Development of students' concept images in analysis

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Students' pre-knowledge and conceptual development in analysis were investigated at a teacher education program to reveal what pre-knowledge endured and how the students perceived the concepts a year after the course had ended. Questionnaires and interviews were used to collect data. Two students' results are presented in more detail in the article. The study was cognitively framed with the influence of situated theories to take as many aspects of concept development into account as possible. The students showed numerous connections between concepts, but they were often unable to discern valid links from invalid links. The perceived richness from many connections causes unjustifiably strong self-confidence which prevents further work with the concept. A tool for classification of the students' connections between concepts resulted from the analysis.

Teachers at upper secondary school need deep mathematical understanding to be able to explain and understand aspects of concepts. Their own interpretations have to be adjustable enough to meet their pupils' interpretations. Students make sense of their mathematics knowledge from representations of concepts which change as they are evoked. The changes may be irrelevant to the overall conception, for example just another experience of a routine operation, or they can have an important impact on related concepts if, for example, a misconception is revealed and rectified. Conceptions that are not evoked may also change over time. The changes, if not sturdily enough integrated to prior knowledge or established in mathematical practice, sometimes revert to former constellations as if they never occurred (Smith, diSessa & Rochelle, 1993).

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Mathematical analysis comprises several challenging concepts to link together. The links vary in character and a categorisation of them deepens an analysis of students' conceptions. The present study deals with changes over time as students, studying to become upper secondary school teachers, are asked to explain their conceptions of functions, limits, derivatives, integrals and continuity before a course and then again a year after.

The research questions posed are:

- What pre-knowledge do students have at the start of a basic analysis course?
- How have the conceptions changed a year after the analysis course?
- How do the students perceive connections between different concepts in analysis a year after the analysis course?
- How can the connections be categorised to form a classification instrument for associations between concepts?

Individual knowledge representations

Mental representations of concepts

The theoretical frame used in this paper is an elaboration of one used in my earlier work (Juter, 2006a, 2006b) based on concept images (Tall & Vinner, 1981) and modes of thinking mathematically. Learning implies complex structuring of cognitive representations activated in social, as well as in individual, settings. A concept image encompasses representations of concepts and processes learned or just briefly perceived. The representations are linked together, for example representations of derivatives and limits through the limit definition. Impressions from activities such as instructions, discussions, solving tasks and reading, which all lead to mathematical thinking, have an impact on the development of the concept image. Tall's (2004, 2008) three worlds of mathematics depict different sorts of mathematical development from perception of a concept through actions in *the conceptual-embodied world*, through *the proceptual-symbolic world* characterised by the duality of symbolic thinking of a concept as a process as well as an entity – *procept* (Gray & Tall, 1994) – , to a formal comprehension of the concept in *the axiomatic-formal world*. Some conceptions develop mainly in one of the first two worlds, and some develop through both, depending on concept and individual experience to reach the axiomatic-formal world.

Understanding a concept means that an individual is able to connect representations of that certain concept to his or her concept image in a significant manner (Hiebert & Lefevre, 1986) in what Skemp (1976) denoted *relational understanding*, which is different from just being able to perform a particular operation, i.e. to have an *instrumental understanding* (Skemp, 1976). Pinto and Tall (2001) described two ways of understanding a concept, trough *formal* and *natural learning*. A formal learner uses definitions and symbols as a ground in the axiomatic-formal world (Tall, 2008), whereas natural learners logically define new concepts from their concept images in the conceptual-embodied world and the proceptual-symbolic world (Tall, 2008). A natural learner has the benefit of a logical understanding of concepts' relatedness in the concept image. A formal learner has, if successful, a reasonably organized structure to build on, but if not, he or she is left with an irrelevant mass of rote learned symbols. Weber (2004) distinguished the different types of formal learners in *formal learners* (successful formal learners in Pinto and Tall's (2001) sense) and *procedural learners* (rote learning without ability of theoretical explanations).

