Affective pathways and interactive visualization in the context of technological and professional mathematical knowledge

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This article reports the findings for a qualitative study on the use of dynamic geometry systems (DGS) and their impact on students' affective pathways. The approach adopted is to view affect through the lens of a representational system. The participants, mathematics teacher trainees, were asked to solve geometric locus exercises using GeoGebra software. The results reveal a number of features that characterize subjects' local and global affect. Future teachers' local affect when using imagery in computerized environments was found to be impacted by the balance between their analytical-algebraic and graphic reasoning and their understanding of the tools at their avail and their use in the instrumental deconstruction of geometric figures. Evidence was observed that linked student teachers' global affect, in turn, to their motivation as defined by their goals and self-concept.

Recent research into the use of technology to learn mathematics confirms that teachers need support and guidance to effectively exploit its classroom potential. Teachers need support to acquire and develop pedagogical technology knowledge (PTK) (Thomas & Hong, 2005), in particular so they and their students can engage in technology-mediated mathematical thinking to interact more fully with the structure of mathematics.

One of the main ways to acquire PTK is in pre-service technology training. As Forgasz (2006) noted, the research implications of such training are far-reaching and pose challenges: "Identifying the needs, adapting courses appropriately, and aptly skilling all pre-service teacher educators are among [the] challenges to be addressed" (p. 464). The nature of PTK and its pre-service acquisition should thus constitute the object of future research.

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This study aims to contribute to PTK in understudied problem-solving areas such as *visualization and affect*, with a view to developing discipline awareness and integrating crucial elements for mathematics education in teacher training. As defined by Mason (1998), teachers' professional growth is regarded here as the *development of attention and awareness*, *structural awareness* of the respective abstract knowledge, i.e., the pathway from the visual to the abstract that typifies the mathematical meaning of visual in DGS.

The research described here was conducted with a group of 30 Spanish mathematics undergraduates (teacher trainees). These future teachers had taken courses in advanced mathematics in differential and Riemannian geometry, but had worked very little with the classical geometry they would later be teaching. They were accustomed to solving mathematical problems with specific software, mainly in areas such as symbolic calculation or dynamic geometry, but were not necessarily prepared to use these tools as future teachers (Lagrange, 2009; Gómez-Chacón & Kuzniak, 2011). This article focuses on the relationship between technology and visual reasoning in problem solving, seeking to build an understanding of the affect (emotions, values and beliefs) associated with visualization in geometric locus exercises in which GeoGebra software is used. The aims were to identify the contexts where visualization and productive affective pathways are found, and the factors that help students stay on or return to an enabling affective pathway rather than becoming overwrought, apprehensive or distressed.

Prior research has yielded significant findings on affective states. Eisenberg (1994), and Presmeg and Bergsten (1995) showed that preference for visualization is unrelated to mathematical aptitude, while other researchers (McCulloch, 2011; Gómez-Chacón, 2011; Gómez-Chacón & Joglar, 2010), studying visualization in technology-mediated problem-solving settings, identified tools that help students maintain productive affective pathways.

The present findings contribute to this line of research by furnishing empirical data supporting the role of both global and local affect in visual processes and affective pathways and proposing methodology that rises to the research challenge of identifying individuals' imagery processes.

Theoretical considerations

Several theoretical approaches to the analysis of visualization and representation have been adopted in mathematics education research. Any analysis of the psychological (cognitive and affective) processes involved in working with (internal and external) representations in reasoning and problem solving requires a holistic definition of the term visualization. Hence, Arcavi's (2003, p. 217) proposal was adopted here:

> the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings.

In the present study the focus was on interactive visualization, a term used to mean an interactive approach to learning concepts such as DGS in which users receive feedback within a few seconds of entering their input. This term stresses the object's changed appearance and the dynamic view of functional dependencies, which are readily attained when the user creates figures using interactive tools in DGS (Kortenkamp, 2007).

