance practical measuring and calculations of areas and spaces ...], Kristiansand, 1859.

– *Regneøvelser for pigebørn* [Exercises in reckoning for girls], Kristiansand, 1867.

– *Regnebog for folkeskolen* [Exercises in reckoning for the primary school], Kristiansand, 1877.

They were all practical books on how to do calculations of numbers and magnitudes, with exercises from daily life. The book *Regneøvelser for pigebørn* [Exercises in reckoning for girls] from 1867 is a collection of exercises connected to practical household.

Many of the textbooks in mathematics for the primary school level, studied by the authors of this article, are of a traditional form, and their approach to teaching mathematics is the so-called ”Rule method” – presenting definitions and rules to be memorized, followed by exercises. The ”Inductive method”, where posing of carefully graded questions lead to the pupils own discovery of the concepts, was based on the theories of Pestalozzi (Michalowizc and Howard, 2003). The first Norwegian textbook where a method similar to the inductive method has been detected, was *Regnebog for folkeskolen* [Reckoning book for the primary school] (Feragen, 1877). This book starts with simple explanations of very basic concepts, followed by guiding questions and specific exercises meant to lead the pupil towards a deeper understanding of the concepts.

A book by J.M. Theiste (1793–1871), officer and teacher at the Norwegian war academy, should be mentioned: *Theoretisk-praktisk regnebog til brug i borgerskoler og ved selvundervisning* [Theoretical and practical reckoning book for use in national schools, and for self-tuition]. Theiste's book is both theoretical and practical reckoning, and was not meant for the general population. The title indicates that it may also be used for self-tuition. Theiste's book was first published in 1829, and came in at least five editions.

H.F. Clausen published in several editions, a synopsis of *Arithmetica Tyronica* by Christian Cramer (1735): *Udtog af Christian Cramers Arithmetica Tyronica: forøget med exempler af kjæde- og vexelregning, udarbeidet i norske speciepenge til brug i norske almueskoler, tilligemed et anhang om algebra* [Excerpts from Christian Cramers Arithmetica Tyronica ...], Christiania 1824. Cramer's Arithmetica Tyronica had many editions even far into the nineteenth century. It is mainly reckoning rules concerning the four arithmetical operations and the Rule of three. Likewise does Clausen's synopsis contain practical reckoning rules and business arithmetic, but it has also a short supplement about algebra.

The most influential person in the development of school mathematics for the learned schools in Norway in the nineteenth century was the mathematics teacher and author of textbooks, Bernt Michael Holmboe (1795–1850). Holmboe was a teacher at Christiania Kathedralskole from 1818 till 1826, and after that professor at the University of Christiania until his untimely death in 1850. As teacher at Christiania kathedralskole, Holmboe earned his reputation as Niels Henrik Abel's teacher in mathematics. After 1826, Holmboe also held a position as teacher in mathematics at the military academy (Christiansen, 2009). The University of Christiania, established in 1811, was in function from 1813. In the first six years at the university, the only use of mathematics was for the *Examen philologico-philosophicum* – a preparatory exam for other subjects at the university. The lectures in mathematics were on trigonometry, stereometry, basic algebra, and later applied mathematics after Christopher Hansteen's appointment. Hansteen (1784–1873) became a teacher in applied mathematics at the university in 1814, and he was professor from 1816 till 1861. Hansteen was very productive, and wrote about terrestrial magnetism, northern light, meteorology, astronomy, mechanics, etc. He received international recognition after an expedition to Siberia in 1828–30 to study the geomagnetism. In 1835, Hansteen wrote a textbook in plane geometry (Hansteen, 1835) that was a clear attack on traditional textbooks in geometry, based on the *Elements* by Euclid (Christiansen, 2009, 2010).

Holmboe wrote several textbooks, four in basic mathematics and one in advanced mathematics:

- *Textbook in arithmetic* (Holmboe, 1825) in a total of five editions.
- *Textbook in geometry* (Holmboe, 1827) in a total of four editions.
- *Textbook in stereometry* (Holmboe, 1833) in two editions.
- *Textbook in plane and spherical trigonometry* (Holmboe, 1834) in one edition.
- *Textbook in the higher mathematics* (Holmboe, 1849) in one edition.

Holmboe stressed the importance of understanding the use of mathematical notation, and he writes in the preface of his textbook in arithmetic (Holmboe, 1825) that this use of signs is an important feature within mathematics, and it is also what separates mathematics from other sciences. The first task of a teacher should therefore be to make the students acquainted with mathematical notation and signs (Holmboe, 1825). Holmboe suggests that the best way of doing this is to first use numerical values for the various magnitudes, before using letters as general values. This advice from the preface is, however, left to the teacher to accomplish, and there are no examples with numerical values in the text of the book. Holmboe concludes the preface by advocating "individually adapted education". His textbooks are meant for teaching in schools, and require an able teacher, that can help the student to understand the subject matter, contrary to Theistes books that were meant for self-tuition. Holmboe writes that

