Learning opportunities offered by a classical calculus textbook

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In this paper we present results of an analysis of what the textbook used by the first year engineering students offers the students, when they take a basic calculus course. The aim of this analysis is to examine as an entirety what students are offered by the book to learn about the concept of derivative. The results show that the presentation of the concept is formal and depends on students' previous knowledge. The treatment of the concept emphasises procedural knowledge. It is not easy for students using the book to make connections between conceptual and procedural knowledge of the concept of derivative.

A group of mathematics education researchers (including the authors) are working at a university college in northern Norway. The university college is rather young, founded in 1994 and it has about 1300 students. One of the main education programmes is engineering, with about 150 new students each year. Some years ago a quality reform was undertaken at Norwegian universities with the intentions to improve quality of all higher education (Kvalitetsreformen, 2003). A closer follow up of students and outcomes was demanded. This has resulted in raised awareness among faculty members about issues related to mathematics teaching and learning. Teachers are asked to work for improved recruitment of students, improved contact with students during courses, improved success rate and fulfilling of studies, and improved learning outcomes. Thus when this university college got an opportunity to hire a research student for doctoral studies in mathematics education, there was a wish to carry out a study that could result in better insights into the

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mathematics components of the engineering education. The area of study was chosen to be engineering students' use of the textbook in a first year calculus course. The main aim was to find out what characterises the students' use of the textbook. The study has several parts and one of them, the analysis of what is offered to the students in the calculus textbook, will be reported in this paper. In another paper the actual use of the textbook by students will be treated and a third paper will investigate some authors' ideas about the calculus textbook. More exploration of what the textbook has to offer to the students and how it is done could contribute to higher awareness in teachers' use of textbooks and to first year's students developing more efficient ways of studying mathematics. Tall (1991, p. 17) writes:

> During the difficult transition from pre-formal mathematics to a more formal understanding of mathematical processes there is a genuine need to help students gain insight into what is going on.

According to Selden and Selden (2001) mathematics education research can not be expected to reveal one "best practice" for how to teach a topic but it "can help develop ways of teaching specific mathematical topics that arise from an understanding of both mathematics and pedagogy" (p. 247). Artigue (2001) also emphasises that it is necessary to improve the links between existing practices, and research about learning mathematics at tertiary level. In particular, future engineering courses could be a possible field of application for the research findings of this study.

Background to the research questions

It is normally expected that students at tertiary level work more individually than students in upper secondary school. Robert and Schwarzenberger (1991, p.128) write: "The students can not learn all new concepts in class time alone. Significant individual activity outside the mathematics class is now an absolute necessity". Much of that work relies on the textbook, which can become an important factor in the process of learning mathematics. In this study we consider the book as a learning tool when students take the calculus course. We search for a holistic picture of what the textbook offers to students. The main focus is on introduction and treatment of the mathematics concept in the calculus textbook used by first year engineering students. The way the concept is introduced to the students is important in the process of acquisition of the concept. Presentation of the concept should encourage interest, create motivation and start the process of learning the concept. The way in which a concept is introduced and treated can also create essential problems for the students. It is of course not possible to investigate the whole textbook in detail. Thus some kind of limitation must be done. We decided to study the chapter that introduces the derivative, because this concept is a basic concept in calculus and understanding of it is fundamental for applications and the future study of other engineering courses. It is one of the first concepts at university level mathematics that is more demanding than earlier concepts, as it is based on the concepts of function and limit, both documented to be demanding (Cornu, 1991; Juter, 2006; Juter & Grevholm, 2007). The concept of derivative is not a new one for the students. According to their pre calculus background, the students should be familiar with it. The concept has been presented in upper secondary school (Oldervoll, Orskaug & Vaaje, 2000)[1]. There it is defined both as a rate of change and as a slope of a curve at a certain point. Analysing the presentation of the concept of derivative by textbooks in the secondary school, we noticed some tendencies to a rather practical approach to the concept. After presentation of the definition, the application of the concept follows very soon. The concept is mainly used to determine some properties of functions, such as whether the function has a maximum or minimum. This fact emphasises the necessity of considering added features in the way the concept is presented and treated in the textbook used during the first tertiary mathematics course. The issue of conceptual and procedural knowledge is also important when considering students' learning of mathematics. Many of the engineering students seem to think that they have to learn only concrete and applied mathematics and not abstract and pure mathematics (Kummerer, 2001).