Image typology and the use of visualization were analyzed as per Presmeg (2006) and Guzmán (2002). Presmeg describes images as both functional distinctions between types of imagery and products (concrete pictorial, kinesthetic, dynamic and patterned imagery; memory images of formulae). In Guzmán they are categorized conceptually: the use of visualization as a reference and its role in mathematization, and the heuristic function of images in problem solving (isomorphic, homeomorphic, analogical and diagrammatic visualization). This final category was adopted in the present study in connection with the use of tools in problem solving and research and the precise distinction between the iconic and heuristic functions of images (more closely related to non-iconic visualization) (Duval, 1999, 2006) in analyzing students' performance (see table 2). According to Duval, *iconic visualization*, the recognition of what forms represent, is based on resemblance to the (real) object represented, or by comparison with a type-model. *Non-iconic visualization*, in turn, is a heuristic series of operations through which geometric properties are recognized when certain configurations cannot be obtained or the configuration obtained cannot be varied.

The theoretical geometric work space (GWS) (Houdement & Kuzniak, 2006) and Artigue's (2002) instrumental approach were deployed to describe the complexity involved in applying technology to geometric tasks. Based on Rabardel's work (1995) on ergonomics, Artigue (2002) stressed the need for instrumental genesis that transforms artifacts into tools in two directions relevant to the present study. Upward transition, from artifact to the construction of geometric configuration, known as instrumentation, describes users' manipulation and mastery of drawing tools. The downward process, called instrumentalization, runs from configuration to the proper choice and use of an instrument and refers to geometric construction. With the instrumental approach, specific technology-related difficulties can be addressed, while GWS is more sensitive to the cognitive and epistemological construction of specific geometric exercises and visual reasoning.

Use of the "Locus" tool calls for an awareness of the properties linked to the operations taking place (non-iconic visualization), either to construct a figure or to transform it, where two types of deconstruction are crucial: instrumental deconstruction (procedural dimension, defined by Mithalal as the identification of a set of independent figural units, the primitives, and a succession of actions performed through the use of instruments, that are used to reconstruct the object itself, or a graphical representation of the object) and dimensional deconstruction (discursive activity on the geometrical properties of the figural units) (Mithalal, 2010).

The reference framework for studying affective processes has been described by a number of authors (Debellis & Goldin, 2006; Goldin, 2000; Gómez-Chacón, 2000a, 2011), who propose regarding affect as one of several internal, mutually-interacting systems of human representation that encode meaning for the individual and can be externalized to communicate meaning to others. Affect includes changing states of emotional feeling during mathematical problem solving (local affect). It also includes more stable, longer-term constructs that establish contexts for and can be influenced by local affect. Known generically as global affect, such constructs include attitudes, beliefs and values.

The present hypothesis is that affect is fundamentally representational, rather than a system of frequently involuntary effects on cognition. Affective pathways are sequences of (local) emotional reactions that interact with cognitive configurations in problem solving. Such pathways provide solvers with useful information, favoring the learning process and suggesting heuristic problem-solving strategies. Prior research (Gómez-Chacón, 2000) identified interactions between affect and reasoning (geometrical visualization as an aspect of mathematical reasoning). The potential for affective pathways was shown to be at least partially inherent in the individual, although the effect of social and cultural conditions was also discussed. The present study focuses on the individual and any local or global affect appearing in classroom mathematical problem solving or observed by interviewees. That same procedure was adapted for use in technological environments, where characterizing affective competencies is meaningful (affective competencies = capabilities that depend on appropriate affective encoding of strategically significant information and instrumental and cognitive skills).

Research methodology

This study is based on design experiment methodology (Cobb et al. 2003) and on educational experiments forming part of a broader research project on the use of GeoGebra in geometric locus exercises.

Data collection

Qualitative research was conducted by observing students during training, performance analysis sessions and video-recorded, semi-structured interviews.

