# Using strands of tasks to promote growth of students' mathematical understanding

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This article reports on the mathematical activity of a group of five high school students (15–16 year olds) who worked together on a series of challenging task in combinatorics and probability. The students were participants in an after-school, classroom-based, longitudinal research on students' development of mathematical ideas and different forms of reasoning in several mathematical content strands. The purpose of the article is to contribute insights into how to promote growth of students' mathematical understanding through problem-solving activities. In particular, the article shows that problem-solving activities involving strands of challenging tasks have the potential to promote growth of students' mathematical understanding by providing opportunities for students to engage in reasoning by isomorphism. This is a type of reasoning whereby students rely on structural similarities, i.e., isomorphism among mathematical tasks to solve or deepen their understanding of the tasks. Implications for classroom teaching, and environmental conditions that promote reasoning by isomorphism are also discussed.

Problem solving plays an important role in mathematics education. In the United States, the *Principles and Standards for School Mathematics* (NCTM, 2000) of the National Council of Teachers of Mathematics state that by the time students complete high school they should be able to solve problems that arise in mathematics or related disciplines, apply a variety of appropriate strategies to solve problems, and monitor and reflect on the process of problem solving. In particular, the standards state that throughout the mathematics curricula, students should be able to build new mathematical knowledge through problem solving.

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The main reason for emphasizing problem solving in the mathematics curricula is the consensus that it has the potential to foster growth of students' mathematical understanding (e.g., Cai, 2010; Lambdin, 2003; Schroeder & Lester, 1989; Van de Walle, 2003). However, problem solving can also promote students' positive dispositions towards mathematics (e.g., Cai, 2003; Carpenter et al., 1998; Veschaffel & De Corte, 1997), enhance transfer, and foster growth of students as autonomous learners (Lambdin, 2003).

The research community has contributed several insights into how problem solving can be used to foster growth of students' mathematical learning and understanding. However, there is a consensus that this issue remains an open question, as more needs to be known about how students build ideas and what conditions support the growth of their mathematical understanding in problem-solving situations (Cai, 2010, 2003). Incorporating problem solving in meaningful ways in the mathematics curricula is still not necessary obvious to mathematics teachers (Cai, 2010). The research community continues to look for contributions on best practices on implementing problem solving in mathematics classrooms. In 2006, the International Commission for Mathematical Instruction (ICMI) organized a study conference in Trondheim, Norway, to discuss the role of mathematical challenge in and outside the classroom. One question proposed for debate was how challenge in mathematical tasks can contribute to mathematical learning (Barbeau & Taylor, 2005). In 2004, the 10th International Congress of Mathematics Education held in Copenhagen, Denmark, included a Topics Study Group on Problem Solving in Mathematics Education. The study group invited contributions on how "to explore the actual mechanisms by which students learn and make sense of mathematics through problem solving and how it can be supported by teachers" (Cai, Mamona-Downs & Weber, 2005).

The purpose of this article is to contribute further insights into how problem solving can be used to promote growth of students' mathematical understanding. The article is based on the analysis of the mathematical activity of five high school students on three challenging tasks from a probability/combinatorics strand. The students were participants in an after-school, classroom-based, longitudinal research on students' development of mathematical ideas funded by the United States National Science Foundation. The tasks were challenging in the sense that the methods used for solving them were not obvious to the students. They had to come up with such methods. The tasks were *problems* rather than *exercises* (Schoenfeld, 1985). This article shows that problem-solving activities organized as strands of challenging mathematical tasks can help promote growth of students' mathematical understanding by way of enhancing students' *reasoning by isomorphism*. This is a form of mathematical reasoning by analogy whereby learners make sense of challenging mathematical tasks by relying on methods used for solving or making sense of other tasks with similar or isomorphic mathematical structures as the given tasks. Implications for classroom teaching and conditions that support reasoning by isomorphism are discussed in the article.

## Theoretical framework

### *Problem solving in mathematics education*

The mathematics education literature distinguishes between two main views on problem solving. One view takes a dualistic perspective on the relationship between problem solving and mathematical learning, isolating the development of problem-solving abilities from the learning of mathematical concepts and procedures. Two similar teaching approaches are associated with this view. In one approach, students are first taught concepts and procedures and then given mathematics problems to practice the content learned. Schroeder and Lester (1989) call this approach *teaching for problem solving* because "Teachers concentrate on how the mathematics being taught can be *applied in the solution* of both routine and non-routine *problems*" (p. 32, emphasis added). Typically, "Students are given many instances of the mathematical concepts and structures they are studying and many opportunities to apply that mathematics in solving the problems" (p. 32). They are expected to gain experience and knowledge of how and when to apply a particular mathematical knowledge. The other teaching approach involves first teaching problem solving as a collection of strategies, such as drawing a picture, guessing and checking, or solving a simpler problem, and then giving students problems to practice the strategies (Cai, 2010). Schroeder and Lester (1989) call this approach *teaching about problem solving* because of the emphasis on problem solving as a body of knowledge. Baroody (2003) describes this approach as "One with its focus on the development of mathematical thinking (reasoning and problem solving)" based on the "assumption that mathematics is, at heart, a way of thinking, a process of inquiry, or a search for patterns in order to solve problems" (p 21). Students are expected to develop expertise in dealing with problematic situations, managing their solving process, and putting forward their thinking. Factors that influence the success or failure in problem solving include (a) heuristic strategies, (b) metacognition or monitoring; (c) control of affects and (d) appropriate beliefs (cf. De Corte, Greer & Verschaffel, 1996; McLeod, Craviotto & Ortega, 1990; Schoenfeld, 1985).

