Different views – some Swedish mathematics students' concept images of the function concept

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This study analyses what kind of concept images a group of engineering and teacher students have of the function concept, and how these concept images are related to the historical development of this concept. The study was conducted using questionnaires, and 34 students at a Swedish university participated. It is found that the students primarily rely on operational conceptions of the function concept, with only a minority of students possessing structural conceptions. The definitions given by the students mostly resemble an 18^{th} or 19^{th} century view of functions. The study also indicates that the character of the definitions given in the textbooks used by the students affect their concept images.

Different approaches have been developed to explain the mechanisms governing concept acquisition. For example, in mathematics education there has been a considerable amount of discussion concerning the distinction between *concept definition* and *concept image*; the concept definition being the formal mathematical definition, while the concept image is seen as something much wider – an individual mental construction representing "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (Tall & Vinner, 1981, p. 152).

Regarding the concepts themselves, Sfard (1991, 1992) speaks of the duality of mathematical concepts, in that they can be viewed both as *processes* and as *objects*. While "there is a deep ontological gap between operational and structural conceptions" (Sfard, 1991, p. 4), the two are not mutually exclusive but rather complementary. Related to this is the

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theory of Gray and Tall (1994) in which the term *procept* is used to represent the "amalgam of concept and process represented by the same symbol." (Gray & Tall, 1994, p. 121). For example, in the context of functions, the symbol $f(x)$ represents the process of "determining the value of the function f at the point x", the corresponding element in the range of the function, as well as the function as an object in its own right. Based on the idea of procept, DeMarois and Tall (1996) have developed a theory for assessing conceptual development where they speak of different *facets* (aspects and forms of representation of the concept) and *layers* (development in depth, from process to object to procept).

Sfard (1991) has also formulated an influential theory of concept formation. According to this model, concept formation consists of three consecutive stages: *interiorization*, where one gets acquainted with the processes behind the concept by performing operations on already familiar mathematical objects; *condensation*, where one gets more insight into the concept, gaining increasing capability of switching between different representations of it; and *reification*, where one gains the ability to view the concept as an object in its own right. For Sfard, this last step is qualitatively different from the first two.

Previous results on the concept of function

When learning the function concept (or indeed most mathematical concepts), the process of reification is by no means an easy one. Several studies (e.g. Hansson, 2006; Norman, 1992; Sfard, 1992; Even, 1990; Vinner & Dreyfus, 1989) indicate that many advanced students, and in some cases even practicing teachers, have not reified the concept of function but merely have a process-oriented view of it. In fact, Sfard claims that the difficulty of the reification process needs to have consequences for the teaching of mathematics. Even though a structural view is necessary to get a deeper understanding of a concept, new concepts shouldn't be introduced in structural terms (Sfard, 1992, p. 69).

When students first encounter the definition of a concept, more often than not they already have concept images that may be more or less developed. Of course, also the concept definition somehow underlies an individual's concept image, but when the concept is used in practice it is almost always the concept image that is evoked (see e.g. Attorps, 2006; Hansson, 2006). Earlier studies (for studies of the function concept, see e.g. Akkoç & Tall, 2002; Tall & Bakar, 1991) indicate that prototypes, i.e. standard examples of the concept used for a pedagogical purpose, tend to contribute strongly to the concept image as a whole, even though they are often chosen only in order to highlight just one particular aspect of the concept. Hence different aspects of the concept image may very well be contradictory since different aspects of the concept image are used in different contexts. This effect is called *compartmentalization* and it has been detected in several studies (see e.g. Eisenberg, 1992; Vinner, 1992; Vinner & Dreyfus, 1989 for examples related to the function concept). Moreover, many studies (e.g. Akkoç & Tall, 2002; Meel, 2000; Vinner & Dreyfus, 1989) show considerable discrepancies between the concept definitions given by students and the concept images they actually use regarding the function concept.