Geometric locus training was conducted in three 2-hour sessions. In the first two sessions, students were asked to individually solve six non-routine geometric locus problems using GeoGebra in accordance with a proposed problem-solving procedure. This included the steps involved, an explanation of the difficulties that might arise, and a comparison of paper-and-pencil and computer approaches to solving the problems. A description of the problems, their graphic solutions and design are given in Gómez-Chacón and Escribano (2011). Two examples (of geometric locus problems) are shown below.

> Problem 2.
>
> Assume a variable line *r* that cuts through the origin *O*. Take point *P* to be the point where line *r* intersects with line $Y=3$. Draw line *AP* from point *A* (= (3,0)) as well as the line *s* perpendicular to *AP*. Find the locus of points *Q*, where lines *r* and *s* intersect as the slope of *r* varies.
>
> Problem 4.
>
> The top of a 5-meter ladder rests against a vertical wall, and the bottom on the ground. Define the locus generated by midpoint *M* of the ladder when it slips and falls to the ground. Define the locus for any other point on the ladder.

Students were also asked to describe and record their emotions, feelings and mental blocks when solving problems on protocols designed for that purpose. The third session was devoted to discussing joint approaches and the difficulties experienced in problem solving. A preliminary analysis of the results reached during the preceding sessions was available during this session.

Analyzing the problem-solving results entailed conducting a thorough study of the subjects' cognitive and instrumental understanding of geometric loci via semi-structured interviews conducted with nine groups of volunteers. The interviews were divided into two parts: problem-based questions, asking respondents to explain their methodologies, and questions designed to elicit emotions, visual and analytical reasoning, and visualization and instrumental difficulties.

Data was collected from the students' problem-solving protocols mentioned above, as well as with two questionnaires, one on beliefs and emotions about visual reasoning completed at the outset and the other on the interaction between cognition and affect in a technological context filled in after each problem was solved.

A model questionnaire proposed by Di Martino and Zan (2003) was adapted here to identify subjects' beliefs about visualization and computers, study their global affect and determine whether a given belief can elicit different emotions from different individuals. In this study, students were asked to define their feelings about a series of beliefs (see figure 1).

Visual reasoning is central to mathematical problem solving.
Visual reasoning is not central to mathematical problem solving.
Give reasons and examples. How do you feel about having to use problem representations or visual imagery?
I like it   I dislike it   I'm indifferent
Explain your feelings.

Figure 1. *Example of questionnaire items on beliefs and emotions*

A second questionnaire, drawn up specifically for the present study, was completed at the end of each problem. The main questions are listed in figure 2.

Data analysis

In the present exploratory, descriptive and interpretative study, data analysis was mainly inductive, with categories and interpretation building on the information collected (previous section). For instance, subjects' visualization, use of imagery and emotions as reported on their solution protocols were identified for each problem. This information is summarized in schemes such as outlined in table 2.

The interviewees' data was also analyzed for the role of affect in the representational system and the considerations described in section two. The data was aggregated and reviewed to identify emerging themes on affect and teacher trainee decision-making in connection with the use of visual reasoning to solve problems with GeoGebra. The post-problem

Please answer the following questions after solving the problem:

1. Was this problem easy or difficult? Why?
2. What did you find most difficult?
3. Do you usually use drawings when you solve problems? When?
4. Were you able to visualize the problem without a drawing?
5. Describe your emotional reactions and specify whether you hit a mental block when doing the problem with a pencil and paper or with a computer.
6. Which of the following routes best describes your emotional pathway when solving the problem? If you identify with neither, please describe your own pathway.

   Affective pathway 1 (enabling problem solving):
   curiosity → puzzlement → bewilderment → encouragement → pleasure → elation → satisfaction → global structures of affect (specific representation, general self-concept structures, values and beliefs).

   Affective pathway 2 (constraining or hindering problem solving):
   curiosity → puzzlement → bewilderment → frustration → anxiety → fear/distress → global structures of affect (general self-concept structures, hate or rejection of mathematics and technology-aided mathematics).
7. Specify whether any of the aforementioned emotions were related to problem visualization or representation and the exact part of the problem concerned.