The other view on problem solving in the mathematics education literature does not separate problem solving from mathematical learning and teaching. Problem solving and mathematical learning hold a dialectic or symbiotic (Cai, 2010; Lambdin, 2003) relationship mutually constituting and/or reinforcing each other as students work on mathematical tasks. Mathematical learning and understanding develop in and through problem solving. Problem solving also benefits from mathematical understanding. Schroeder and Lester (1989) call the teaching approach associated with this view *teaching via problem solving*. It typically begins with "A problem situation that embodies key aspects of the topic, and mathematical techniques are developed as reasonable responses to reasonable problems" (p. 33).

The dualistic view on problem solving is the traditional method for teaching mathematics, whereby teachers show and/or tell students knowledge (concepts and/or problem-solving strategies) and students' practice the given knowledge. It does not promote mathematical understanding (Schroeder & Lester, 1989; Van de Walle, 2003) and no research is currently being conducted with this approach as an instructional intervention (Cai, 2010, 2003). In particular, Baroody (2003) claims that *teaching about problem solving* makes learning of mathematical content incidental since "Learning content, such as the formal procedure for multiplying fractions, is secondary to developing children's thinking processes" (p. 22). The dialectic view on problem solving, however, is consistent with the reform movement in mathematics education. It emphasizes students' *building* of knowledge through problem solving, which help promote students' mathematical understanding and "adaptive expertise" (Baroody, 2003). The design of the longitudinal research that provided the context for this study was based on the dialectic view on problem solving. The problem-solving sessions in the longitudinal study became sources of insights into the processes and conditions that support students' development of different forms of reasoning and growth of their mathematical understanding. This article shares some of such insights.

### *Building mathematical understanding in problem solving*

In [teaching via] problem solving, learning takes place as students attemp to solve problems in which relevant concepts and skills are embedded (Lester & Charles, 2003; Schoen & Charles, 2003). As students solve problems, they can use any approach they can think of, draw on knowledge they have previously learned, and try to justify their ideas in ways that they think are convincing (Cai, 2010). They can also come up with mathematical representations (Uptegrove, 2005) and heuristics

(Powell, 2003) that help solve complex problems and thus promote their mathematical understanding. When the problem-solving activity takes place in a social environment, students have the opportunity to share solutions with others, and thus learn mathematics through social interactions, meaning negotiation, and attempts to reach shared understanding. In particular, students can challenge each other's principles and standards of reasoning, which forces them to engage in higher levels of mathematical thinking as they try to clarify, modify or even drop incorrect ways of thinking (Weber, Powell, Maher & Lee, 2008). The result is more sophisticated perspectives on the concepts or ideas that they are building.

When working on mathematical problems, students can go beyond the acquisition of isolated ideas and move towards the development of increasingly connected and complex systems of knowledge (e.g., Cai, 2003; Carpenter, Franke, Jacobs, Fennema & Empson, 1998; Cobb et al., 1991; Hiebert & Wearne, 1993; Lambdin, 2003). This suggests a relational definition of mathematical understanding whereby it is construed as the ability to recognize connections among mathematical ideas. Growth of mathematical understanding is then characterized as the ability to make sense of increasingly connected and complex web of relations among mathematical ideas (e.g. Dörfler, 2000; Lambdin, 2003). In particular, recognizing connections among mathematical ideas can involve the ability to realize that two or more seemingly different mathematical tasks have similar underlying mathematical structures. This enhances reasoning by isomorphism, a particular form of reasoning that allows students to rely on structural similarity among tasks to solve or make sense of a challenging task. Reasoning by isomorphism can be summarized in three main steps. In step one, a student faced with a challenging mathematical task looks for another one with a similar mathematical structure. In step two, the student unveils the structural similarity by mapping the structure of the given task onto the structure of the other task, i.e., by building an isomorphism that relates the structures of both tasks. This is equivalent to translating one task into the other task, i.e., making sure that solving one task is equivalent to solving the other task. Indeed, in step three, student tries to use the methods for solving or making sense of one task to solve or make sense of the other task. This is precisely the power of reasoning by isomorphism: allowing students to solve challenging tasks by relying on methods for solving or making sense of other [isomorphic] tasks that the students are aware of.

At a more refined level, reasoning by isomorphism can help students develop problem-solving schema, an abstract knowledge of underlying similar or isomorphic mathematical structures of a common class of problems (Nunokawa, 2005; Weber, Powell & Maher, 2006). A schema involves three processes that enhance problem-solving skills: *categorizing problems into types* or classes based on their common mathematical structure after reading just a few words of the problem statement, *identifying important aspects of a problem* that should be addressed first, and *retrieving a mathematical technique* (e.g., equations or procedures), suited for solving (parts of) the problem (Weber, 2001; Weber et al., 2006). For example, the Pythagorean theorem *formula* or equation is often used to a solve class or category of problems whose mathematical structure can be reduced to finding sides of right triangles. Similarly, Pascal's triangle is also often used as procedure to solve several counting/combinatorial tasks. One distinction between experts and less successful problem solvers is that the former usually display more sophisticated forms of reasoning by isomorphism or schemata. Therefore, scholars call for more opportunities in the curricula for students to develop powerful problem-solving schema (De Corte, Greer & Verschaffel, 1996; Nunokawa, 2005; Reed, 1999). This article shows that strands of challenging tasks can promote students' reasoning by isomorphism and foster growth of students' mathematical understanding.

