Interference of subtraction strategies

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This study concerns a particular kind of mistake that a number of pupils made when subtracting two positive whole numbers. The aim was to analyse the cause behind this particular mistake. According to the pupils, the difference was equal to the subtrahend. It was found that the pupils counted down to the subtrahend. But instead of finding the answer as the number of steps between the two terms, the pupils applied the last-number-word rule and gave the subtrahend, which was the last mentioned number word, as the result. When seeing subtraction as a concept, it could be assumed that the lack of experience of subtraction as a comparison and as equalization played a decisive role for this mistake. A comparable mistake described in previous research is also analysed.

Primary-school-children's mistakes when adding or subtracting positive whole numbers are not as random and unsystematic as they may seem. It has been shown that the causes behind these mistakes can be hidden by consistent reasoning, which is, however, based on misconceptions. This circumstance could make formative assessments of the causes behind these mistakes relatively much easier. A 500-interviewee-study comprising two schools on the west coast of Sweden showed that a number of pupils believed that 16 – 6 = 6. Given a number of similar items, these pupils came up with the same answer. Therefore, it is possible to assume that the mistakes are not fortuitous but are instead based on misconceptions. In this article the roots of this kind of mistake will be analysed and compared to the result of a study with quite a different approach.

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Research review

The development of the understanding of the number concept passes several crucial stages. Fuson (1988, 1990, 1992) has reviewed and described these stages of which there is a consensus among most mathematics educational researchers. She characterised the first stage as "Many children learn the sequence of words to twenty as a rote list of meaningless words, much like the alphabet" (1992, p. 132). One of the next stages in the development of the understanding of numbers is the transition from the meaning of counting words to the cardinal meaning. According to Fuson (1992), it could be assumed that this transition (from the ordinal to the cardinal meaning) initially is learnt procedurally as a rule, the last-number-word rule.

Hitherto the child has been able to add two sets of objects by using the *counting-all strategy*, which according to Fuson (1992) only requires a transition from the counting meaning to the cardinal meaning. To be able to start counting from one of the terms in an addition, the child first needs to go from the cardinal to the counting meaning. Secondly it needs to count on and finally to return to the cardinal meaning. At this stage, which is characterized as a counting-on strategy, the child does not necessarily experience the two aspects, ordinality and cardinality simultaneously. The child masters, however, the transition in both directions (Fuson, 1988, 1990, 1992). When for instance adding 5 and 3, the cardinal meaning of 'five' is converted into its counting meaning. Then the counting continues with 6, 7 and 8. The counting stops at eight and a transition from the counting meaning into the cardinal meaning takes place anew and the sum eight is arrived at. This stage, *Breakable chain*, in Fuson's description of children's development of number sense, includes addition and subtraction with positive integers.

The *Numerable chain* stage is characterised by the fact that the number words not only continuously but also simultaneously have both a counting (ordinal) and a cardinal meaning. This insight makes keeping track possible. As a track-keeping method, finger-counting is frequently used. At this stage, it is also possible for the child to leave the concrete level and perform calculations mentally (Fuson, 1988, 1990, 1992).

Subtraction strategies

It is possible to distinguish three different subtraction strategies, two counting-down strategies and one counting-on strategy. In these strategies it is helpful to distinguish between the *number sequence* and the *tracking sequence.* The number sequence is equal to the sequence of the number words, in contrast to the tracking sequence, which emerges when the child needs to keep track.

Counting-down strategies

When the first version of the counting-down strategies is being applied, the tracking goes in the negative direction. The counting-down starts at the minuend 11 and goes 4 steps down to seven by means of the tracking sequence and the result 4 is to be found in the tracking sequence.

$$11 - 7 = \left[11 \stackrel{4}{\rightarrow} 7\right] = 4$$

Thus the counting-down starts at 11 and goes down four steps to 7: 11_1, 10_2, 9_3 and 8_4. Consequently the result 4 is reached in the tracking sequence. So, the child experiences the correct answer in the tracking sequence by letting the subtrahend represent the last number in the number sequence.

However, counting-down process, could be carried out in another way, in which the answer is found in the number sequence:

$$11 - 7 = \left[11 \stackrel{7}{\rightarrow} 4\right] = 4$$

Anew the counting-down starts at 11 and now it goes down to the difference, 4, 11_1, 10_2, 9_3, 8_4, 7_5, 6_6 and 5_7. The result is to be found in the number sequence reached by the tracking. In this case, the counting (ordinal) and the cardinal aspects in the number sequence are taken advantage of, since the answer is found in the number sequence.