Figure 2. *Student questionnaire on the interaction between cognition and affect*

protocols and interview data were compared for each case and among cases throughout to determine the consistency of the emerging themes.

Results

The construction of the reported affective pathways for all six problems revealed that two themes consistently emerged in the group of 30 future teachers. First, the local affect involved in the use of imagery in computer-based problem solving was observed to be influenced by:

1. The balance between analytical-algebraic and graphic analysis in the subjects' reasoning process
2. Subjects' understanding of the tools and instruments at their avail and skill in instrumentally deconstructing geometric figures through non-iconic visualization for the visual identification and handling of images in a DGS.

And, second, evidence was found that teacher trainees' affect was linked to motivation in the form of goals and self-concept.

Visualizing activity and productive affective pathways: contexts

The three cases chosen for the detailed description of the findings that follows illustrate the consistency of the thematic analysis. The introduction to each case includes an explanation of why it was chosen. The criteria used are shown in the top row in table 1.

Table 1. *Three case studies: characteristics*

Case	Gender	Mathematical achievement	Visual cognitive style	Beliefs about computer learning	Feelings about computers	Beliefs about visual thinking	Feelings about visualization
TT 19	Male	High	Visualizing	Positive	Like	Positive	Like
TT 20	Female	Average	Non-visualizing	Positive	Dislike	Positive	Dislike
TT 26	Male	High	Style not clear	Positive	Indifference	Positive	Like

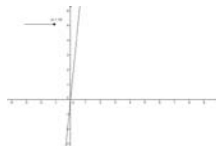
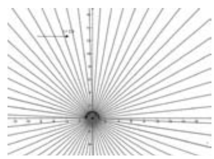
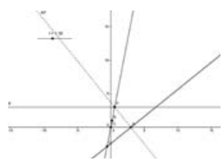
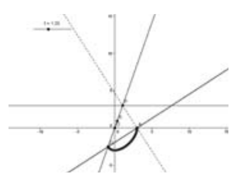
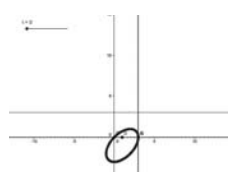
Teacher trainee 19 (TT-19) is a visualizer. In the interview he said that the pleasure he derives from visualization is closely associated with a mathematical vision. He regards visual reasoning as essential to problem solving to generate in-depth learning, contribute to the intuitive dimension of knowledge and form mental images. A summary of the protocol he followed to solve problem 2 is given in table 2.

Teacher trainee 20 (TT-20) is a non-visualizing thinker with positive beliefs about the importance of visual reasoning, whose case is discussed at greater depth in the next section.

Teacher trainee-26's visual thinking style could not be clearly identified. He expressed a belief in the importance of visual reasoning, confirming his liking for visualization and representation because it made it easier to understand problems. While he thought the use of axioms and theoretic formal symbolism to process the mathematics was helpful for solving it, he added that he felt indifferent about applying technological software to mathematics, although he believed GeoGebra, specifically, to be useful.

A deeper understanding of the relationships between instrumental dimension and visualization was sought by analyzing problem 2 as broached by TT-19 and TT-26, who did not use the "locus command" to solve it. The reason was that they found it difficult to articulate their instrumental and visual processes. The instrumental deconstruction strategies and starting points observed had different consequences for the subjects' emotional reactions.

Table 2. *Analysis of TT-19's solution process, use of imagery and affective pathway according to his own protocol*