### *Conditions that promote growth of mathematical understanding*

The research literature distinguishes between two types of conditions that promote students' mathematical understanding in problem solving. The first typ is the nature of the mathematical tasks used in problem solving. There is agreement that, if tasks are to promote students' conceptual understanding, foster their ability to reason and communicate mathematically, and capture their interests and curiosity, they must be intellectually challenging,(e.g., Cai, 2003, 2010; Marcus & Fey, 2003). Lappan and Phillip (1998) developed a set of 10 criteria that help tasks promote growth of students' mathematical understanding. Cai (2010) argues that researchers and curriculum developers alike agree on the first four criteria: important mathematics in the task, higher-level of thinking, conceptual development, and opportunity to assess learning.

The second type of condition that help promote mathematical understanding in problem solving is the classroom discourse during problem solving. Classroom discourse refers to "ways of representing, thinking, talking, and agreeing and disagreeing that teachers and students use to engage in instructional tasks" (Cai, 2010, p. 3). In particular, Cai (2010) lists three main discourse factors that can promote growth of students' mathematical understanding: (a) providing students with enough time for them to work on the tasks, (b) avoiding removing the challenges from the tasks by telling or showing students how to solve tasks, and (c) listening and asking thought provoking questions.

Research has suggested several ways in which the conditions above influence growth of students' mathematical understanding. Cai (2010) argues that story or word problems have limited impact because they are often not problematic enough for students. However, he recognizes that teachers can modify standard textbook problems in ways that help them promote students' mathematical understanding and problem-solving abilities. Nunokawa and Fukuzawa (2002) call for problem-solving tasks that ask students to justify their solutions, particularly through "proofs that explain" (Hanna, 1995), i.e., proofs that justify why a solution is or is not correct. Nunokawa (2005) suggests selecting situations that bridge the old and the new knowledge, scaffolding students' activities in problem solving, and bringing in appropriate sociomathematical norms such as the expectation that mathematics is a sense making activity (Cobb & Yackel, 1998; Yackel & Rasmussen, 2003).

Research suggests that students develop schemata and related forms of reasoning such as isomorphisms through induction, i.e., "through solving many problems that are related to the targeted mathematical knowledge" (Nunokawa, 2005; p. 329). Reed (1999) recommends greater time spent showing students how problems with different story content may have the same solution. However, he warns that, while this approach can be successful in helping students categorize problems by common solution, it is not as successful in enhancing their ability to use an example solution to solve isomorphic problems. Other researchers have tried to induce schemata through superficially different problems, hoping that students will build schemas as they learn to transfer solution strategies (e.g., Novick & Holyoak, 1991). However, there is evidence that in such cases students have difficulties engaging in such forms of reasoning when *non-superficial problems* are involved (e.g. Lobato & Siebert, 2002). As a result, more recently, there have been suggestions for using complex tasks (see e.g. Weber, 2005; Weber, Powell & Maher, 2006). This was the case in this study which used challenging tasks as a context for a research on conditions influencing students' development of mathematical reasoning and growth of their mathematical understanding.

## Research context

### *Research setting*

The longitudinal research that provided the context of the study reported in this article took place in a working class community in the United States. The study began in second grade and continued through high school and college (e.g., Francisco & Maher, 2005; Maher, 2004, 2005; Powell, 2003). Approximately twenty-five students started the study.

However, over time, the research team followed the mathematical behavior of a focus group that included the five students in this study. The one female and four male students were known as Romina, Ankur, Brian, Jeff, and Mike. The goal of the longitudinal study was to understand how students build mathematical ideas and different forms of reasoning and justification while completing challenging tasks in several mathematical strands. The study differed from teaching experiments where researchers have particular concepts that they want students to construct, often by traversing an anticipated learning trajectory (e.g., Simon, 1995). In the longitudinal study, there were no specific ideas that students were expected to learn nor were there particular stages that they were supposed to traverse. Students' ideas and ways of reasoning were the result of investigations, not of preconceived goals. For this reason, the researchers called the research sessions *learning* rather than *teaching* experiments. The informal, after-school setting with no fixed curriculum allowed such a focus.

The students worked on the mathematical tasks in particular conditions. They were encouraged to work collaboratively in small group and to always justify their solutions. Researchers received their contributions positively and avoided making judgments about their mathematical validity. Instead, they encouraged the students to be arbiters of the validity of each other's ideas based on whether they thought that the arguments presented "made sense." The students were given extended time to work on the tasks, and opportunities to revisit and refine their mathematical ideas and reasoning. The researchers saw their role as facilitating the students' mathematical activity within a constructivist approach. They avoided telling or showing the students, spent most of the time listening to the students' ideas, and tried to promote the students' mathematical reasoning through interventions such as "Tell me more", "What do you mean?", "Why do you think that way?" or "Convince me you got the right answer?" The researchers often moved out of the students' view and learned from studying the videos of the sessions the problem-solving strategies that students had used.