As seen above, students' conceptions of the function concept have already been studied by several researchers internationally. However, not that much research on the subject has been done in the Nordic countries. Apart from Hansson (2006), one should also mention the work of Pettersson (2008). These studies differ from the present study both in the methods used and in their foci. Hansson uses concept maps to investigate students' understanding of the function concept, while Pettersson focuses on how students use their conceptions of concepts from calculus when doing mathematics. Moreover, Repo (1996) touches upon the subject; this work deals with the construction of the concept of the derivative using the Derive software.

An outline of the history of the function concept[1]

As the function concept is one of the fundamental concepts in modern mathematics, it is perhaps somewhat surprising that it has been studied systematically only for about 300 years. Functional ideas had been used even in ancient times, for example to construct astronomical charts and tables. However, ancient mathematics lacked the necessary algebraic prerequisites (Kleiner, 1989, p. 283) as well as the interest in motion and change (Sierpinska, 1992, p. 31) needed to develop a concept of function. From 1450–1650, mathematics developed greatly in these areas culminating in algebraic geometry and calculus. But the calculus of Newton and Leibniz was a calculus of curves, not functions (Kleiner, 1989, p. 283), and it was not until the work of Euler in the mid–1700s that calculus began to be seen as a study of functions.

In 1718 Johann Bernoulli gave the first definition of the function concept (Rüthing, 1984). This definition was very imprecise and was somewhat improved by Euler. These early definitions all required the function to be given by an algebraic expression. A lively debate concerning the problem of describing the motion of a vibrating string (see Kleiner, 1989, p. 285–288) caused Euler to change his view regarding this, and in 1755, Euler gave the following definition of the function concept: "a quantity should be called a function only if it depends on another quantity in such a way that if the latter is changed, the former undergoes change itself." (Sfard, 1992, p. 62f) This definition is clearly process-oriented, using Sfard's terminology.

When Fourier published his work on trigonometric series in 1822, it was obvious that his proofs were lacking conceptual precision. A rigorous re-treatment of the concepts of calculus was needed, and this endeavor was undertaken in the first half of the 19th century by such mathematicians as Cauchy, Dirichlet and Weierstrass. In fact, it was Dirichlet who gave what can be seen as the first modern definition of the function concept: "If a variable y is so related to a variable x that whenever a numerical value is assigned to x there is a rule according to which a unique value of y is determined, then y is said to be a function of the independent variable x." (Sierpinska, 1992, p. 46). In one sense this definition is less general than the one of Euler, speaking of "rule" rather than "dependence", but, more importantly, this is the first definition mentioning *one-valuedness*. According to Even (1990)[2] this is one of the essential features of the concept of function in the modern sense, the other being *arbitrariness*. Arbitrariness means that the value of a function at any point is independent of the value at other points but also that the domain and range can be arbitrary sets; specifically they need not be number sets. One-valuedness simply means that for each element in the domain there is a unique element in the range.

In the early 20th century, development in analysis, topology and algebra paved the way for a thoroughly abstract, set-theoretical definition of the function concept, as the following one, given by Bourbaki in 1939:

> Let E and F be two sets, which may or may not be distinct. A relation between a variable element x of E and a variable element y of F is called a functional relation in y if, for all x in E, there exists a unique y in F which is in the given relation with x. We give the name of function to the operation which in this way associates with every element x in E the element y in F which is in the given relation with x; y is said to be the value of the function at the element x, and the function is said to be determined by the given relation. Two equivalent functional relations determine the same function.
>
> (Rüthing, 1984, p. 77)

Here we find explicit reference to domain and range, and no reference is made to number sets. Moreover, the definition is totally static. In Sfard's terminology, the reification has now become complete. It can be seen that the historical development of the function concept follows the pattern described by Sfard quite well, going from an operational to a structural view of the concept. Also, it is worth noting, that more operationally oriented definitions, like the one given by Dirichlet above, are still often used, for example, in calculus.