These last two strategies are basically different. In the first, the answer is to be found in the tracking sequence, while in the second it is to be found in the number sequence.

Counting-on strategy

When calculating 11 – 7 it could also be seen as an addition 7 + ? = 11. The counting-on starts at 7 and the child keeps track after having started from seven, 8_1, 9_2, 10_3 and 11_4. The result is four and the tracking sequence is 1, 2, 3 and 4. It is worth noticing that the counting and the cardinal aspects of the first term concern the number sequence and of the second term four, the tracking sequence.

Subtraction: situation and procedure

Sometimes subtractive situations in text problems are seen as equivalent to subtraction as a concept. Subtraction could then be understood in three different ways (Fuson, 1992). Let some text problems illustrate this.

Tom has 51 zeds and looses 2. How many zeds has he left?

This problem represents a subtractive change-take-away situation (Fuson, 1992) and subtraction is understood accordingly. The 2 zeds are taken away from 51 and the calculation is consequently performed by a counting-down strategy, by two steps. The answer is to be found in the number sequence.

The second way of seeing subtraction as a concept is called comparison (Fuson, 1992). Such a situation is represented in the following problem:

> Tom has seven marbles and Peter has nine. How many more has Peter?

In this text problem the focus is on the difference between the two numbers, actually the number of steps between them. This could correspond to a counting-down strategy starting from the minuend and the number of steps between the two numbers is the answer, found in the tracking sequence.

The third way of seeing subtraction conceptually is termed equalization (Fuson, 1992), which is reflected in the following text problem:

> Tom has seven marbles and Peter nine.
> How many more must Tom get to have equally many?

In this situation the calculation can be performed as a counting-on situation. The counting starts at seven and goes on to nine and the number of steps between the two numbers is the answer.

This kind of identification between procedures and concepts is, however, not unproblematic. Sometimes the counting strategy becomes highly inadequate.

> Tom has 51 zeds and looses 49. How many has he left?

This is a change-take-away situation and according to the identification the calculation should then be carried out as a counting-down strategy, the number of steps equal to the subtrahend, which means counting down 49 steps. A more adequate strategy would be to count down to 49 and the answer is then the number of steps between 51 and 49, which lies in the tracking sequence. Consequently, in this calculation, subtraction is seen as a comparison despite the fact that the situation in the text problem is a subtractive change-take-away situation.

In a comparison situation the calculation can differ from the situation in the text problem. If Tom has 51 marbles and Peter 3. How many more has Peter? A more adequate calculation than counting the number of steps between 51 and 3 would be to count down tree steps from 51.

Also in an equalization situation the calculation strategy can differ. If Tom has 51 marbles and Peter 3. How many must Peter have to an equal number of marbles? An identification of the situation and the calculation strategy implies that a counting-on strategy will be applied from 3 up to 51. It is probably much easier to count down three steps from 51. According to the analysis of the TIMSS-results (Skolverket, 2008), conceptual teaching seemed to be rather rare in Swedish schools. Therefore, disregarding the complication mentioned above, subtraction as a concept can cast light over this kind of mistake.

A different experiment problem – explained in a similar way

In the analysis of quite a different situation the result seems to end up in quite a different explanation. However, it will be shown that the problems could be explained in similar ways. Neuman (1987) described a situation in her attempt to develop a Phenomenographic theory of how children acquire the number concept she interviewed about 60 seven-year-old children. The children were encouraged to guess the numbers of marbles in two boxes. Altogether the two boxes contained 9 marbles, a fact that the children were fully aware of. One child replied: ”2 in one box and 9 in the other”. In her analysis, this child was found to envisage the 9 marbles in a row and to count them. Neuman found the transition from the ordinal to the cardinal aspect for the first part of marbles unproblematic. When the child continued to count the second part of the marbles, however, the number word ”nine” was attached to the last marble. So, the number of marbles the child arrived at in the second part was nine. According to Neuman´s interpretation, the child focused the counting (ordinal) aspect of the number concept, even if still aware of the part-whole relation. She labelled the category ”Numbers as names” and regarded it as a qualitatively low category, since only two of the aspects, ordinality and part-whole relation, was in focal awareness.

So, the two different counting-down strategies, the one in which the counting goes down to the subtrahend and the one in which the counting goes down to the difference are of major interest for the analysis of the problem mentioned above and for the mistake described in the present study. Thus the analysis applied to this kind of mistake could also be applied to the quite different mathematical problem in the study carried out by Neuman (1987).