## *The mathematical tasks*

In the longitudinal study, the tasks were challenging, open-ended, well defined, and often involved manipulative objects such as unifix cubes, dice, Cuisenaire Rods, and educational software. The students worked on each task for approximately one hour and a half. However, often they chose to stay longer. Over the course of the study, a large database of videos, students' written work, researcher notes, still photos, and other forms of information about the students' work on the tasks was collected by the research team. The video data, now in digital format, and accompanying metadata are currently archived at the Robert B. Davis Institute for Learning at Rutgers University. As part of the design of the longitudinal study, the tasks were organized into strands, i.e. a sequence of tasks that looked different but had similar underlying mathematical structure. This was intended to enhance the researchers' ability to trace the students' development of particular ideas and forms of reasoning over time (e.g., Maher, 2005; Maher & Martino, 1996). There were several strands. The probability and combinatorics strand included the World Series Problem and two other tasks: Tower Problem and the Pizza Problem. "World Series" is an American name for the playoff games that lead to the champion of baseball league in America. Below is the statement of the three problems in the order they were implemented in the longitudinal study:

**Five-Tall Tower Problem**
Work together and make as many different towers five cubes tall as possible when selecting from three colors. See if you and your partner can find a way to convince yourself and others that you have found all possible towers five cubes tall. Extensions of the problem included increasing the number of colors while keeping the height of the tower and vice versa or establishing lower and upper bound for the number of colors in towers of fixed height (suggested by a student).

**The Pizza Problem**
A local Pizza shop has asked us to help them keep track of pizza sales. Their standard "plain" pizza contains cheese with tomato sauce. A customer can then select from the following toppings to add to the whole plain pizza: peppers, sausage, mushrooms, and pepperoni. How many different choices for pizza does a customer have? List all the possible different selections. Find a way to convince each other that you have accounted for all possibilities. Extensions of the problem included asking students to consider additional pizzas where half of the pizza is a topping and the other half of the pizza is another topping.

**The World Series Problem**
Two teams played each other in at least four and at most seven games. The first team to win four games is the winner of the World Series. Assuming that the teams are equally matched, and there are not ties, what is the probability that the World Series will being won in a) four games b) five games c) six games and d) seven games.

### *Data analysis*

The analysis of the students' work on the three tasks above proceeded in a manner consistent with the Powell, Francisco, and Maher's (2003) methodology for studying students' development of mathematical ideas in problem-solving situations. A central aspect of the methodology is the identification and subsequent articulation of the significance of *solution-critical episodes*. These are instances of students' mathematical behavior that provide insights into their progress or lack of in solving or understanding the task (Maher, 2005; Maher & Martino, 1996). The analysis focused on the students' work on the World Series Problem. The students' mathematical activity on the Tower and Pizza Problems was examined whenever the problems were referenced in the World Series Problem. Four main steps were involved in the analysis: (1) viewing the videos of the students' work on World Series Problem several times to have strong sense of the content; (2) identifying solution-critical episodes; (3) describing the particular isomorphisms, if any, used by the students to try to address mathematical challenge, and (4) discussing analyses of the students' mathematical behavior until disagreements were resolved among the researchers.

## Results

The sections below describes the critical episodes in the chronological order in which they occurred along with accounts of the extent to which the students used reasoning by isomorphism to address challenges they encountered in trying to solve the World Series Problem. In the text, $p(n)$, ($n = 4, 5, 6$, and $7$), stands for the probability of the series ending in $n$ games.

### *Critical episode one: Romina's guess*

| | |
|---|---|
| Romina: | They can go all seven or they could go all four. So, it would be A, A, A, A and B, B, B, B –Team A and Team B? |
| Jeff: | Wait, what's the – wait – wait – |
| Romina: | So those are the only possibilities for four? |
| Jeff: | Mm hm. |
| Romina: | So, in four games, would it be, like, one-half of a chance? Or would we have to write it out with -- using all seven? |
| Jeff: | See, I think that it's the hardest to win it in four games. Definitely the hardest |
| Romina: | Yeah, exactly. |
| Jeff: | So, it wouldn't be one-half. |

In this episode, Romina calls one Team A and the other Team B and suggests that the probability of the series ending in four games, $p(4)$, could be one-half because there are only two ways in which the series can be won in four games: AAAA and BBBB. Jeff, however, argues that winning the series in four games is "hardest," suggesting that one-half is too high a value for $p(4)$. The students agree with him and drop Romina's suggestion. The students used reasoning by contradiction to decide to drop Romina's suggestion. Indeed, if $p(4)$ is one-half as Romina suggested, then the other probabilities would have to be smaller than $p(4)$. This is a contradiction since, intuitively, the students believed that is harder to win the series in four games than it is to win the series in five, six or seven games. So, they decided to abandon Romina's idea.