Thus, in the part of the study reflected in this paper we pose the following research questions:

1. What characterises the introduction of the derivative and the further treatment of the concept in the calculus textbook for first year engineering students?

2. What kind of knowledge does the textbook emphasise?

We wanted to explore the holistic impression of what learning opportunities the book offers to the students. In trying to consider the introduction of the derivative concept in the textbook we studied the context used and the way in which the concept is introduced. Considering the treatment of the concept we focus on emphasis of conceptual and procedural knowledge in the examples and exercises proposed to the students.

Below we will first present some research results relevant for our study and the theoretical framework we have used. The following constructs are part of our framework: mathematics and textbooks, the concept of derivative, the mathematical definition, and conceptual and procedural knowledge. Then we go into the methods used and the methodological considerations. This is followed by analysis and main results and we end by a discussion and some conclusions.

Mathematical textbooks and the concept of derivative

Many research studies about textbooks in mathematics have been done. However most of the research is about textbooks used on lower levels. Pepin and Haggarty (2001) analyse the ways the textbooks are used in classroom contexts and how this influences the culture of the mathematics classroom. Johansson (2005, 2006) considers the textbook as the potentially implemented curriculum. Her focus is on how teachers use the textbook and she concluded that the teachers depended on the textbook. Juter (2006), in her study about students' problems with limits, claims that textbooks used at upper secondary schools do not provide much theory or many tasks in that area, and thus most students do not have a well developed concept image about limits. Because of this the transition from high-school textbooks to university level textbooks can be difficult. A few studies have also been done about textbooks at tertiary level. Raman (2002) discusses difficulties students could have with informal and formal aspects of mathematics when they use pre-calculus and calculus textbooks. Lithner (2004) analyses exercises in different calculus textbooks with a main focus on mathematical reasoning. He also studies students' reasoning when they are working with textbook exercises (Lithner, 2003). Contrary to Lithner, who only investigates the textbook exercises, we intend to explore the entirety of what the book offers in one specific topic. Also, we did not find any studies that answer the questions about what engineering students learning mathematics are offered from a more holistic perspective on the book.

Students' problems with learning of the notion of derivative are explored in well-known and wide ranging previous research (e.g Orton, 1983; Tall, 1992a, 1992b). Orton (1983) showed that students had problems with questions that required explanation of the meaning of the derivative. Some new studies about derivative have also been done. Viholainen (2008) examined informal and formal understanding of the concepts of derivative and differentiability. The study shows students' problems with connecting formal and informal reasoning and in particular that students avoid using the definition of the derivative in problem solving situations. Hähkiöniemi (2006) has developed a model of a hypothetical learning path for the concept of derivative. According to him the learning in the conceptual-embodied world means perceiving the rate of change, local straightness and increase, steepness and horisontalness of a function. Learning in the proceptual-symbolic world (Tall, 2005) could be experienced through calculating average rate of change over different intervals. According to Hähkiöniemi (2006), this creates a natural need for the limiting process of the difference quotient and thus for the formal definition of derivative. The suggested learning path illustrates, as we see it, the need for variation in the learning process and the need to highlight different properties of the derivative.

Theoretical framework

The analysis of the introduction and treatment of the derivative concept presented in the textbook for engineering students is based on some selected theories about learning mathematics. The notions of concept image and concept definition, meaningful learning and conceptual and procedural knowledge are essential in this analysis. Below we present the different theoretical constructs that are of importance in the study.

Mathematics and textbooks

The role of the mathematics textbook seems to be varying according to the different levels in mathematics education. The textbook used at primary and secondary school usually covers the topics defined in the curriculum that students should work with during a particular school year. The textbooks used by students at university level usually cover more topics than those encountered in a single course unit. At tertiary level the curriculum is often given in a short text and the course content is defined by the list of literature. For example, the University college department where the study was conducted emphasises that the mathematics course should ensure a theoretical foundation that can be applied to engineering subject matter and that ensures that students are able to work with professional literature based on mathematics.

The textbook, recommended by the teacher, gives some important messages about what topics are expected to be learnt during the particular course and about the nature of mathematical knowledge. Formal mathematics is represented by definitions, theorems and proofs. Informal mathematics can also use definitions but they are often of a more descriptive character and refer to the intuitive understanding of the concept. The issue of interplay between formal, informal and intuitive aspects of mathematics has been and is still discussed in mathematics education research (Fischbein, 1994, 1999; Raman, 2002; Pettersson, 2008). According to Dreyfus (1991, p. 27) mathematics is often presented to the students as

> the finished and polished product, even though historical mathematics was created through error, intuitive formulations, etc. This way of presenting may work well for students who major in mathematics, but it can be difficult for students majoring in science or engineering and taking mathematics as a required service subject.