Research questions

The primary purpose of this study is to get a preliminary understanding of mathematics students' ideas of the function concept, by deriving a categorization of the conceptions of this concept displayed by a group of Swedish engineering and teacher students and, if possible, comparing this with the results of relevant studies conducted elsewhere. Hence, the first research question is: *What kind of conceptions of the function concept do the participating students have?* More specifically, this question is divided into two sub-questions: *How do the students define the concept of function?* and *How developed are their concept images for the function concept with respect to the dual nature of mathematical concepts?*

Similar categorizations to the one we seek already exist in the literature. With respect to this study, those presented by Vinner and Dreyfus (1989), Sfard (1991) and DeMarois and Tall (1996) are of the greatest interest.

Another purpose of the study is relating the conceptions of the participating students to the historical development of the function concept. As we have seen, historically the function concept has developed from a process-oriented view towards a more reified one. Also, earlier studies (e.g. Akkoç & Tall, 2002; Meel, 2000; Vinner & Dreyfus, 1989) indicate discrepancies between students' concept definitions and concept images; the definitions are often object-oriented while students' conceptions about them are mostly process-oriented. Our second research question then is: *What differences and/or similarities can be found between the students' conceptions and definitions of the function concept, and the formal definitions as they have developed historically?*

Method

The study was conducted at a Swedish university, and the participants were teacher students attending their first course in calculus (14 students), and first-semester (in 5-year program) engineering students, also attending their first course in calculus (20 students). The courses the two groups were taking were different, but were intended to cover mostly the same topics. Moreover, the students had recently begun the course at the time the study was conducted. The teacher students had studied more than one semester of mathematics (except for the 3 students aiming at upper secondary school who had only taken a course in algebra) while the engineering students had only taken a course in algebra. The participants were selected among students encountering the concept of function in a calculus context for the first time at university, and to represent two major groups among those studying mathematics at the tertiary level.

The data were gathered by questionnaires. The students were asked to associate freely regarding the concept of function, and to construct a ”mind map”. No specific instructions were given on how to construct these maps. Hence they should not be considered concept maps in the sense described e.g. in Hansson (2006). The students were then presented with 12 mathematical expressions [3] and 4 figures, and were asked to determine which of these represented functions, and to rate the degree of certainty of their answers. Furthermore, they were asked for their opinion on the possibility of constructing a function with certain given characteristics – having an integer value for every non-integer, and a non-integer value for every integer; this example is also found in Vinner and Dreyfus (1989) – and finally they were asked to state their own definition of the concept of function. The students were asked to fill out the questionnaire as part of an ordinary lecture, and the time allotted for this was about one hour (half of an ordinary two-hour lecture).

The data were then analyzed, aiming at finding a categorization of the students' definitions and concept images of the function concept. To that end, we to some extent made use of the categories given by Vinner and Dreyfus (1989), Sfard (1991) and DeMarois and Tall (1996). When analyzing the mind maps, we looked for mathematically relevant expressions or symbols as a measure of richness, and links between them as a measure of depth. In the answers to the problems, we looked for the amount of correct answers, and more specifically for internal consistency in answers. The analysis was made by the three authors independently, and then compared, showing a high degree of agreement. A more detailed description of how the categorizations were made is given in the examples below.

Results

The students' definitions of the function concept

The first part of the first research question deals with the students' definitions of the function concept. Our data reveal that the definitions given by the students can mostly be described as process-oriented and that only a small minority present structural definitions of the concept. Furthermore, nearly a third of the students in the study failed to provide any meaningful definition whatsoever.

Our classification of the students' definitions of the function concept has been made using categories somewhat inspired by those in Vinner and Dreyfus (1989). We make use of five categories, of which one covers meaningless and non-existent answers. The categories are the following (each followed by a few examples of the definitions given by the students).

1. **Correspondence/dependence relation.** A function is any correspondence or dependence relation between two sets that assigns to each element in the first set *exactly* one element in the other set. Domain and range may or may not be mentioned.