### *Critical episode two: Brian's multiplicative strategy*

| | |
|---|---|
| Brian: | Isn't it the odds of winning one game, times the odds of winning one game, times the odds of winning one game? |
| Ankur: | It's a fifty percent chance of winning the first game. |
| Romina: | [It's] One-half |
| Brian: | So, it's like, half times a half – no, wait – remember the odds get harder to win two [games] in a row, like a coin flip? |
| Romina: | Yeah, that's how you do it |
| Brian: | Yeah. |
| Romina: | [Computing $p(4)$] Four – hold on – four times – |
| Brian: | That's one-sixteenth. |
| Romina: | [computing $p(5)$] Is it one thirty-two? Oh, never mind, I get it. Now, would you have, for five games, like, would it be like that [$\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}$]? [pauses and looks at her paper]. Wouldn't you have easier odds of winning in six games than in four? |
| Jeff: | Yeah. |
| Romina: | Doesn't it get less, though? |
| Jeff: | That's why it's wrong. |
| Romina: | Okay [crosses out what she has written]. |

In this episode, Brian suggests computing the probability of a team winning the series in $n$ games, $p(n)$ ($n$ = 4, 5, 6, and 7), by multiplying the odds [sic probability] of a team winning one game $n$ times. After Ankur and Romina add that the probability a team winning a game is one-half or fifty percent, respectively, Brian suggests computing $p(n)$ by multiplying one-half $n$ times. However, he reminds his colleagues that, ”like in a coin flip,” it should be harder to win games in a row than winning the same number of games if losses are allowed in-between. The students try Brian’s suggestion. However, when Romina announces after computing *p* (4) and *p* (5) that the odds of winning the series are not getting easier, the students decide to drop Brian’s suggestion.

Mathematically, the statement ”like a coin flip” suggests a form of reasoning by isomorphism, i.e. an implicit mapping or isomorphism between the World Series Problem and the Coin Problem. Indeed, if we make the event ”Tossing heads” [one side of coin] equivalent to ”Team A wins a game” and ”Tossing tails” [the other side of coin] equivalent to the event ”Team B wins a game,” then the probability of team A winning the series in *n* games is equivalent to the probability of getting four heads in *n* tosses of [a fair] coin with the last toss being heads. This probability can be computed as $A_4^n(\frac{1}{2})^n$ where $A_4^n$ denotes the number of ways we can get four heads in *n* tosses of a coin with the last toss being heads. Now, in *n* tosses of a coin, there are certainly more ways of getting four heads if some tails are allowed in-between than there are ways of getting four heads in a row. So, Brian is right to remind his colleagues that the odds of winning games *in a row* should be harder. However, his multiplicative strategy above suggests that he retrieved only the part $(\frac{1}{2})^n$ of the formula above. Now, by this formula, the odds of winning the series do not get easier with the number of games. Consequently, the students decide to drop Brian’s suggestion. So, an isomorphism helped promote growth of students’ mathematical understanding. More specifically, the isomorphism with the Coin Problem helped the students understand that the probabilities of a team winning the World Series are equivalent or isomorphic to the probabilities of tossing four heads in the coin problem. This allowed the students to reject the formula $(\frac{1}{2})^n$ as a way of computing the probabilities of a team winning the World Series because it does not make the odds of winning the series easier with the number of games, as should be the case by the isomorphism with the Coin Problem.

### *Critical episode three: the brute-force strategy*

Romina: You know how we do this thing [indicates strings on her paper]? Wouldn’t we just do that? Say we did that, right? Whatever the probability would be like, say, the probability of someone winning and then it would be like B, B, B, B.

Jeff: Oh. Yeah. Then that would be that number and that number. Yeah, that’s what I was thinking, but –

Ankur: So, then we got to do it like that

| | |
|---|---|
| Jeff: | Well, wait. Before we do that, let's look at, um, how do you get to that point in the first place? To finding out 'Cause there's like a lot of different combinations – two to the seventh. Is that two to the seventh? |
| Romina: | Isn't it – yeah, two *n*? |
| Jeff: | Yeah. All right, so say it's two to the seventh |
| Ankur: | For this, you've gotta find all possibilities with – |
| Brian: | Yeah, it's the order you win, though, too. |
| Romina: | Yeah, I know. |

In this episode, Romina suggests listing game combinations as strings of As and Bs and then computing the probabilities by dividing the number of series-winning game combinations (such as BBBB) by the total number of possible game combinations (such as ABAA). Jeff points out that there are "a lot of combinations" and they might not be able to actually list *all* of them. He suggests that there could be as many as $2^7$ game combinations. Romina agrees that there would be "a lot of game combinations," but thinks that there would be around $2^7$ combinations, instead. An impasse follows. Ankur intervenes to state that the series-winning game combinations cannot be computed as either $2^n$ or $2^7$. Brian explains that this is because the order of wins matters when computing series-winning game combinations, but not when computing all possible number of game combinations. The students agree and decide to actually list the series-winning game combinations (see figure 1). However, they still lacked a method for determining the total number of game combinations, i.e., the sample spaces or denominators of the probability

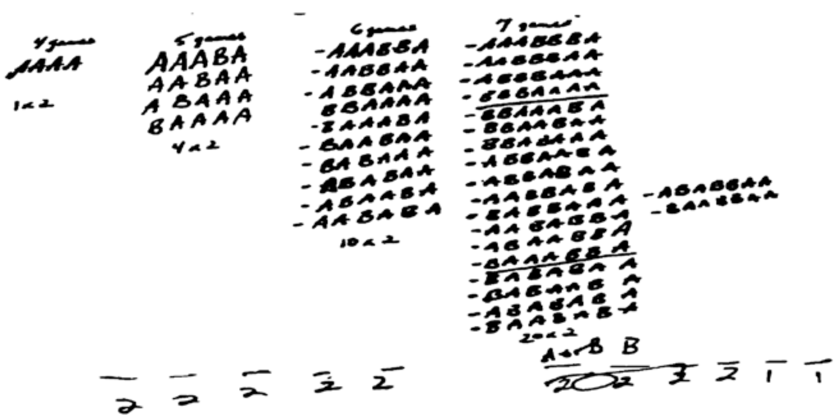


Figure 1. *The students listed favourable game combinations.*

ratios. They came up with a method in the next episode and it involved an isomorphism.