The theoretical frame of the three worlds neatly encompasses students' development from intuitive to formal phases. Terms of embodiment, objectification from processes, generalisation, abstraction (Dubinsky, 1991), and deduction can be used to characterise their conceptual change in the three worlds (Juter, 2006a). Students' mathematical flexibility, for instance transfer between the worlds, is a measure of how accessible their knowledge is to them under different circumstances which I will come back to.

Conceptual relations and transformations

Research exposes students' struggle to link intuitive representations to formal representations (e.g. Cornu, 1991; Juter, 2006a, 2006b; Sirotic & Zazkis, 2007; Williams, 1991). Sirotic and Zazkis claimed that underdeveloped intuitions often are due to flaws in formal knowledge and an absence of algorithmic experience. Links between intuitions, formal knowledge and algorithmic procedures are necessary for understanding as depicted by Hiebert and Lefevre (1986). Even if students experience a need to link intuitive representations to new formal representations, they sometimes recognise the efforts required to understand formally stated mathematics. They may then settle with the intuitive representations that have been sufficient so far, which was the case for some students in my earlier work (Juter, 2005, 2006a). The students were asked to state and explain the limit definition and describe their learning of it. One student's strategy was to have an intuitive representation to work with in problem solving since it had been working for her at upper secondary school. She thought the definition was too much trouble learning and that it was not worth the time required. This is an example of a student who is well aware of her own motivation and her learning processes. The student was going to be a mathematics teacher at upper secondary school, but she felt that the emphasis on theory at the university drove the joy out of mathematics and she was not prepared to adjust her conceptions at that expense. The impact of her awareness was, in this particular case, satisfaction with the models of explanation she already got, i.e. no conceptual transformation. Explanation models used at upper secondary school often differs from the models used at university, with stringent formalism, and students need to re-arrange their concept images to fit the new models of explanation. Vosniadou (2007) claimed that a consequence is that students' conceptions are shallow and hence easily forgotten. Several students in the study (Juter, 2005, 2006a) learned the definition by rote at the end of the course to pass the exam on theory. Such *disjunctive generalisation* (Tall, 1991) may work in routine situations, e.g. learning a definition or a method for problem solving, and students can give the false impression of understanding a concept. The students are then in a *pseudo-conceptual mode of thinking* (Vinner, 1997). A conception, once suitably integrated in a concept image, which time rendered disjoint from most other conceptions can leave an individual unknowingly in a pseudo-conceptual thinking mode. I argue that if the individual is able to use the concept, in a, for him or her, acceptable manner, through symbols or otherwise, he or she will remain unaware of the need for rejoining the conception to the concept image. For the students to acquire relational understanding, they need to be in a *conceptual mode of thinking*, which means that they consider concepts' core features and relations to other concepts (Vinner, 1997). Links between different concepts can then be created and strengthened from mathematical experience as long as they are suitably interpreted and consciously integrated in the concept image, i.e. the students are learning *intentionally* (Vosniadou, 2007). Vinner (1997) stated that pseudo-conceptual thinking modes do not belong to a constructivist framework as "[t]he person is looking for a satisfactory reaction to a certain stimulus" (p. 121). The notion can however be used as a complement to cognitive explanations of students' actions. Vosniadou (2007) and Merenluoto and Lehtinen (2004a) claimed that cognitive and situated theories both are needed as complements to eachother in research about learning, which the example above illustrates.