Method described by the subject	Representation/image typology	Affective pathway reported by the subject
(1) The problem piqued my curiosity. With the change in slope on line r, the r-s intersection would form a conic. But what kind? A circle or an ellipse?	Analytical (search for mental image: specific figure/illustration and dynamic image)	Curiosity
(2) My first mental block came when I tried to figure out how to represent the line with a variable slope. Initially I established a numerical slider with high end values. But then I realized that the line would never be vertical and wouldn't take in all the possible slopes. I remembered the preceding exercise and decided to use a point that runs along a circle as support.	 Kinesthetic technological manipulation. "Variable line" concept (specific illustrations)	Confusion Not knowing how to continue Puzzlement
(3) I got my confidence back. If I built a circle with its center at the origin, ran a point around it and drew the line that passes through that point and the origin, I'd have worked through this first mental block. A half-circle sufficed, i.e., the slider values ranged from 0 to π.	 Interactive image generation, analogue slider	Encouragement Confidence
(4) I represented point P where line r intersects with line y=3. I drew line AP from point A=(0,3), and s, the line perpendicular to it.	 "Variable line" concept	Confidence
(5) I identified point Q as the intersection between lines r and s. I activated the tracker for point Q to view its path as the slope on line r varied.	 Interactive image generation, analogue slider	Joy
(6) The geometric locus for points Q was an ellipse, as I knew intuitively from the start.	 Specific illustration with interactivity (Analogical)	Satisfaction

Problem 2 (see examples above), must be translated into a functional model to be solved with GeoGebra, although it is not expressed in those terms. Such a functional approach calls for implementing a DGS. Students must be given clues about the need to use instrumental deconstruction and non-iconic visualization to solve the problem. The difficulty in problem 2 is determining what is meant by a ”variable line”. It could be assumed to be a sheaf of lines passing through the origin and an external point. To use the locus function in GeoGebra, this external point must lie on an auxiliary object, a line or a circle (or in GeoGebra 4.x, a slider): it may not be a free point. This non-iconic object is needed to induce real mathematical reasoning about the locus. For example, taking a point on the line $y=-1$, the family of lines would be the lines running through the origin and the moving point. A ”variable line” is not just an unstructured set of lines, but must be determined precisely and systematically to use a DGS. In this example, the lines pass through the origin and a point on an auxiliary line. With the ”free” point so confined, the rest of the construction is straightforward: the locus tool can

Table 3. *TT-26's description of his affective pathway for problem 2*

Affective pathway	Cognitive processes
Curiosity	When reading the problem
Confusion	When calculating the point Q locus using paper and pencil and GeoGebra
Puzzlement	When unable to solve instrumental and dimensional deconstruction with GeoGebra
Stimulus, encouragement	When drawing with GeoGebra
Pleasure	When drawing and constructing
Joy	When calculating with pencil and paper
Puzzlement	When unable to get the right solution after operating During diagrammatic visualization
Anger, frustration	When not finding the locus analytically and not knowing what GeoGebra displayed
Stimulus, encouragement	When recognizing the solution from the track drawn by the software, which was an aid to calculating the locus
Frustration	When not finding the locus analytically and not knowing what was displayed
Anxiety	When not finding the locus analytically and not knowing what was displayed
Distress	When not reaching an analytical solution or a solution with the "locus" command

be used and yields an ellipse. By contrast, if this problem is broached from the iconic/non-iconic visualization standpoint, the imprecisely defined ”variable line” concept yields no meaningful result.

These teacher trainees' cognitive-emotional processes are listed in tables 2 (TT-19) and 3 (TT-26). An analysis of the relationship between these two students' affective pathways and their cognitive visualization shows that visualization-averse feelings stem essentially from their lack of familiarity with the tools, a want of resources in their pursuit of computer-transferable analogue images and their switch from a paper and pencil to a computer environment to interpret mathematical objects.

In TT-19's experience, a physical form was reproduced with explicit visual/perceptive control and implicit theoretical control. The specific auxiliary objects were controlled both via a mathematical and technical-instrumental command of the software. The second teacher trainee, TT-26, used mainly symbolic representation for the geometric object and non-visual representation (figure 3). TT-26's negative feelings emerged not only during interactive construction, but also when he tried to progressively schematize visualization. He thought he could systematically