> A function always gives just one value when you insert a value. If you have one set which is the domain and insert one of those values into the function you get one of the values in the range. (T 2)[4]

> A function depends on a variable. Depending on what value the variable has you get a unique value of the function. (E 11)

Characteristic of this category is the use of words like *depends* or *corresponds* and mention of domain, range and/or one-valuedness. The association between domain and range is made clear without using procedural language suggesting the processing of elements. Also no mention is made of specific or representative descriptions, like formulas or expressions.

2. **Machine**. A function is a ”machine” or one or more operations that transform variables into new variables. In this case no explicit mention of domain and range is made.

> A ”machine” that assigns to any input-variable a specific number or something similar. (E5)

> A ”device” where you insert an input, and after the ”process” you get an ”output”. (E 19)

This category is characterised by the use of words like *device*, *machine* or *tool*, suggesting a reworking of elements. However, the specific workings of this process are not made explicit. The machine is seen as a ”black box”, inside of which the reworking is done.

3. **Rule/formula.** A function is a rule, a formula or an algebraic expression. Compared to the second category, the difference is that now regular behaviour is expected whereas the machine could conceivably perform totally different transformations of different elements.

> A description of a pattern, which varies depending on different variables. (E 7)

> A function is a formula for which value y assumes for any given value of x. (T 1)

Here the specific features are words like *expression*, *pattern*, *formula*, *algorithm* and the like. Sometimes explicit examples of formulas are given, like $s = v \cdot t$. The regularity of the function is emphasised, for instance through the expression ”varying in step with”. In some cases reference is made to courses of events in the physical world.

4. **Representation**. The function is identified with one of its representations.

> A curve where one x-value has one y-value. (T 3)

> A number acting on another number. The value of that number is totally dependent on the value of the other number. (T 10)

In this category, functions are defined as curves, numbers or graphs. We are aware of the fact that in set theory functions are often identified with their graphs, viewed as subsets of the Cartesian products of domain and range. However, in the context of this study, we feel confident in assuming that the participating students have not been acquainted with this kind of definition. Also, in using the word *curve*, piece-wise continuity is implicitly assumed, imposing severe restrictions on the definition.

5. **Nonsense**. A meaningless answer or no answer at all.

> A function is an explanation of how something works. (E 4)

To this category we have referred non-existent answers, or answers not containing mathematical language.

Of these categories, the fourth and the fifth cannot be said to represent definitions in a strict mathematical sense. The other three categories more or less trace the historical development of the function concept outlined above, with the second and third categories having an operational and the first one a structural character. The early definitions of Bernoulli and Euler fit into the category 3, while Euler’s later definition could perhaps be said to belong to the category 2. The definition of Dirichlet should belong to category 3 although it is also related to the category 1 (one-valuedness). The Bourbaki definition fits squarely into category 1.

Finally, it is also worth noting here the definitions given in the textbooks used by the students. The majority of the teacher students use a textbook presenting the following definition: "A variable y is a function of x, if to every value which x can assume, is assigned only one of the values which y can assume." (Rodhe & Sigstam, 2000, p. 88) This definition is somewhat loosely formulated, and it is not obvious to which category it should belong. Nevertheless, since it speaks of assigning unique values, without using procedural language or referring to formulas or rules, we wish to assign it to category 1. On the other hand, the textbook used by the engineering students and the three teacher students aiming at upper secondary school gives the following definition: "A function f on a set D into a set S is a rule that assigns a unique element $f(x)$ in S to each element x in D." (Adams, 2006, p. 24) As we can see, this definition is less general in that it speaks of the function as being a *rule*, but on the other hand it makes explicit mention of domain and range, and uses a clearer, more formal mathematical language. We have assigned this definition to category 3. The categorization of the participating students' answers is shown in table 1.