Students are often unaware of the deficiencies of their prior knowledge (Merenluoto & Lehtinen, 2004a) making them feel confident about their abilities to master concepts (Juter, 2006a). There is hence no stimulus for the students to reconsider their conceptions. Reflection on prior knowledge is necessary regardless of changes as new parts are integrated in concept images (Tall & Vinner, 1981). *Enrichment* of a concept image is when new parts of knowledge are added to existing parts (Vosniadou, 1994) requiring awareness of evoked parts of the concept image, but not any structural reconsiderations. Conceptual change of this kind is characterised by *continuous growth* (Merenluoto & Lehtinen, 2004a). When newly achieved parts are incompatible with prior knowledge the learner can become aware of the problem and revise their concept images through restructuring, i.e. *discontinuous change* (Merenluoto & Lehtinen, 2004a). If a student is unable to see errors or identify crucial links, new parts of a concept image can be created and exist in parallel with erroneous or similar parts. Two parallels may be evoked in different situations depending on the origins of their creations (Juter, 2006a, 2006b).

New concepts are sometimes introduced intuitively, perhaps with an image, which set the ground for more strict representations later on as the learner is able to link the intuitive representation to a stricter one or a complete one. Images of concepts can however work in a way opposed to the intended as Aspinwall, Shaw and Presmeg (1997) found in their case study on mental imagery. A concept image, if incoherent with formal concept definitions (Tall & Vinner, 1981), can confuse rather than ease making sense of concepts and links between them. Unsuitable parts of the concept image are then evoked possibly leading to dead ends for the students' mathematical work. Vital parts for dealing with a mathematical situation may exist, but if they are not brought to mind when needed, they become useless. A mathematics student consequently needs several abilities, i.e. conceptual awareness, mastery of methods for coping with various situations (e.g. proving, problem solving and discussing), and capacity to evoke and select suitable methods for different situations (see for example Kilpatrick, Swafford & Findell (2001) for an additional description of proficiencies for mathematical activity).

Links between topics of analysis

Functions, limits, derivatives, integrals and continuity are tightly linked together. Derivatives and integrals are defined by limits of different kinds (limits of difference quotients and sums of infinitely thin rods respectively). Derivatives and integrals have a quality of being each others inverses with the possible exception of constants. Continuity is closely linked to limits by their definitions, and also to derivatives since differentiability is a stricter condition than continuity of a function's smoothness. Merenluoto and Lehtinen (2004b) studied 538 students' conceptual changes on the concept of number at upper secondary schools in Finland. The concepts density, limit and continuity were studied in connection to number. The students showed almost no links relating the different concepts. The endurance of prior knowledge was one reason for the students' disjoint concept images. Established conceptions are not easily changed (Merenluoto & Lehtinen, 2004a; Smith, diSessa & Rochelle, 1993) and prior representations based on intuitive or otherwise vague arguments are likely to interfere with aquisition of new mathematical concepts (Vosniadou, 2007).

Recent research (e.g. Hähkiöniemi, 2006; Juter, 2006a; Tall, 2008; Viholainen, 2008) shows students' efforts to cope with analysis concepts in the different modes of thinking in Tall's three worlds (2004). Hähkiöniemi concluded that students could have a procedural understanding of the limit of the difference quotient in the derivative definition, but still lack a conceptual understanding of the limit process. He stated that the students used a variety of different representations of limits in their learning of derivatives, but the students had difficulties to link them to formal mathematics. Their concept images were not enough developed to formally abstract the procedural representations. A similar result was drawn from the formerly mentioned study of students learning limits of functions (Juter, 2006a) where students' intuitive perceptions often did not cohere with their formal concept definition leaving the students with two incoherent representations, one for theory and one for problem solving. Students' struggle with separated concept images, from disability to formalise the intuitive representations, and their lack of links to other concepts form a cognitive impediment for the students to overcome.

The study

The following section presents a study of students' conceptions of functions, limits, derivatives, integrals and continuity. Outlines of the student sample and the course are followed by a presentation of the methods used.

The course and the students

The students investigated were enrolled in a ten weeks analysis course. The part presented here is part of a larger study on 15 students' pre-knowledge (Juter, 2009). They were aged 19 years or older. Three students were selected for further investigations from their responses to a questionnaire at the beginning of the course to represent different explanation models for the concepts. The course was part of their teacher education programme, but it was also given outside the program. All students had, at least, had an introduction to the concepts at upper secondary school.