The concept of derivative

The concept of the derivative is one of the fundamental concepts in calculus. The concept is particularly important for engineering students because of its application in other subjects. At the same time the concept is complicated; it relies on the limit concept which creates many problems for the students (Cornu, 1991; Juter, 2006). The concept of differentiation is graphical in its origin and was arithmetised in the 19th century through the work of Cauchy, Riemann and Weierstrass.

In the calculus textbooks the concept of derivative is usually first defined at a fixed point as follows

> [...] we considered the derivative of a function *f* at a fixed number *a*:
>
> 1. $f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$
>
> Here we change our point of view and let the number *a* vary. If we replace *a* in equation 1 by a variable *x*, we obtain
>
> 2. $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$
>
> Given any number *x* for which this limit exists, we assign to *x* the number *f*'(*x*). So we can regard *f*' as a new function, called the derivative of *f* and defined by equation 2. We know that the value of *f*' at *x*, *f*'(*x*), can be interpreted geometrically as the slope of the tangent line to the graph of *f* at the point (*x*, *f*(*x*)). (Stewart, 2003, p.165)

Thus the definition consists of two different definitions. First the definition of the derivative in a point and then the definition of the new function *f*', the derivative function of *f*. These two aspects, local and global, of the definition have to be distinguished when the concept is introduced.

The issue of mathematical definition

A mathematical definition is designed to describe a mathematical idea. The definition can be of formal or informal character. Previous research shows that formal definitions can create serious problems in the concept formation of students (Vinner, 1991; Cornu, 1991; Juter, 2007). To study this issue we find the theory of concept image and concept definition useful. These notions were introduced by Tall and Vinner (1981). They distinguish between the formal definition, often presented in the textbook, and the complete set of ideas that a learner has about a particular concept. This *concept image* is built up from previous experiences of all kinds and can be changed as the individual meets new situations. This draws attention to the issue of previous knowledge of the students that has to be considered, when a new concept is presented. New concepts in many textbooks are presented by a definition (Vinner, 1991, p. 66). To be able to work with the concept students need to achieve a rich concept image. To present the concept by a formal definition is useful only if the definition is meant to be used actively by the students (Fischbein, 1994). Otherwise, the definition will be stored in the memory as an isolated piece of information, not linked to any other conceptual structure. Dreyfus (1992, p. 25) emphasises that it is not sufficient to define and exemplify an abstract concept. Students have to use the definition to construct the properties of the concept through deductions. The definition has to be given meaning in order to be useful. Vollrath (1994) mentions some abilities that help students to develop meaning of the definition: to give examples and counterexamples, to test examples, to know properties, to know relationships between concepts, and to apply knowledge about the concept.

Conceptual and procedural knowledge

According to Hiebert and Lefevre (1986) it is difficult to give a precise definition of conceptual and procedural knowledge: "Not all knowledge can be usefully described as either conceptual or procedural. Some knowledge seems to be a little of both, and some knowledge seems to be neither" (p. 3). Conceptual knowledge is described as knowledge that is rich in relationships. It grows through the creation of relationships between existing knowledge and new information or between two pieces of information that the learner already knows. Ausubel (2000) used the term *meaningful learning*, defined as a process through which new knowledge is assimilated by connecting it to some existing relevant aspects of the individual pre-existing knowledge structure. Other researchers, for example Novak and Gowin (1984), have elaborated on the concept of meaningful learning and emphasise that the students themselves decide if the learning will be meaningful, that is richly connected to the already existing knowledge structures. Hiebert and Lefevre (1986, pp. 7–8) defined procedural knowledge as follows:

> One kind of procedural knowledge is a familiarity with the individual symbols of the system and with the syntactic conventions for acceptable configurations of symbols. The second kind of procedural knowledge consists of rules or procedures for solving mathematical problems. Many of the procedures that students possess probably are chains of prescriptions for manipulating symbols.