If $r \equiv y = mx \qquad m\epsilon R, \{0\}$

If $Y = 3 \to r \cap Y = \{(x, y/3 = y = y = mx\} = \{(x,y)/3 = mx\} = y = 3$

$= \{(x,y)/x = 3/m\} = \left\{(\frac{3}{m}, 3)\right\} \quad y = 3$

$\to P = r \cap Y = (3/m, 3)$

Now if $A = (3,0)$, $\overrightarrow{AP} \equiv y = \frac{3-0}{\frac{3}{m}-3}x + c \qquad \to y = \frac{1}{\frac{1}{m}-1}x + c$

Entering $A \to 0 = \frac{1}{\frac{1}{m}-1} + 3 + c \quad \to \overrightarrow{AP} \equiv y = \frac{1}{\frac{1}{m}-1}x - \frac{3}{\frac{1}{m}-1}$

Calculating perpendicular line $s \equiv y = -\left(\frac{\frac{1}{m}-1}{1}\right)x + k$

$\to S \equiv y\left(1-\frac{1}{m}\right)x + k$, entering $A \to \; 0 = \left(1-\frac{1}{m}\right) 3 + k \to$

$\to S \equiv y = \left(1-\frac{1}{m}\right)x - 3\left(1-\frac{1}{m}\right) \quad \to S \equiv y = \left(1-\frac{1}{m}\right)x - 3\left(1-\frac{1}{m}\right)$

$\to r \cap s \equiv \left\{(x,y)/y = mx = y = \left(1-\frac{1}{m}\right)x - 3(1-1/m)\right\} =$

$mx = \left(1-\frac{1}{m}\right)x - 3\left(1-\frac{1}{m}\right) \to m^2 x = (m-1)x - 3(m-1)$

$\to (m^2 - m + 1)x = -3(m-1) \to \quad x = \frac{3(1-m)}{m^2-m+1}// \qquad y = mx = \frac{3(m-m^2)}{m^2-m+1}//$

$Q = \left(\frac{3(1-m)}{m^2-m+1}, \frac{3(m-m^2)}{m^2-m+1}\right)//$

Figure 3. *TT-26's solution as reported in the protocol*

and analytically transform the drawings into meaningful structured representations. Frustration and anxiety ensued due to a lack of balance between the instrumental and visual dimensions. While axiomatic control initially existed (figure 3), there was no visual control or any real control of the dynamic phenomenon. In addition, in the interview he said:

> Then I … I found … point Q, but no … I got totally … frustrated because I couldn't see how it would form an ellipse. Then I tried to prove it theoretically, but couldn't. That led me to stimulus-encouragement because I'd proven it here and I thought, "OK, the ellipse" and I discovered … which ellipse it was. But even then I tried again and again to prove theoretically that it formed this ellipse … but I failed. I knew it was right, because I proved it, but … in the end I couldn't prove it theoretically. I only … only calculated point Q. And I knew that it formed an enclosure, with 5 points … 5 points, but then I couldn't …

This transformational visualizing context would enable TT-26 to use visual images "to see the unseen", but as he did not find the solution he felt distressed because he was unable to replace his initial static with abstract dynamic perception.

Global affect and productive affective pathways

The belief expressed by the group that visual thinking is essential to problem solving and that dynamic geometry systems constitute a visualization aid influenced their pathways considerably. The data showed that all teacher trainees felt that visual thinking is essential to solving mathematical problems and 80 % believed it to be reliable, speedily executed and useful for developing their intuition and spatial vision. They added that the approach helped them work through mental blocks and enhanced their confidence and motivation. As future teachers they stressed that GeoGebra could favor not only visual thinking, but help maintain a productive affective pathway. They claimed that working with the tool induced positive beliefs towards mathematics itself and their own capacity and willingness to engage in mathematics learning.

That belief enabled students to maintain a positive self-concept as mathematics learners in a technological context and to follow positive affective pathways while solving each problem, despite their adverse feelings at certain stages along the way and their initial lack of interest in and motivation for computer-aided mathematics.

Student-20 exemplifies these attitudes. She was found to be a non-visualizing thinker with positive beliefs about the importance of visual reasoning (table 4). Neither her motivation nor her emotional reactions to the use of computers were favorable, although she claimed to have discovered the advantages of GeoGebra and found its environment friendly. She also felt more confident with GeoGebra's dynamically visible solution than with manual problem solving.