The course was given fulltime over ten weeks. The students had two lectures (40 minutes each) and two sessions for problem solving (40 minutes each) twice every week, which gives a total of 80 lessons and problem solving sessions. The syllabus of the course included limits of functions, continuity, derivatives, and integrals, with derivatives and integrals as main parts of the course. Differential equations, parametric equations, polar coordinates, infinite sequences and series (Taylor and Maclaurin series) were also taught. The course included some group tasks for the students to work with between lectures. The tasks were designed to encourage the students to think about their conceptions of the theory presented at the lectures and hence become aware of possible misconceptions or deficiencies in their concept images. The students could, for example, get a task on limits to determine if it is the same thing to say

> For every $\delta > 0$ there exists an $\varepsilon > 0$ such that $|f(x) - A| < \varepsilon$ for every x in the domain with $0 < |x - a| < \delta$
>
> as
>
> For every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - A| < \varepsilon$ for every x in the domain with $0 < |x - a| < \delta$.

They were asked to justify their claims with an explanation to what the differences of the two descriptions mean. I used this particular task in my prior research (Juter, 2006a) where one striking result was that several students thought that the two statements were equivalent since ε and δ come in pairs. These students had a procedural approach (Weber, 2004) in thier learning. The group work in the present study focused on explanations of changed details or certain parts of proofs to improve the students' conceptual thinking (Vinner, 1997), for them to gain relational understanding (Skemp, 1976).

Methods and instruments

Different types of tasks were used in written questionnaires and interviews to encourage the students to describe the concepts in various contexts. Given that concept images of a concept may be created differently in two dissimilar contexts, e.g. formally and practically (Juter, 2006a), it was important to give the students opportunities to evoke their concept images in several contexts to acquire as much detail as possible of their conceptions. The tasks to the students were open at the beginning of the interviews and then gradually more detailed to make sure that the students had opportunity to describe the concepts in relation to other concepts. The study included the previously mentioned five concepts, which may seem too many for a qualitative study. They were all included for their strong relations that can be used to describe concepts from other perspectives, revealing links otherwise overlooked.

On their first session of the course the 15 students filled out a questionnaire where they were asked to describe the concepts functions, limits, derivatives, integrals and continuity and also to write what the first four concepts are used for. The reason for this open approach was to encourage the students to write from their concept images, evoked by the words representing each concept, without restrictions from other formulations. Strong parts of the concept image, i.e. parts with numerous and, for the student, valid links are presumably what students select to describe. A drawback of briefly formulated open tasks is that parts of the students' concept images may not be evoked, with the consequence that students seem to know less than they actually do.

The three selected students' perceptions of the concepts were deeper investigated in the individual interviews one year after the course ending. They got the same open tasks about functions, limits, derivatives, integrals and continuity as at the beginning of the course but orally this time. The students were exposed to the same stimulus to enable comparisons of what representations they used in their descriptions at the beginning of the course and a year after. Parts of concept images endure whereas others do not, particularly parts procedurally learnt (Weber, 2004). Pre-knowledge have a tendency to remain, or reappear, despite possible inconsistencies and instruction (Merenluoto & Lehtinen, 2004a; Smith, diSessa & Rochelle, 1993; Vosniadou, 2007) and one aim with the tasks was to discover what pre-knowledge endured. The students were also asked to formally state definitions of the concepts using their own words and symbols, i.e. not necessarily using the formulations of the textbook.

As a complement to the tasks formulated in words, four graphs were presented to the students for them to determine differentiability, integrability, limits and continuity at all points. One graph was smooth, next had a peak, the third had a discontinuity but a limit at all points, and the last had a discontinuity and no limit at that point. Links between the concepts and the special features of each concept, e.g. sufficient and necessary conditions, become exposed when they are evoked at the same time.