> as much as possible, to prevent the reluctance to the science that easily becomes the consequence with the novice after too much effort, one should in the mathematical lectures omit the most difficult statements until the remaining are learned. If one does not omit such statements, one should not demand that the student from the start can give an account of them, but during the examination, guide him, and by repeated explanation of the coherence in the deductions, aim at making them as evident as possible.
>
> (Holmboe, 1825, p. VI–VII)

He also stresses the necessity to the students' own ciphering books (notebooks) as he writes that

> the students' ciphering books should, in my opinion, contain examples and practical applications, and they have to keep up with the theoretical teaching in mathematics. (Holmboe, 1844, fortale)

Holmboe's textbooks may have been the first textbooks in mathematics for the learned schools that were published in Norway. They were, according to Holmboe, nearly the only ones used before 1850, and some of them still came in several editions, even after Holmboe's death. In the preface of the second edition of the textbook in arithmetic (Holmboe, 1844), Holmboe writes that his textbook has now been in use for nearly 19 years at Christiania kathedralskole and, to his knowledge, with one exception at the other learned schools in Norway. Holmboe's textbooks were eventually succeeded by the textbooks written by Ole Jacob Broch (1855, 1860). Ole Jacob Broch graduated from a learned school at the age of 17 in 1835, and was considered to be "a new Abel". He was engaged at the university in 1842, and succeeded Holmboe as professor in 1850. Broch was very active in the political life, and had several posts as cabinet minister. He became in 1879 director of the Bureau International des Poids et Mesures in France. Broch published sets of exercises for Holmboe's textbook in arithmetics.

There are reasons to believe that Holmboe was influenced by the Norwegian philosopher and politician Niels Treschow (1751–1833). He was the first rector at the university in Oslo, at the time when Holmboe was a student. Treschow published a textbook on logic (Treschow, 1813) where one finds basically the same conceptualization that Holmboe used, and Holmboe's use of concepts is comparable to the descriptions found in Treschow (1813) who in turn was influenced by the logical systems described by Immanuel Kant (1724–1804).

The first half of the nineteenth century was in many ways a breaking point for the higher education in mathematics in Norway. The position of mathematics as a school subject was strengthened through school reforms at the turn of the century, and the first university established in 1811. Bernt Michael Holmboe's textbooks in mathematics were the ones that were predominantly used in the learned schools at that time (Holmboe, 1844). His textbooks were, however, not without opposition. The core of that debate was the use of *proofs* in elementary mathematics, and whether the introduction of geometry should be in a traditional Euclidean way using logical deductions and theoretical thinking – as in the case of Holmboe – versus a more "informal" way using everyday language and terms. An analysis of these books, and the responses they caused, may help to give an understanding of the development of mathematics education and the didactical debate in this period (Christiansen, 2009).

Discussion of similarities and differences

Our aim in this paper was to study similarities and eventual differences in the development of public mathematics education in the nineteenth century in Iceland, Estonia and Norway. More precisely we aimed at studying how these developments were reflected in the arithmetic textbooks written in the vernacular in these countries during this period.

All the countries, Estonia, Iceland and Norway, experienced foreign rule by other Nordic countries at the vulnerable time of religious reformation into the Lutheran faith. All the societies in Estonia, Iceland and Norway/Denmark were marked by the reformation which brought education closer to the general public by the abolishment of Latin and introduction of the vernacular at Christian services, the main organized social gatherings, and by its promotion of literacy and nurture of arithmetic education for its ethical role. In Norway's case though, services were conducted in the related Danish, which caused a lasting schism in the mother tongue of the large country.

In the eighteenth and still more in the nineteenth century, the Enlightenment had a lasting impact in all the countries; in the sense of "enlightening" of the general public but preserving the social structure of the elite and the common people. In Iceland, Estonia and Norway the secondary level education for the elite had relatively long traditions already from medieval times. Cathedral schools were established in the catholic era for educating the clergy, while protestantism strengthened their role of general education in the form of latin schools, or learned schools, gymnasia, of the Prussian model; in the case of Norway and Iceland under Danish ordinances, and in the Estonian case by the influences from the Baltic German elite. Gradually, secondary mathematics education in those countries gained firm grounds; in Norway in the early 1800s, in Iceland in 1822, and in Estonia by the end of the century. Estonia had its own university already in 1632 and Norway in 1811, while the University of Iceland was established only in 1911 and then without mathematics instruction.

In Estonia, Iceland and Norway, as in other Northern-European countries, the need for schools to educate common people was generally recognised but it took time before such schools became mandatory. In the first decades of the nineteenth century, public education remained undeveloped and home schooling dominated. In the mountainous Iceland and Norway with their difficulties in communication also the form of ambulatory schools was used, what was not the case in Estonia.