### *Critical episode four: the Tower Problem*

| | |
|---|---|
| Romina: | Should it [the probability ratio] be over seven, though? |
| Ankur: | It'd be over, like, total possibilities of – |
| Jeff: | Yeah, the total possibilities is eight, right? |
| Ankur: | They have eight ways of winning but it'd be over I'd be over – the total possibilities of two, like two – two colors and five things. |
| Mike: | It should be over – over seven, 'cause it's four out of seven games. |
| Ankur: | But this one wouldn't be over seven. |
| Jeff: | It wouldn't be. |
| Ankur: | It wouldn't. None of this would be over seven. |

In this episode, Romina asks "Should it be over seven, though?", i.e., if the denominator of the probability ratio $p(5)$ should be seven. Mike claims that the denominator should be seven "because it's four games [a team needs to win] out of seven [possible games]." Ankur, however, argues that the denominator should be equal to "the total possibilities," i.e., the number of all possible game combinations in the five game series. Jeff asks if the number is eight, referring to the eight game combinations that the students had listed for a series ending in five games. Ankur explains that the eight games represent the number of ways the series can be won in five games, i.e., the number of favourable outcomes, the numerator of the ratio $p(5)$, and not the number of all possible game combinations or denominator of the ratio. In particular, Ankur adds that the denominator of $p(5)$ is "equal to the total possibilities like in *two colors and five things*" (emphasis added). This statement suggests an isomorphism between the World Series Problem and the Tower Problem. If we make the event "A red unifix cube" equal to "Team A wins a game" and "A yellow unifix cube" equals to "team B wins a game," then the number of all possible game combinations in an $n$-game World Series is equivalent to the number of all possible towers five cubes tall when choosing from two colors. In previous years, the students had come up with a convincing argument that there are $2^n$ such towers (see Maher & Martino, 1996). In this episode, Ankur implicitly suggests computing the denominators of the probability ratio $p(5)$ as $2^n$.

### *Critical episode five: Pascal's triangle*

The students listed the series-winning game combinations by brute force, determined the number of all possible game combinations as $2^n$ and came up with the solution: $p(4) = \frac{2}{16}$, $p(5) = \frac{8}{32}$, $p(6) = \frac{20}{64}$, and $p(7) = \frac{40}{128}$. They liked that the probabilities added up to one. However, Ankur immediately pointed out that, "We can't prove that we have [listed] all [favorable] the possibilities." Romina also noted that they could not explain why $p(6)$ and $p(7)$ were equal. The students found this counter-intuitive. They expected $p(7)$ to be larger than $p(6)$ because the "odds should get easier with the number of games played". However, they still decided to present their solution to researchers:

Researcher: How do you know you're not double-counting?

Jeff: That's the big question.

Mike: All right, I just found, like, if you take the fourth number in each one [circles these entries]- that way, if you double each number, 'cause you have two teams, you can get the possibilities of four games. Four games, um, equals two, right? You got eight, twenty and forty, like they said [see figure 2]. I don't know how I'm going to explain it

Researcher: You're – you're doing fine.

Mike: But, um – Do you guys see anything?

Jeff: Well, obviously, there's something going on with the one, four, ten and twenty.

Ankur: Yeah.

So, the researcher asks the students how they can be sure that they had not double-counted when listing series-winning game combinations in

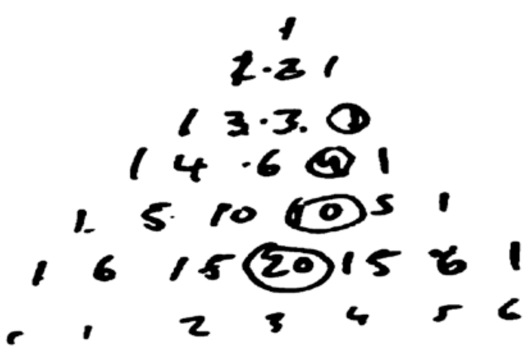


Figure 2. *The students noticed a connection between Pascal triangle and World Series Problem.*

their solution. This is an issue that Ankur raised earlier and which the students could not explain. Jeff acknowledges the issue by saying, "That's the big question". As the students showed signs of not knowing what do, Mike suddenly intervenes to say that he had noticed a connection with Pascal's triangle: the numbers of series-winning game combinations in series ending in four, five, six, and seven games were the same as twice the fourth numbers on the third, fourth, fifth, and sixth rows of Pascal's triangle, respectively. He circles the numbers and shows them to researcher (figure 2). However, he also says, "I don't know how I am going to explain it," thus admitting that he cannot explain/justify the connection. In the next episode, the students eventually explain the connection and once again rely on an isomorphism.