Theoretical framework

In the present study, phenomenography was the theoretical framework extended into the post-positivist paradigm. The ways phenomena are experienced can be studied within phenomenography. Since phenomena can be both a concept and a procedure, the understanding or the application can be focused. It is the differences in the ways a phenomenon is experienced and not the phenomenon per see that is of particular interest. This is seen as the second order perspective in contrast to the first order perspective, in which the phenomena themselves are focused. So, within phenomenography, the world around us is seen from the learner's perspective. The variation in the different ways of understanding a concept or of applying a procedure is in the centre of attention. The descriptive categories give a description of the different ways a concept is experienced. Studies are carried out at group level, since the data are more stable there. The categories are seen as not yet falsified hypotheses and are the best descriptions so far. This makes it possible to see phenomenography as a theoretical framework within the post-positivist paradigm. According to the epistemology of this paradigm, scientific knowledge is seen as probable but not yet falsified hypotheses. Phenomenography, however, does not hold a complete set of basic philosophical assumptions. For instance, the description of the role of the researcher is not complete.

By seeing phenomenography as a framework within the post-positivist paradigm, a complete set of the necessary philosophical assumptions becomes available. It is also possible to return from group level to individual level and to describe each individual's exposed understanding of a concept or his application of a procedure. An individual can hold more than one way of understanding a concept including fragments of ways of understanding. It is assumed that an individual's description of the ways he understands a concept or applies a procedure reflects his ways of thinking. For that reason, the transcribed interviews are seen as the basic data. The data are analysed by means of comparative analysis and the differences in the ways of experiencing aspects of the world are focused (Marton & Booth, 2000; Bentley, 2008).

One of the key issues of the ontological assumptions is that the world is not divided into two parts, one world possible to experience and the other that is experienced. So, the one and only world contains both the experienced world and the world possible to experience. The ontological assumptions focus on the ways the world is experienced and the epistemological assumptions comprise the descriptive categories of the ways the world is experienced. Consequently, the ontological and the epistemological assumptions are close but still possible to distinguish (Marton

& Booth, 2000; Bentley, 2008). Each individual's exposed understanding of a concept and exposed application of a connected procedure can therefore be seen as a case (Stake, 2000).

Method

The interviewed grade-two-pupils belonged to five different classes on the west coast of Sweden. The present study was part of a larger interview study of all the pupils of two schools. As a whole, the larger study aimed at discovering misunderstandings that could constitute obstacles in the pupils' mathematical development. Traditionally, theoretical sampling is the basis for selecting interviewees within the phenomenographic framework. In the present study, the main interest was the ways of understanding of the particular cases that is each one of the pupils who got the difference of a subtraction equal to the subtrahend. Those pupils, who gave this kind of answers, were selected for the analysis.

The interviews, which were semi-structured, took place in a calm separate room familiar to the pupils. The participating pupils, who were asked to solve specific items like 16 – 6, 16 – 7, 21 – 19, 19 – 8 and 11 – 8, were told to describe their ways of reasoning when solving the items. Follow-up questions were put in order to get as much information as possible about the pupils' ways of reasoning. Plenty of time was given to the pupils to explain their thinking. The interviewer was well-trained and did not reveal any expected answer by mimics or questions. Each interview lasted for about one hour on average.

Having transcribed the recorded interviews, they were analysed regarding exposed procedures and understanding of the number concept. The pupils were given fictitious names.

According to Stake's (2000) terminology, a collective case study, as in the present study, aimed at investigating a phenomenon in a number of cases in order to provide insight.

Result

The difference of a subtraction was found to be equal to the subtrahend. Those pupils, who exposed this way of reasoning, seemed to follow a consistent pattern. Three pupils' reasoning will illustrate the phenomenon. Being asked to calculate a number of subtractions like 16 – 6 the answer given was six. When asked to describe their ways of reasoning, three of the pupils demonstrated how they counted down to six and did not pay any attention to the fact that they counted down 10 steps. Instead, they believed the last mentioned number word, the subtrahend six, to be the correct answer.

Two pupils, Gina and Siv, gave consistent answers to several test items during the interviews. When asked about the subtraction 16 – 6, Gina explained: "It is six. I start on 16 and count down to six: 16, 15, 14, 13, 12, 11, 10, 9, 8, 7, and 6". Gina counted down to six in the number sequence. If she had performed the calculation correctly, she would have found the answer 10, which are the steps of the tracking sequence. On the contrary, she seemed to apply the last-number-word rule and got the answer six, which is the last mentioned number word in the number sequence.