Procedures may or may not be learnt with meaning. Procedures that are learnt with meaning are procedures that are linked to the conceptual knowledge. The relationship between concepts and procedures is an important issue in learning of mathematics (Hiebert & Lefevre, 1986; Silver, 1986). Students with only procedural knowledge can receive correct answers when they are working with tasks, but they do not understand what they do and why they are acting in a specific way. According to Fischbein (1994) solving procedures that are not supported by formal, explicit justification are forgotten sooner or later. And Hiebert (2003, p.17) claims that students who practice procedures before they understand them have more difficulties to make sense of these procedures later. On the other hand, students with good intuitive sense for mathematical concepts can have problems with using procedures. It is not enough to understand a system of concepts to become able to use them in solving problems. According to Star (2005) each type of knowledge, both conceptual and procedural, can be either deep or superficial. He considers flexibility, comprehension and critical judgment of use of particular procedures as indicators of deep procedural knowledge (p. 408). Some procedural operations can also contribute to establish more confidence with treating the concept and more conceptual understanding of the concept. Tall and Ali (1996) use the term ”conceptual preparation” to describe some operations or simplifications used in order to make the algorithm easier to apply. Their study shows that the more successful students were more likely to use some form of conceptual preparation.

From our interviews with and observations of students and teachers in the engineering programme (Randahl, 2010) we experience that the teachers are eager to teach for conceptual knowledge but the students are more interested in a quick fix, through learning algorithms and procedures. This makes it important for us to use the theory on procedural and conceptual knowledge as our framework here.

Methods and methodological considerations

This study is an exploratory case study. The textbook studied is ”Calculus – a complete course” written by Robert A. Adams (1991). It has been used in 2006 and 2007 in a basic calculus course for engineering students at the university college. The book is used at several universities and universities colleges in Norway (and in Scandinavia) as a main textbook in basic and more advanced calculus courses. According to the author "The text is designed for general calculus courses, especially those for science and engineering students" (preface, p. xv). We analyse the 6th edition (2006) of the book. The book consists of seventeen chapters and five appendixes. In our study, parts of chapter 2 "Differentiation" have been analysed. The chapter "Differentiation" consists of eleven sections. We analyse the first three of them (pp. 93–113):

2.1 Tangent lines and their slopes.

2.2 The derivative.

2.3 Differentiation rules.

In every section we examine the introduction and treatment of the concept, definitions, examples and exercises which are proposed to the reader.

The analysis started with an exploration of the structure of the sections in the book. We notice that the structure of the presentation of topics in the textbook is almost the same in every section: introduction, definitions and results, examples with some explanations and exercises in the end of the section. The part with exercises is strictly separated from the rest of the text. The three kinds of building blocks, introduction, definitions and results, examples and exercises were further analysed in detail.

Methods for analysis of the introduction of the concept

In the introduction of the derivative in the textbook, we investigate how the concept is presented, what context and kind of definition is used and what previous knowledge is required. We also consider what position the definition has in the treatment of the concept. We make distinction between informal and formal definition. When we use the term formal definition we mean the concept definition accepted by the mathematical community (Tall & Vinner, 1981). By informal definition we mean the verbal explanation of the concept without using mathematical symbols. We consider also how global and local aspects of the concept are treated in the text.

Methods for analysis of examples

With examples we mean mathematical problems presented together with a solution in the text. When presenting a new concept, many textbooks give some examples to illustrate properties of the concept. The formation of a concept requires examples that have something in common in order to notice the characteristics of the concept (Skemp, 1987; Zazkis & Leikin, 2007). The examples can help the student to develop better concept images and add to the students' experiences because in the examples certain aspects of the concept are highlighted. We explore how the local and global perspectives on the derivative concept are treated in the examples.

Some functions are differentiable and some are not and this might not be clear to students. An analysis of whether a particular function is differentiable or not, and if not, why not, can contribute to new experiences of the concept.

The students need both procedural and conceptual knowledge. The textbook should offer examples to illustrate both kinds of knowledge. One way to assist the students to create connections between conceptual and procedural knowledge is to focus more on justification (Fischbein, 1994). It is also important that the students are able to use knowledge in different contexts and situations. Thus, in the analysis of the examples exposed to the students we study the justification aspect and new context aspect, and we use the following categories:

1. Worked examples (only explicit solutions are given and can be used directly to find correct answers when working with exercises; no focus on justification).

2. Examples which intend to increase understanding of the concept (by using different contexts or where justification is required).