### *Critical episode six: the Pizza Problem*

In previous sessions in the longitudinal Study, Mike used Pascal's triangle to solve the Pizza Problem. Using a binary notation where "1" stood for adding a topping and "0" stood for adding no topping, Mike was able to list all 16 pizzas as permutations of zeros and ones. He eventually discovered that the number of pizzas with $k$ toppings ($k = 0, 1, 2, 3 \ldots n - 1$) when choosing from n toppings was $C^n_k$, i.e., the entry on the $n$th row and $k$th column of Pascal's triangle. Mike was also able to explain the addition rule in Pascal's triangle in terms of pizzas. So, to stimulate Mike's thinking about the connection between Pascal's triangle and the World Series Problem (see previous episode), the researcher decides to ask Mike to explain again the addition rule in terms of pizzas. Mike starts by saying that the row 1 3 3 1 represents the three-topping pizzas, and the numbers represent the numbers of pizzas with three, two, one, and zero toppings, respectively:

Researcher: Mike showed me something last time that I guess you all didn't hear. Mike, you see that addition of ten, you know, the six and four? Or the twenty? Why do you add them together? You had an explanation and you were using pizzas to explain it to me. You were talking about toppings on pizzas. Any of you ever heard this before?

Mike: Yeah, I remember. Right. [Pointing to the row with entries 1 3 3 1] This is like a three-topping pizza. There will be one with, uh – [Ankur and Romina "Plain"] Plain, right? Three with just two toppings, three with, uh, just one topping, three with just two and one with all toppings.

The researcher insists on the addition rule and asks Mike to explain how the "1" and the "3" in the row 1 3 3 1 combine to make four in terms of pizzas. Mike chooses to explain how the "3"s in the same row add to make a 6 in the following row of Pascal's triangle. He explains that this is because one can make pizzas with two toppings (when choosing from four toppings) by either adding or not adding topping to pizzas with one and two toppings (chosen from three toppings), respectively. Since there are 3 pizzas on each group they add and make 6 pizzas:

Researcher: Show me the one and the three giving you the four, in terms of pizzas. Can you tell me that?

Mike: I'm trying to think. I had it last time I talked to you. I had it so good. All right. You're going to add a topping to every single pizza on there, right? There's going to be twice as many pizzas. But these three pizzas – three of them got a topping, went there, and three of them didn't, went there. One of them had a topping, right there, and one of them didn't, went there. 'Cause these three pizzas are going to turn into six pizzas. – Now I got it, right? That's why they add.

All of sudden Mike is able explain the connection between Pascal's triangle and the World Series Problem which he had not be able to in the previous episode. He starts by saying that the numbers 1 3 3 1 represents the number of ways of winning zero, one, two, and three games in the world series, respectively. He also suggests that if we divide one of the numbers by the sum we can get the probability (For example, if we divide "1" by 8 (1+3+3+1) we get the probability of a team winning the series in three game, i.e. $p(3)$). Ankur builds on Mike's reasoning and adds an explanation of Pascal's triangle in terms of strings of As and Bs which they used to list series-winning game combinations:

Mike: Now with the one, three, three, one [entries 1 3 3 1], that circled one is, I guess, you win those three games in a row. There's only one possibility. You know what I'm saying? Like, how many is up there? One plus three- No. That's just three games. All right. Your probability of winning three times in three games. The first one you have a one out of eight chances of losing all three. And the second one, you have three possibilities of winning one: you could win it the first time, the second time or the third time. The third one would be winning twice. And there's only one other, one way to win three times.

Ankur: Actually, I was going to say that the "1" represents winning three games in a row, or like three A's. And then, if you go to the right, that's like getting another A, and there's only one way to get four A's. If you go to the left that's like getting a B, and that's like three A's and a B, and there's four different ways you can write that.

This episode shows that revisiting the connection between the Pizza Problem and Pascal's triangle helped Mike explain the connection between Pascal's triangle and the World Series Problem. This suggests an indirect isomorphism between the World Series Problem and the Pizza Problem built via Pascal's triangle. In the isomorphism, a win or loss of a game in the World Series Problem corresponds to adding or not adding a topping in the Pizza Problem, and different types of game combinations correspond to different types of pizzas. The isomorphism helped Mike arrive at an explanation of why they could use Pascal's triangle to compute series-winning game combinations.

## Discussion

The previous section described several episodes in which students relied on isomorphisms with other problems to make sense of or solve [parts of] the World Series Problem. In episode two, an isomorphism with the Coin Problem helped the students realize that the probability of winning the World Series in $n$ games is equivalent to the probability of obtaining four heads in $n$ tosses of a fair coin with the last toss being heads. This helped the students understand that the probabilities could not be computed as $\left(\frac{1}{2}\right)^n$. In episode four, an isomorphism with the Tower Problem helped the students realize that there are as many game combinations in an n-game World Series as there are towers $n$ cubes tall when choosing from two colors. This allowed the students to compute denominators of the ratio $p(n)$ as $2^n$. In episodes six, an isomorphism with the Pizza Problem allowed the students to understand that in an $n$-game series, there are as many series-winning game combinations as there are pizzas made when choosing from $n$ toppings. This allowed the students to explain why the series-winning game combinations and the probabilities in the World Series Problem could be computed through Pascal's triangle. These episodes support the main claim in this article that problem-solving activities that involve strands of tasks have the potential to promote growth of students' mathematical understanding by way of fostering reasoning by isomorphism.