Two other pupils, Britt and Gill, reversed 21 and got 12 and perceived the test item 21 – 19 as 19 – 12. They did not seem to discriminate the order of the numbers in the subtraction. For the girls 12 – 19 was the same as 19 – 12. A quotation from Gill is given to illustrate this way of thinking. It needs to be pointed out that Gill also applied the last-number-word rule as a transition from the counting meaning to the cardinal meaning. She said: "It is 12. I had 19 and counted down to 12: 19, 18, 17, 16, 15, 14, 13, 12". So, Gill gave the last counted number in the number sequence as her answer. Gill, Gina and Siv seem to perform the calculation in the same way.

In one of the other classes, an application of the last-number-word rule seemed to be implicitly exposed. When invited to solve the subtraction 16 – 7, Gunnar, one of the pupils, said: "When the text says seven, I know that the result is seven". Being asked to solve 19 – 8, he continued: "It is eight. I told you before. When I see that there are eight (pointing at the subtrahend), it is eight". Given the task 11 – 8, he said: "It is eight. I can see that it is eight, and then it is eight more". In these three consecutive test items, the subtrahend was seen as the answer. Gunnar did not expose any counting-on or -down process. He seemed to expose an automatized way of calculation the items.

Analysis and discussion

Two possible explanations are possible to find, one in a procedural perspective and one in a more conceptual perspective. Taking a procedural perspective, that some of the interviewed pupils did, was to mix the two different ways of counting-down. Gina counted down to the subtrahend 6. Instead of realising that the answer was to be found as the number of steps between the two terms (tracking sequence), she arrived at 6, which is the last mentioned number word (number sequence).

Condensing the outlines in the research review, the subtraction 16 – 6 is possible to calculate in two procedurally different ways. In both ways a counting-down strategy is applied, as already accounted for. In the first way the counting-down goes down 6 steps, the subtrahend and the difference 10 is to be found in the number sequence:

$$16 \overset{6steps}{\longrightarrow} 10$$

Hence, the application of the last-number-word rule is correct. The second way is characterized by an application of the other counting-down strategy:

$$16 \overset{10steps}{\longrightarrow} 6$$

In this case, the subtrahend 6 is in the number sequence and the difference 10 is to be found in the tracking sequence. The pupils seemed to mix the two strategies and did not arrive at the number of steps between the two terms but at the last mentioned number word 6.

Departing from this procedural perspective and taking a more conceptual one, the mistake could also be seen as a lack of experience of subtraction as a comparison and as equalization (Skolverket, 2008). It could be assumed that the pupils more frequently have experienced subtraction as a change-take-away situation (Skolverket, 2008). If the pupils see the problem 16 – 6 as a comparison and the comparison is not fully grasped. It is possible that the two ways of understanding, subtraction as comparison and as change-take-away, may interfere. It seems as if the pupils start with a comparative view, by which the answer is to be found as the number of steps between the minuend and the subtrahend that is in the tracking sequence. But when they try to find the answer they seem to switch to the change-take-away understanding and get the answer from the number sequence instead of in the tacking sequence. So, the lack of experience of especially subtraction as comparison may cause that the two ways of understanding interfere and the final part of the calculation is performed as if it were a change-take-away understanding (Skolverket, 2008).

Reanalyzing Neuman's (1987) theory based on her experiments with marbles in two boxes (9 marbles), it could be suspected that the child came to the pattern (2 and 9 marbles) with the help of the last-number-word rule. Applying this rule is not necessarily the same as understanding the cardinal aspect of numbers. When envisaging the nine marbles placed in a row, the child seemed to apply the last-number-word rule without hesitation. Counting the rest of the marbles, the child started at two and stopped at nine. Since the rest was found to be 9, the application of the last-number-word rule seemed to be more or less mechanical. As the last number word was 9, the child gave that as an answer for the number of marbles in that part. Swedish children of this age are probably more used to find the answers of additions and subtractions in the number sequence in contrast to the tracking sequence. However, as the child managed to count the first four marbles correctly, a beginning understanding of the cardinal aspect was exposed.

Another way to reanalyze Neuman's problem is to apply an alternative conceptual perspective. First let's assume that the pupils saw the problem as an equalizing situation and accordingly they saw subtraction as an equalization and they consequently applied a counting-on strategy. It should be noticed that Swedish pupils have little or none experience of subtraction as equalization (Skolverket, 2008). Neither in this case can it be excluded that this understanding interferes with subtraction as the change-take-away conception. Accordingly, the answer is to be found in the number sequence instead of in the tracking sequence as would be the case with subtraction as equalization.

From the present study follows that this reanalysis of Neuman's experiment admits an alternative interpretation of the pupils' reasoning. The most characteristic features of the interpretation are the procedural applications of last-number-word rule and the interference of subtraction as equalization and as change-take-away.

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