To indicate how the categories were used we offer some illustrations of the analysis. A worked example could be like the following (Adams, 2006, p. 95):

> Find the equation of the tangent line to the graph of the function $f(x) = x^2$ at the point (1, 1).

Solution:

> Here $f(x) = x^2$, $x_0 = 1$ and $y_0 = f(1) = 1$. The slope of the required tangent is:

$$m = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{(1+h)^2-1}{h} = \lim_{h\to 0}\frac{1+2h+h^2-1}{h} = \lim_{h\to 0}\frac{2h+h^2}{h} = \lim_{h\to 0}(2+h) = 2$$

> Accordingly, the equation of the tangent line at (1, 1) is $y = 2(x - 1) + 1$ or $y = 2x - 1$.

In this example, a particular procedure is given and can be directly "copied", when students work with similar problems in the exercises. Here is an example in the second category (Stewart, 2003, p. 170):

> Where is the function $f(x) = |x|$ differentiable?

Solution:

> If $x > 0$, then $|x| = x$ and we can choose h small enough that $x + h > 0$ and hence $|x + h| = x + h$. Therefore, for $x > 0$ we have
>
> $$f'(x) = \lim_{h\to 0} \frac{|x+h| \; - x}{h} = = \lim_{h\to 0} \frac{(x+h) - x}{h} = \lim_{h\to 0} \frac{h}{h} = \lim_{h\to 0} 1 = 1$$
>
> And so f is differentiable for any $x > 0$.

Similarly, it is shown that f is differentiable for any $x < 0$. Further the differentiability for $x = 0$ is considered. The left and right limits are computed. Since these limits are different, $f(0)$ does not exist and f is differentiable at all values except zero. This example needs special justification and is important, because many students assume (as soon as the definition of the derivative is introduced) that all functions are differentiable. It is also crucial to consider special points (here $x = 0$), where f is not differentiable.

Methods for analysis of exercises

Exercises mean mathematical problems given for the students to solve by themselves. We examine them with the intention to locate the emphases on conceptual and procedural knowledge and how the possible links between these kinds of knowledge could be created. Starting from the theoretical framework about procedural and conceptual knowledge we decide to use the three following categories:

1. Exercises which mainly require the use of particular procedures.
2. Exercises which require some conceptual preparation before one can use a procedure.
3. Exercises in which justification of the solution is required or new context is used.

Exercises in the first category are often called ”drill exercises”. They help the learner to develop skills in calculation. The following exercise illustrates the first category:

> Calculate the derivative of the function $f(x) = \dfrac{1}{x^2 + 5x}$.

The student is only expected to follow the specific procedure to obtain the answer.

We assume that the second and third categories of exercises promote the conceptual knowledge and help the learner to develop connections between the concept and procedures. For example consider the following exercises: (inspired by similar examples in Stewart, 2003 and Adams, 2006)

1. Find the points on the curve $y = x^4 - 6x + 4$ where the tangent line is horizontal.
2. For what values of x is the function $f(x) = |x^2 - 9|$ differentiable? Find f' and sketch the graphs of f and f'.
3. Sketch the graphs of the function $f(x) = 3x - x^2 - 1$ and its derivative $f'(x)$. What feature of the graph of $f(x)$ can you infer from the graph of $f'(x)$?

The exercises are not difficult to solve. But some preliminary reflections are required to give the correct answers. In exercise 1 one has to take into account that the tangent has to be horizontal. In exercise 2 the notion of absolute value has to be considered before the differentiation can be discussed. We consider exercise 1 and 2 to be of category 2. Sketching the graphs of both f' and f', and analysing them, in exercise 3, gives the opportunity to obtain better understanding of the derivative concept. Exercise 3 is of category 3. In the investigation of the exercises and examples, we also consider if and how the definition of the derivative is used.

Analysis and main results

Introduction of the concept

The aim of the introduction of the concept of derivative is quite clearly stated in the section. The problem of slopes is defined by the author as one of two fundamental problems which are considered in calculus. Its solution is the topic of differential calculus (Adams, 2007, p. 93).