The latter half of the nineteenth century was a period of rapid transformation and modernisation in Norway and Estonia. Changes in Iceland by the turn of the twentieth century were less extensive. Education, and mathematical education in particular, was considered as an important driving force in growing national movements in all the countries, still dependent of other nations; Norway in a union with Sweden, Iceland striving for autonomy from Denmark, and Estonia from Russia. Mathematics education contributed to capability to manage own resources and increased self-esteem, as did the use of the vernacular, the language spoken by the general public.

Norway had public school legislation already in 1827 but primary schooling became mandatory from 1889. In Estonia general compulsory primary education was adopted in 1860s. In Iceland home-education in writing and arithmetic was added in 1880 to the eighteenth century ordinance on knowledge in reading and christendom before the confirmation. School legislation for age 10–14 was issued there in 1907. The vernacular as a subject became significant in the teaching, as well as arithmetic. Thus, while teaching arithmetic was rare in all the countries at the beginning of the nineteenth century, arithmetic was becoming a compulsory subject by its close.

The content of mathematics has been determined by social practices and developed in relation to social and technological challenges. Mathematics became a necessary craft knowledge for many. The increasing importance of trade and commerce demanded extensive computation skills in trade, commercial and banking companies. Also manufacturing and distribution necessitated new mathematical tools and hence appropriate schooling was demanded. The first arithmetic textbooks published in the late eighteenth century and mid-nineteenth century in Estonia, Iceland and Norway were influenced by German arithmetic textbooks based on the commercial textbook tradition originating in Italy in late medieval times. The content handled in these textbooks shows many correspondences. The topics referred to the domains of problems relevant in everyday life in the respective times. Textbooks dealt with integers, fractions and quantities (measuring numbers) with the elementary arithmetical operations addition, substraction, multiplication, division, in some cases also with powers and roots. Also proportions and proportionalities were presented. In connection with that, the most frequently used method was the Rule of three. As most of the problems dealt with situations of daily life, also magnitudes played important role in these early textbooks. In consequence of that, transformation of different measuring units into one another, using tables of measuring units, were found in all textbooks.

Similarities have also found with respect to the style of presentation of the books reflected in influences by the ideas of Comenius and

Pestalozzi, in case of Estonian and Norwegian textbooks, and Comenius and Spencer, in case of Icelandic textbooks. The textbook authors avoided suggesting mechanical memorization of calculation algorithms, but emphasized memorization as a support to drawing own conclusions and stressed meaningful learning and mathematical reasoning. That idea was represented most strongly in the Estonian textbook compiled by Kallas who claimed: "reckoning should be thinking, keep from the old dead machine reckoning!"

All these nineteenth century arithmetic textbooks are also clearly influenced by ideas of Enlightenment: the authors made considerable efforts to include general educational values in their texts, for example via the context of story problems.

The situation in more centrally situated and densely populated countries, such as France, England and the United States, may not easily be spoken of in general terms. However, also in those countries, public education, including arithmetic education, became more structured during the nineteenth century. The pedagogy of mathematics teaching was increasingly paid attention to. Thus, the peripheral and sparsely populated societies in Estonia, Iceland and Norway were following general trends in Western culture, subject to their capacities and their situation constrained by limited infrastructure and/or foreign rule and foreign languages.

Comparison of nineteenth century events in mathematics education in Iceland, Estonia and Norway reveals the following:

1 schooling was about to become mandatory for ordinary people and arithmetic was considered as an obligatory part of it;

2 mathematics was considered to be a pragmatic discipline aiming at providing the population with calculation skills demanded by rapidly developing societies; the main purpose of primary level mathematics was to prepare pupils for practical tasks needed in their lives and not for secondary courses;

3 mathematics seen as mental discipline, meant to train logical thinking skills, was generally considered an important secondary aim;

4 the first arithmetic textbooks written in the vernacular had been published by the mid-nineteenth century in all three countries;

5 those arithmetic textbooks typically covered rather unified content, characteristic for all countries;

6 as some of the authors were influenced by the pedagogical ideas of Comenius, Pestalozzi or Spencer, the presentation style of those textbooks stressed meaningful learning and developing learners reasoning skills. Memorizing procedures and repetition were however considered necessary to develop deeper understanding, to show the already known from a new perspective and as a support to the ablility to draw own conclusions;

7 public schools were still not mandatory in Iceland by the end of the nineteenth century and arithmetic textbooks were aimed at self-instructing adolescents rather than teachers or school children, which may be counted as one of the main differences between the three societies;

8 in Norway and Iceland secondary level arithmetic textbooks, written in the vernacular, were also composed, while many groups of those societies didn't have access to secondary mathematics. The first secondary level mathematics textbooks written in Estonian language were composed only in the beginning of the twentieth century; another area of differences between the three countries.

It is surprising that notwithstanding different socio-historical background and geographical isolation, these three countries have adopted similar practices in mathematics teaching in many respects even if differences may also be detected. In spite of the remoteness of the three countries, which all were marginally situated in a sense, they all belonged to European culture and adapted the educational currents and ideas to their own national needs and autonomy.

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Notes

1 All translations are provided by the authors of the paper.

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