Table 1. *The number of students' answers in the five categories* ($n = 34$).

Category	1	2	3	4	5
Number of students	3	10	9	7	5

Three students fall into the category 1, the category resembling the structural definition of function, while 12 students end up in the categories 4 or 5 failing to give a useful definition.

The students' concept images of the function concept

The second part of the first research question concerns the students' concept images of the function concept. In classifying the students' concept images of the function concept, as expressed in the questionnaires, we made use of some ideas from the classifications of Sfard (1991) and DeMarois and Tall (1996). We ended up classifying the students' conceptions [5] of the function concept as *pre-operational*, *operational* or *structural*, cf. DeMarois and Tall (1996) who speak of pre-action and action. A student with a pre-operational conception has a rudimentary and inconsistent concept image. A students' conception of function is operational if he or she clearly views a function as a process, and structural if he or she is also able to view the function as an object in its own right.

When analyzing the students' answers to the questionnaire, we were looking for signs of their conceptions in their use of language, handling of tasks etc. If a student handled tasks in an inconsistent manner, answering differently to problems of a similar kind; and was unable to give a useful definition and constructed a poor mind map, showing difficulty in handling mathematical terminology, we have classified the student as having a pre-operational conception. If the student gave a definition of a procedural type, used procedural language in the mind map and solved problems with some consistency, we took these as signs of a procedural conception. Signs of structural understanding would be, for instance, giving structural definitions, constructing a rich mind map containing also more object-like aspects of the function concept (for instance arbitrariness and the importance of specifying domain and range), and being able to handle less procedural problems, like the Dirichlet function and the integer-noninteger function.

Using the above classification we found that the majority of students expressed pre-operational or operational conceptions of the function concept, while only a few expressed something resembling a structural conception. However, there was also one interesting exception, to which we will return in the last section. Table 2 below shows the distribution of the students' conceptions. At the end of this section we will also discuss a few examples in more detail in order to show how the analysis of the data was carried out.

Table 2. *The distribution of students' conceptions of the function concept* (n = 34).

Category	Structural	Operational	Preoperational	Other
No. of students	3	18	12	1

The most common concept to appear in the mind maps was the concept of graph or curve (20 students). Yet only one student mentions the vertical line test, a consequence of the one-valuedness property on the graph of a function. Common were also such concepts of calculus as derivative and integral as well as the function machine and terms like formula, expression and operation. Regarding the essential features mentioned above, eight students specified domain and/or range and four mentioned one-valuedness. Notable by their absence were such concepts as inverse function and composite function, as well as examples of standard functions. Only a few of the students referred to any of these concepts in their maps.

The students were also asked to determine whether given expressions and graphs could be said to represent *y* as a function of *x*. Some of these expressions together with the distribution of "yes"- and "no"-answers are presented in tables 3 and 4.

Table 3. *The distribution of students' "Yes"- and "No" -answers* ($n = 34$)[6].

Expression	$x^2 + y^2 = 4$	$xy^2 = 5$	$x = 3$	$y = 3$
Yes	24	23	4	13
No	10	10	29	20

Table 4. *The distribution of students' "Yes"- and "No" -answers* ($n = 34$)[6].

Expression	$f(x) = 3$	$f(x) = \begin{cases} -3 & x < 0 \\ e^x & x \geq 0 \end{cases}$	$f(x) = \begin{cases} 1 & x \text{ rat.} \\ 0 & x \text{ irr.} \end{cases}$
Yes	22	32	24
No	12	2	10

We see that a majority of the students considered both of the first two expressions as being functions despite the fact that such "functions" would not be univalent. Also, a substantial number of students rejected the constant function $y = 3$. Finally, an overwhelming majority of the students accepted split domain functions. Here some interesting inconsistencies appear.