The author makes no visible connection to students’ previous knowledge about the derivative and the way in which the concept could have been introduced in the upper-secondary school. There is no practical context/situation in the text, which clearly points out the necessity of extending existing knowledge about the derivative. The introduction of the concept has a highly formal mathematical character. The definition of the derivative concept is built up developmentally; it starts with a mathematical problem of finding a straight line *L*, which is tangent to a particular curve C at a point *P*. The Newton quotient (also called differential quotient) is introduced and the definition of the tangent and the slope of the tangent are stated in terms of the limits. The definition of the derivative is presented as the limit of the Newton quotient (p. 98):

> The derivative of a function *f* is another function *f*' defined by
>
> $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
>
> at all points *x* for which the limit exist (i.e., is a finite real number). If *f*' (*x*) exists, we say that *f* is differentiable at *x*.

The concept of the derivative is presented by the formal definition. No informal, intuitive alternatives or graphic illustrations are given. As mentioned previously, the definition relies strongly on the concepts of function and limit and they are known to be difficult for the students. The derivative of a function *f* at a fixed value is not explicitly defined. The given definition starts with the global view, the derivative as a function. Differentiability at one value *x* is mentioned after the global view. The local perspective is also treated in an implicit way as a remark, where two different kinds of notation are exposed (p. 99):

> *Remark*. The value of the derivative of *f* at a particular point x_0 can be expressed as a limit in either of two ways:
>
> $$f(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

Further the term of differentiation is introduced as follows (p. 99):

> The process of calculating the derivative *f*' of a given function *f* is called differentiation.

Sketching the graph of *f*' is described in the book as a procedure and is called graphical differentiation. Differentiation is thus related both to algorithmic and graphic treatment. Later in the text the students are guided to do algebraic calculations of derivatives from the definition of the derivative. The derivatives of elementary functions are expected to be memorized. The author writes: ”Derivatives of some elementary functions are collected in table 1 later in this section and are recommended to be memorized" (p. 100). We noticed two ways of relating to the definition:

1. Direct use; for example (ex. 11–22, p. 105):

> Calculate the derivative of the given function directly from the definition of the derivative.

Here the students were expected to use directly the procedure exposed in example 2, p. 100–101. And on the other hand:

2. Indirect use; as for example in the following (ex. 28–31, p. 105):

> Using the definition of the derivative, find equations for the tangent lines to the following curves at the indicated points.

The main aim is to find the equation for the tangent.

Results about examples

There are seven examples in section 2.1, five examples in 2.2 and ten examples in section 2.3. The majority of the examples are worked examples. They mainly present a procedure to solve a problem. It can be illustrated by the following (ex. 7, p. 111):

> Differentiate the functions:
>
> a) $f(x) = \dfrac{1}{x^2+1}$ and b) $f(t) = \dfrac{1}{1+\frac{1}{t}}$.
>
> Solution: Using the Reciprocal rule.
>
> a) $\dfrac{d}{dx}(\dfrac{1}{x^2+2}) = \dfrac{-2x}{(x^2+1)^2}$.
>
> b) $f'(t) = \dfrac{-1}{(t+\frac{1}{t})^2}(1-\dfrac{1}{t^2}) = \dfrac{-t^2}{(t^2+1)^2}\dfrac{t^2-1}{t^2} = \dfrac{1-t^2}{(t^2+1)^2}$.

and (ex. 5, p. 109):

> Let $y = uv$ be the product of the functions u and v. Find $y'(2)$ if $u(2) = 2$, $u'(2) = -5$, $v(2) = 1$ and $v'(2) = 3$.
>
> Solution: From the Product rule we have
>
> $y' = (uv)' = u'v + uv'$.

Therefore

$y'(2) = u'(2)v(2) + u(2)v'(2) = (-5)(1) + (2)(3) = -5 + 6 = 1.$

Only one worked example points out two ways to solve the problem (ex. 3, p. 109):

Find the derivative of $(x^2 + 1)(x^3 + 4)$ using or without using the Product rule.

But we also identify other types of examples: those which emphasise justification or show other possible contexts in which the concept can be used. One illustration is (ex. 6, p. 110):

Use mathematical induction to verify the formula $\frac{d}{dx}x^n = nx^{n-1}$ for all positive integers n.

Solution:

For $n = 1$ the formula says that $\frac{d}{dx}x^1 = 1 = 1x^0$, so the formula is true

in this case. We must show that if the formula is true for $n = k \geq 1$,

then it is also true for $n = k + 1$.

Therefore assume that $\frac{d}{dx}x^k = kx^{k-1}$. Using the Product rule we calculate

$$\frac{d}{dx}x^{k+1} = \frac{d}{dx}(x^k x) = (kx^{k-1})(x) + (x^k)(1) = (k+1)x^k = (k+1)x^{(k+1)-1}.$$

Thus the formula is true for $n = k + 1$ also. The formula is true for all integers $n \geq 1$ by induction.