Table 4. *Summary of TT-20's representation of her affective pathway for problem 4*

Affective pathway	Cognitive processes
Curiosity	Reading the problem
Frustration	Global visualization of problem Pictorial image
Confusuion	Search for mental image Inability to visualize the ladder as the radius of a circle
Puzzlement	Dynamic and interactive image No command of instrumental technique
Stimulus, encouragement	Technological manipulation Pictorial representation with GeoGebra
Satisfaction	Pictorial representation with the GeoGebra tracking function Full construction from scratch

In addition to the affective pathway reported in table 4, TT-20 described her difficulties as follows:

> This problem struck me as unusual and interesting, although I didn't know where to start. I tried a couple of approaches [like trying to fit a segment with a fixed length of 5], but I couldn't solve the problem. The idea of using a circle was the key to getting the solution. Suddenly I could see the ladder as one of its radii. I used the 'geometric locus' option and tracked M to get the solution, one-fourth of a circle ... I was frustrated because even though I liked the problem I didn't know how to solve it. Elated when I found I could solve it with the circle.

Convincing trainees such as TT-20 that mathematical learning is important to teaching their future high school students helps them retain a positive self-concept, even though they feel unsure in some problem-solving situations. TT-20's is an exemplary "productive pathway" which, despite negative feelings, keeps students' global affect in positive self-concept mode in the learning process as a whole. A clear relationship among objectives, purposes and beliefs was found in the affect-cognition interaction (Cobb, 1986; Hannula, 2002). In her own words:

> I think that computers, not only GeoGebra software, provide excellent support for any mathematics student. Today the two are closely linked. I mean, if you study mathematics you use a computer at some point ... mathematics is closely linked to computers and, specifically, to programs like GeoGebra [if you want to teach high school math, for instance].

Conclusion

The results of this study suggest that various factors are present in visual thinking. Firstly, an "appropriate" geometric work space for locus exercises appears to be particularly unstable and dependent upon students' visual cognitive style and beliefs about mathematics learning in computerized environments.

Secondly, certain tendencies were observed in the cognitive–affective states in the visual solution of geometric locus problems using GeoGebra in the study group. According to the data, students experienced positive feelings in the initial stage of problem solving, when seeking an image that would describe the structure sought; or of satisfaction and happiness when they were able to construct an interactive image. However, the attempted generation of interactive images, the use of analogue visualization and progressive schematization based on the balance between dimensional deconstruction (expressed as the analytical-algebraic analysis underlying the representation) and an understanding of the instruments used in the instrumental deconstruction of geometric figures (expressed as non-iconic visualization for the visual identification and handling of images in a DGS), prompted confusion, mental blocks and frustration. A fair share of students lacked a heuristic strategy for the geometric interpretation of visualizations that led them to a functional view of geometric locus.

In addition to providing helpful hints on the support that can be established for students learning to use technology, the data in this study reflects the instructor's role as epistemic mediator in visual mathematical thought, contributing to the development of PTK (pedagogical technology knowledge). The findings suggest that one way to help students stay on productive affective pathways is to provide the ancillary support needed to favor visualization and a heuristic strategy for visual reasoning. This calls for an understanding of the properties that link the operations involved, either to build or transform the figure in question.

The present educational experiment enabled teacher trainees to develop structural awareness associated with abstract thought and learn how to help others progressively master visual thought (Rivera, 2011). That in turn raises awareness of the transitions taking place from icon to symbol, and changes mindsets via the acknowledgement that affective representation can favor success in mathematics. For the geometric locus problems posed, each transition can be associated with explanations and symbolic mathematical notation as well as the efficient use of a visual tool for reifying mathematical abstraction.

One question stemming from this study that merits further research is the definition of the constituents of an overarching theory of visualization for solving problems in technological environments where, requisite to a progressive command of the visual approach, is an explanation of its cognitive and affective features.

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