For example, the function $f(x) = \begin{cases} -3 & x < 0 \\ e^x & x \geq 0 \end{cases}$ is constant on part of its domain but is still accepted as a function by more than twice as many students as the function $y = 3$. In fact, as many as 18 students accept one but not the other. It is also interesting to compare this with another question in the questionnaire where the students were asked about the possibility of constructing a function which is integer-valued for all non-integers, and non-integer-valued for all integers. About half of the students considered the existence of such a function possible and 12 students constructed an example of it. But quite a few of those who rejected it did so based on an assumption that a function must be defined by one and only one formula on the whole of its domain; despite having had no problem accepting the piecewise defined function above. On the other hand, of those students who accepted this type of function, only two rejected the Dirichlet function. It seems that the students who have grasped the idea of arbitrariness appear to have done so in a consistent manner.

For further exemplification and in order to give an outline of how the classification of the students' conceptions was carried out, we shall now consider in more detail three students, intentionally chosen as clear representatives of each category.

Teacher student T 10

25-year-old woman studying to become a lower secondary school teacher in mathematics and English and with 1½ semester (45 ECTS points) of studies in mathematics, rates her own mathematical ability as average. Her mind map contains two mathematical notions – the word "graph" and the symbol $f(x)$ – and several disclaimers: "difficult – complicated", "Don't know if I have fully grasped what a function is." When asked to determine whether given expressions or graphs represent functions [7], her answers were inconsistent – the equation of a circle represents a function, but not the expression $xy^2 = 5$, further, $x = 3$ represents a function but not $y = 3$ etc. She also expressed uncertainty about her answers (on average 2 on the scale 1–5). When asked whether it is possible to construct a function which is integer-valued for every non-integer and non-integer-valued for every integer [8], her answer was conspicuous: "No, I don't think so (?). An integer · another becomes a new integer. Didn't sound completely right when you read it [...]" There is no mention of multiplication in the formulation of the problem, so the idea that the function has to be constructed using multiplication of integers is her own invention. Now compare this to her definition of the function concept: "A number acting on another number. The value of that number is totally dependent on the value of the other number". This indicates that she considers functions as numbers operating on one another. Taken together, these statements suggest that her concept image of the function concept is built on multiplication by numbers. From our own experiences teaching about functions we know that it is not an unknown misunderstanding for students to interpret the symbol $f(x)$ as "f times x", analogous to for example $a(b+c)$. This student's conception of the function concept has been categorized as pre-operational.

Engineering student E 15

20-year-old female engineering student, has taken one course (6 ECTS points) in mathematics at university and considers her mathematical ability to be relatively good. At first glance, her mind map appears to be rich but closer investigation reveals that it contains only vague and general concepts: ”reworking of information”, ”machine-input-output”, ”description of reality”, ”how different things are related to one another”, ”Could be unreasonable, e.g., a connection between the number of storks and the number of newborns could be found without the one necessarily depending upon the other” and so on. Her definition is also vague: ”Function = a process where you take certain information and, with the help of specified methods, transform it into new information”. All that she has written is very diffuse. No references to domain, range or one-valuedness, to representations such as graphs or formulas, or to operations such as differentiation or integration appear. Nonetheless, her procedural knowledge of functions is quite good. She is one of only two students who answered correctly to all statements in questions 2, 3 and 4. Her concept image appears to be well developed but unarticulated. Her comment about functions being ”unreasonable” is very interesting. She appears to view functions very much as being models of processes in reality. In her view, a functional relation not reflecting a causal connection might be a function, but it is not a reasonable one. Her conception of the function concept has been categorized as operational.

Teacher student T 2

Woman of age 21 studying to become a teacher of mathematics and English in upper secondary school, has taken one previous course (10 ECTS points) in mathematics at university, rates her mathematical ability as relatively good. Her mind map is very rich, including a multitude of concepts connected to show conceptual relations. Practically every mathematical concept appearing in any of the other students' mind maps also appears in this one. Moreover, her definition is the only one to mention domain, range and one-valuedness. Furthermore, little in her answers suggests that she requires the domain or range to be number sets. She does use the term ”value” but this probably reflects the fact that in Swedish the term ”range” is called ”värdemängd” (”set of values”). Her ability to work with the function concept is good. Her answers to questions 2, 3 and 4 were almost totally correct. Especially her answer to question 4 is very thorough, discussing points of discontinuity, domain and so on. Her conception of the function concept has been categorized as structural.