Another example (ex. 4, p. 102):

Verify that: If $f(x) = |x|$, then $f'(x) = \frac{x}{|x|} = \operatorname{sgn} x$.

Out of twenty-two examples proposed to the students, we find that seventeen of them can be described as worked examples with emphasis on procedures. Only five examples have emphasis on justification. Emphasis on justification could support development of conceptual knowledge. It seems that the main role of the examples is to demonstrate the use of particular procedures. The students are not challenged to give examples of their own. The difference between the derivatives f' of f at a *fixed value a* and f' as a new function with *x as variable*, is not taken up as a problem to be discussed.

Results about exercises

In total, 140 exercises are proposed to the students in the three first sections. We categorise the problems according to their emphasis on procedural or conceptual knowledge.

Examples of exercises with main emphasis on procedures (category 1):

1. Find an equation of the straight line tangent to the given curve at the point indicated. (ex. 1–12, p. 98)

In order to reach to the expected answer it is only required to use the definition of the slope of the curve. No conceptual preparations are required. The procedure to receive the correct answer is demonstrated in detail in ex. 7, page 97.

2. Calculate the derivatives of the given function. (ex. 1–32, p. 113)

Only use of differentiation rules is required to receive the correct answer. Thus those exercises demand only procedural knowledge from the students.

Examples of exercises which require some conceptual preparation (category 2):

Find the coordinates of points on the curve $y = \frac{x+1}{x+2}$ where the tangent line is parallel to the line $y = 4x$. (ex. 46, p. 113)

Here the students are expected to make some interpretations of the task, like that the tangent line must have slope equal to 4. The exercises are quite easy to answer but one has to take into account some additional conditions and analyse the situation before using the procedures.

Examples of exercises that require some justification (category 3):

1. Show that $f(x) = |x^3|$ is differentiable at every real number x, and find its derivative. (ex. 52, p. 113)
2. Show that the curve $y = x^2$ intersects the curve $y = \frac{1}{\sqrt{x}}$ at right angles. (ex. 48, p. 113)
3. Show that the derivative of an odd differentiable function is even and that the derivative of an even differentiable function is odd. (ex. 49, p. 106)

These exercises are more demanding. Being able to apply the concept of differentiability is required and the issue of the absolute value has to be considered. Even if the carrying out of the solution of example 49 is not difficult, the derivative is used in a new context.

We found that there are 76 exercises (54 %) in the first category. The exercises are either ”drill” exercises or exercises which emphasise the application of different techniques.

There are 34 exercises in the second category and only 21 problems of type: ”show”, ”verify”, ”prove” which require some justification. Some of them are marked with the symbol *, which indicates that they are on a more difficult level.

We did not find exercises which require explanation of the meaning of the derivative concept, like for example ”what does it mean that the derivative to the function *f* in a particular point has the value 5”. There is rarely a focus on situations where the functions fail to be differentiable.

The set of exercises is graded in a particular way. The exercises which require mostly knowledge of easy procedures for obtaining correct answers are placed in the beginning of the set. Exercises with emphasis on conceptual knowledge are placed later. This fact can contribute to a situation (for example little available time) in which a majority of the students never work with more challenging tasks. The observations of how students work with the textbook (from the other part of this study) confirm the statement that many students never work with the tasks in the end of the exercises section (Randahl, 2010).

We find that the textbook emphasises learning of algorithms and procedures, which seems to be what the engineering students prefer. For a more long-lasting and substantial learning outcome there is a need for more emphasis on conceptual learning in the textbook and more varied examples and exercises, which illustrate the properties of the derivative in a richer way.

Discussion and conclusion

For the teacher, the textbook offers a source of aspects to teach, of examples to go through and of exercises to ask students to work with. From our interviews and observations in class we know that this also happens (Randahl, 2010). Calculus is quite different from the mathematics the students are used to from before. To give the students an overview of the main ideas of calculus, having more focus on the connections between ideas could be useful. The text could explain better the necessity of an introduction of the derivative concept, which would contribute to improve the motivation to learn the concept with understanding, in order to later use it in different fields of application. When the students do understand and master the main concepts of calculus like limit, continuity and derivative, they "will have established the foundation for a great deal of very useful mathematics" (Martin, 1969, book 7, p. 1). The book by Martin illustrates a quite different approach than the one we found in Adams' textbook.