The results offer insights into the processes that were involved in the students' mathematical reasoning. First, the isomorphisms came as the students tried several problem-solving strategies or approaches, drew on prior knowledge such as tasks with similar mathematical structures as the World Series Problem, and tried to prove that their solutions were correct. This supports claims that these processes help promote students' mathematical learning and understanding in problem-solving contexts (Cai, 2010). In particular, the isomorphisms were the result of the students' application of one particular heuristic- solving a similar problem. This is further evidence that heuristics help promote students' mathematical reasoning (Powell, 2003). Second, as mentioned above, coming up with a binary notation, where ”1” stood for ”adding a topping” and ”0” stood for ”adding no topping,” helped Mike list all 16 pizzas as permutations or strings of zeros and ones. In turn, the strings helped Mike ”see” the isomorphism between the Tower Problem and Pizza Problem. This supports the idea that mathematical representations can help foster students' mathematical reasoning and understanding (Uptegrove, 2003). Third, this study clearly supports the idea that students build schemata and isomorphisms through induction, i.e., by working on several tasks involving the same mathematical knowledge (Nunokawa, 2005). However, the results do not support findings that suggest that students have difficulty using example solutions to solve isomorphic problems (Reed, 1999). Indeed, they used the formula $2^n$, example solutions for the tower and pizza problems, to solve parts of World Series Problem.

The results of this study also suggest insights into *conditions* that enhanced the students' mathematical reasoning. First, the tasks were obviously challenging to the students. This supports suggestions for using strands of *complex* rather than just simple or easy tasks in trying to promote students' development of schemata and reasoning by isomorphism (e.g., Weber, 2005; Weber, Powell & Maher, 2006). Second, social and not only ”purely” mathematical conditions also influenced the students' reasoning. As described, the researchers gave the students plenty of time to work on the tasks; avoided removing the challenges involved in the tasks by telling or showing students how to solve the tasks; and spent most of their time listening to the students and asking questions to stimulate their thinking. The influence of the researcher's questioning on the students' thinking is particularly evident. In episode four, the researchers' insistence that the students prove that they had not double counted their series-winning game listings stimulated the students' pursuit of a justification for the claim. In Episode 6, the researcher invited Mike to ”retrieve” his explanation of the Pascal's triangle addition rule in terms of pizzas. Mike was later able to build on his explanation to come up with an explanation for the connection between Pascal's triangle and World Series Problem. The influence of social factors on the students' thinking is evidence that the classroom discourse is essential for fostering growth of students' mathematical understanding in problem-solving settings (Cai, 2010). The key role played by the researcher is an example of the importance of scaffolding students' activities in problem solving. The researcher's insistence on a proof promoted the idea of mathematics as a sense making activity, a key sociomathematical norm for promoting students' mathematical reasoning and understanding (Cobb & Yackel, 1998; Yackel & Rasmussen, 2003). The Pizza Problem worked as a springboard for the students to eventually get to a valid explanation of the connection between Pascal's triangle and the World Series. This is an instance of how selecting situations that bridge the old and the new knowledge is an important scaffolding intervention that help promote students' development of schemata and reasoning by isomorphism (Nunokawa, 2005).

There is one potential implication for classroom teaching that follows from this study. K-12 mathematics teachers often complain that they do not have enough time to cover the great amount of content they are often supposed to teach (Francisco & Maher, 2011). The extent to which this is a valid complaint depends on several factors. However, this study suggests that teachers may want to consider organizing the mathematical content that they are supposed to teach in *strands* as opposed to isolated material. This has the potential to help teachers reduce the amount of material to cover, which may help teachers address the problem of too much content to cover. It can also provide opportunities for students to revisit the same ideas several times in different mathematical contexts, which help students build durable forms of understanding.

Finally, three key lessons or insights can be drawn from this study. The study shows that general factors that support students' mathematical learning in problem-solving situations also help promote students' reasoning by isomorphism. The study also shows that strands of *challenging* tasks can indeed help promote students' development of schemata and reasoning by isomorphism, but the classroom discourse is crucial. In particular, the study suggests that if the conditions in which students engage in mathematical activity and the perspective taken on problem solving emphasize students ability to show what they can do in challenging mathematical situations as opposed to simply how well they can repeat back what they have been told or shown by teachers/researchers, students are capable of powerful forms of reasoning such as reasoning by isomorphism. Finally, this study suggests that reasoning by isomorphism is a complex process. The isomorphism between the World Series and the Tower Problem was *partial* as it helped determine only the sample spaces, and not the actual probabilities. The isomorphism between the World Series Problem and the Pizza Problem was built *indirectly* via Pascal's triangle. In the isomorphism with the Coin Problem, the students correctly recognized the similarity of structure between the problems but retrieved the wrong formula or technique. This suggests that the field of mathematics education would benefit from more research on the conditions and processes involved in students' building of reasoning by isomorphism and how it can help promote growth of students' mathematical understanding.

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