The students' conceptions in a historical light

The second research question concerns the relation between the students' concept images and definitions of the function concept and the formal definitions of function, as they have developed historically. As we have seen, most of the students (19, to be precise) gave definitions of an operational character (categories 2 and 3 above), resembling the formal definitions used up until the early 20th century. More precisely, 9 students give definitions belonging to category 3, matching the 18th century view of equating functions and algebraic expressions, while 10 students give definitions belonging to category 2, taking into account the concept of arbitrariness, new to the 19th century mathematicians' view of functions, and central to the modern function concept. The ideas behind the modern, set-theoretic definition are not so commonly displayed by the students in the study. Three students give definitions belonging to category 1, and a further 5 students mention the concepts of domain and/or range elsewhere in their questionnaires. A similar picture is seen when looking at the participating students' conceptions of the function concept, with a modern, structural conception being less common, expressed by three students in the study.

In this context, it is interesting to consider how the participating students are distributed in the categories according to what definition they have encountered in their textbooks. We have seen above, that one group of students in the study used a textbook giving a definition that we classified as belonging to category 1 (Rodhe & Sigstam, 2000), while the other group used a textbook giving a definition classified as belonging to category 3 (Adams, 2006). However, the picture is more complex than that; Adams' definition explicitly mentions domain and range, while Rodhe and Sigstam's does not. Table 5 shows the students' definitions grouped according to the textbook used.

Table 5. *The distribution of the students' definitions of the function concept ordered by textbook used* ($n = 34$).

Textbook	Category				
	1	2	3	4	5
Rodhe & Sigstam	0	1	1	7	2
Adams	3	9	8	0	3

We can see that the students using Rodhe and Sigstam (2000), where a definition is given which at the same time is of a structural character and is expressed in less formal mathematical terms, generally give less developed definitions. Also, looking at the students' conceptions, among these students nine expressed pre-operational and two operational conceptions of the function concept, as opposed to the group using Adams (2006), where we see a wider range of conceptions, with three students expressing structural, 16 expressing operational and four expressing pre-operational conceptions.

Discussion

This study shows that the participating students primarily have operational and, in some cases, only pre-operational conceptions of function. This is compatible with earlier research on the subject, which has indicated that a reified concept of function is rare among mathematics students. In our study the students' conceptions generally agree relatively well with the definitions they have given, in the sense that students expressing operational conceptions tend to give operational definitions. This contradicts, at least to some extent, earlier studies (for example, Akkoç & Tall, 2005; Meel, 2000; Vinner & Dreyfus, 1989), where students show discrepancies between their definitions and concept images of the function concept.

A possible reason for this is the difference between the definitions of function the students have encountered in textbooks during their studies. It is explicitly stated in Akkoç and Tall (2005), and it can be concluded also from Vinner and Dreyfus (1989), that in Turkey and Israel (at least at the time when the respective studies were conducted) the strictly structural Bourbaki definition of function is used in upper secondary school, something which is not at all the case in Sweden. However, the definition given in the textbook used by the prospective compulsory school teachers is of a structural bent, though not as abstract as the Bourbaki definition in any sense, and as we have seen, the definitions given by these students were the least developed in the study. This finding lends support to Sfard's (1992) view, that introducing new concepts in structural terms should be avoided. We find this to be an interesting result, and a topic for further research could be to investigate the possible connection between the character of the formal definitions encountered by students in their textbooks on the one hand, and the compatibility of their concept definitions and concept images on the other. Also, since many of the previous studies of students' conceptions of the function concept have been conducted in countries whose teaching traditions differ somewhat from those in the Nordic countries, further studies in a Nordic context might be useful.