The issue about presentation of concepts using different kinds of mathematical knowledge should be a main concern in future research about textbooks. As mentioned before the concept of derivative is not new to the students. But the students entering the basic calculus course in engineering education have rather poor concept images about the derivative (Randahl, 2010). It is mostly created by a procedural approach to the concept in the upper-secondary school. The research questions of this study show that our aim was to investigate the presentation and treatment of the derivative concept in the textbook. We find the presentation of the derivative concept offered to the students very formal. The introduction is clearly mathematical and depends on students' knowledge of the limit concept. There is no practical context or situation which explains the necessity of extending the existing student knowledge. Strict, pure mathematical contexts can contribute to the fact that students see the textbook as hard to use. The formal approach to calculus is discussed in different research papers on mathematics education (Tall, 1986, 1991; Cornu, 1991). Cornu (1991, p. 165) pointed out the problem of context in which the learning is taking place. The students have to see the concept as a useful tool and not only the presentation of a new concept by definition, a sequence of examples and exercises. In the textbook by Adams some examples are given to show applications of the derivative to represent and interpret changes and rates of changes: velocity and acceleration, dosage of the medicine and economics (for example marginal cost of production). But they are considered later in the section 2.11 and in chapter 4: "Some applications of derivatives". To point out earlier in the text the application aspect of the concept could make it more interesting for future engineers. Presentation of the concept through discussing (not necessarily very complicated) problems from different fields like physics, economics, and biology could create more motivation and interest for the concept. The emphasis on previous knowledge seems to be an important issue for the author. In the preface Adams stated "[...] success in mastering calculus depends on having a very solid basis in pre-calculus mathematics (algebra, geometry, and trigonometry) to build upon" (Adams, 2006, p. xiii). It means that the author of the book has some expectations of the students' knowledge.

But the author makes no reference to the ways in which the derivative might have been treated in the upper-secondary school. To help students to make connections between their previous knowledge and the new mathematical ideas is one of several challenges for the book. The role of the definition in the book is explicitly pointed out in the following way:

> As is often the case in mathematics, the most important step in the solution of such a fundamental problem [to find a tangent line to a curve at given point] is making a suitable definition.
>
> (Adams, 2006, p. 93).

But the introduction of the concept by using a formal definition (and some examples) is not enough to support students' learning. We claim, with Fischbein (1994) and Vollrath (1994), that the definition should be used more actively in the process of concept formation. We find that both examples and exercises have a strong focus on using procedures. In this way the textbook emphasizes the procedural knowledge more than the conceptual. To achieve meaningful learning and build a rich concept image the students can be helped by for example working with different kinds of exercises, which highlight different properties of the concept.

The specific structure of the part with exercises (with more demanding exercises at the end) does not make it easy to work with "justification" tasks. It requires that the students use the textbook in an efficient way and this can be difficult for first year students or that the teacher explicitly guides them. Ability to use rules correctly is important for engineering students. But equally important is to have learnt to use them in different contexts and to know exactly why a particular procedure is needed. It is also important that the students develop some procedural flexibility when they work with tasks (Star, 2005). By proposing tasks which require more than one way of solving the problem, the students could be challenged to make the choice and be more creative. Solving mathematical problems by using an appropriate approach and strategy and evaluating the proposed solution not only require but also contribute to develop a richer concept image. Thus, turning back to our research questions to summarise:

> What characterises the introduction of the derivative and the further treatment of the concept in the calculus textbook for first year engineering students?
>
> What kind of knowledge does the textbook emphasise?

We find that the introduction of the derivative is formal and purely mathematical with few signs of motivation or explanation of the background of the concept. Applications are not given in the introduction, and we find no intuitive explanations that could help students' reasoning. The author does not help students to overcome for example the problem to see the difference between the derivative as a function and the value of that function for a given value of the variable, which is important in order for students to create a rich concept image. The further treatment has an emphasis on procedures and memorisation and the worked examples are straightforward and easy. Few of the examples and exercises support conceptual development and knowledge and students are not challenged to justify, prove or reason more deeply using the concept. The concept images of the students are not given much opportunity to be expanded. We conclude that the textbook has much potential to be improved to meet the needs of students' meaningful learning of mathematics.

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Notes

1 Reference is given to the textbook which is most frequently used in the upper secondary schools in Norway.

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