Even though the concept images of the students in this study seem to agree rather well with their concept definitions, they are not very rich. This agrees with the results of Hansson (2006) where it is shown that the function concept is not so well integrated into the general conceptual structure of mathematics students. Also the lack of more specific concepts and examples of standard functions may appear to be contradicting earlier results regarding the importance of prototypes in the formation of concept images, cf. Akkoç and Tall (2002) and Tall and Bakar (1991) but this can just as well reflect the students' aim at generality. Furthermore, several examples of compartmentalization were found. For example, almost all students stated that the diagram showing a curve with a loop did not represent a function, but at the same time, a majority of the students claimed that the expression $x^2 + y^2 = 4$ represented a function. As mentioned above, one interesting exception to the classification of the students' conceptions was found. One of the engineering students displayed a poorly developed concept image but of a structural rather than a procedural character. He almost exclusively wrote about relating numbers from different sets to one another and didn't use procedural language at all. It is almost as if he views functions as objects but not as processes. This doesn't fit too well with the fundamentals of the model of DeMarois and Tall (1996) where the process layer is to be attained before the object layer. It would perhaps be more descriptive to say that this student possesses a pre-structural conception of function.

As noted above, differences in the character of the formal definitions encountered by the students in their textbooks appear to have affected the type of definitions they give in the questionnaires. Students encountering a more structural and, at the same time, less explicitly mathematical, formal definition seem to produce less developed definitions. In other parts of the questionnaire, no such systematic differences were seen, with one notable exception – the construction of the integer/non-integer function described above. None of the prospective compulsory school teachers were able to take a position. Indeed, only one of them even tried. The rest just answered "I don't know" or claimed not to have understood the question. On the other hand, almost all of the students in the other group were able to give a correct construction. And of these students, even those who believed that such function cannot exist provided some kind of argument in favour of this view. On the basis of our data only, however, we are not able to ascertain whether there is any connection; we can merely note the systematic difference.

Finally, what good can our present results provide for enhancing conceptual learning of the fundamental concepts of mathematics? We believe that knowing thoroughly students' already existing conceptions regarding the matter to be taught is an absolutely necessary (but, of course, not yet sufficient) condition for designing efficient teaching approaches to the topic. We have seen from a number of studies, including the present one, that we cannot expect from our students, even at the university level, that they have a reified conception of the function concept. Moreover, this unfortunate state of affairs is not restricted to this concept alone, cf. Attorps (2006). On the other hand, being more aware of university students' procedural understanding of this concept might help us to focus on finding relevant tools for elevating their understanding closer to a structural level. For example, focusing on clarifying the role of the formula – as "an operating rule of a machine that is intended for a special kind of purpose, for joining domain and range in a certain way" – in the examples of the function concept could help students reach a more thorough understanding of the actual relationship between the domain and range, and further, through that to see the whole function as an individual object. Furthermore, studies of the actual teaching of the function concept at the university level could help clarifying what opportunities for learning different approaches to the concept might provide.

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Notes

1 For a more thorough treatment, see for example (Kleiner, 1989).

2 Even uses the term *univalence*, but since this is a term used in a different sense in other mathematical contexts, it is perhaps less appropriate.

3 For examples, see tables 3 and 4.

4 The participants are identified by a letter and a number. The letter describes the category: T for teacher, E for engineer. All quotes from the questionnaires have been translated from Swedish by the authors.

5 Sfard (1991) defines *conceptions* similarily to the *concept images* of Tall & Vinner (1981). Hence we use the terms operational and structural *conceptions*, since these analytical constructs are borrowed from Sfard.

6 In some cases the number of answers doesn't add up to 34. This is because not all students gave an answer to every item.

7 Questions 2 and 3 in the questionnaire.

8 Question 4 in the questionnaire.

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