At the end of the interview the students got a table with the words listed in tables 4 and 5. The students read each word in connections to each of the five concepts investigated and decided whether they thought there was a connection or not. Then they explained how they perceived the connections. The words were selected from the students' prior descriptions in the questionnaire and from formulations in the textbooks used in the course and from lectures. The aim was to evoke different characteristics in the students' concept images of the different concepts and from that see how they linked them together. Accurate links can be categorised in relevant and irrelevant links. Integrals can, for instance, be linked to sums with the explanation that you can sum up different integrals, which would be an irrelevant link. I define invalid links as links that reveal a misconception or an erroneous counter perception to previous statements. In an interview situation the students might feel the pressure to deliver links when asked to, even if they feel that there are none, and therefore come up with farfetched connections. The connections are however done through links in their concept images and can thus shed light on the students' perceptions.

Pseudonym names, Alex, Ian and Kitty, are used to retain anonymity for the students. Kitty was achieving a bit higher than average, Alex somewhat lower and Ian was a typical average student judged by the course exam.

Results

The first part of the results focuses on the students' conceptual development from the first to the second time of data collection. Then their links between conceptions at the time of the interviews are presented. Alex's and Kitty's results are presented in more detail whereas Ian's results only are presented in shared tables to give room for deeper discussions.

Students' conceptual change after a year

Table 1 shows the three interviewed students' descriptions of what the concepts are used for, at the beginning of the course, compared to the other 12 students in the group. All students are denoted with a letter from A to O with Alex (A), Ian (I) and Kitty (K).

Students in the first category had a mathematical approach to their responses, without any context from real life. They wrote about exploring the functions. On functions, Alex wrote: "To examine how different graphs behave". In the second category, on the other hand, the students linked to events and sometimes to examples. One such example, again on functions, was given by student M: "Solve problems of various kinds where values vary from dependent variables, for example, how far one can

Table 1. *Categories for the 15 students' descriptions of what the four concepts function, limit, derivative and integral are used for at the beginning of the course.*

Category	Function	Limit	Derivative	Integral
1. Examine graphs/functions	**A**EJ	**AK**O	**A**BDJ	N
2. Study developments of events	BCDF**I** **K**LMO	J	CG**K**MO	GO
3. Find function values (max/min or zero)	**A**C		H**I**	
4. Change functions to other functions			**A**N	**A**
5. Differential equations		B		
6. Mean values			E	
7. Determine area or volume				BH**K**
8. Determine intervals of function values		CFGH		
9. Study relations	GHN			

go in a certain time depends on the velocity". The third category links to exploring functions' values. Similar descriptions were given for functions and derivatives. Students A and N claimed that derivatives were something to use to change functions in the fourth category. The answers in categories 5 and 6 were simply: "Differential equations" and "Check different mean values". Alex and Kitty responded similarly to other students in the group. Alex had more than one answer on what functions are used for which was, as will be evident further on, typical for him.

Tables 2 and 3 show Alex's and Kitty's responses, before and a year after the course, to the tasks: Describe the concept of function/limit/derivative/integral/continuity in your own words.

Alex's perceptions at the beginning of the course of function, derivative and integral endured the year including the course. A severe misconception is clear from his descriptions of derivative and integral as he saw them as means to simplify or change functions, which he emphasised at both times reported in table 2 (see table 1 also). He was unable to explain the concepts in more detail. The changes he made on limits remained for the year with an emphasis on the process of the limit definition with the illustration used in the course literature and in the lectures. The illustrations showed how the limit was enclosed by intervals ($|f(x) - A| < \varepsilon$ and $|x - a| < \delta$) and the meaning of the quantifiers in the definition was stressed aided by these illustrations at the lectures. Illustrations worked

Table 2. *Alex's responses to the five tasks before and one year after the course*

Task	Before the course	A year after the course
Function	A function is an approximation like an equation with the difference that you can picture a function on a graph.	$y = kx + m$ is a function for me, you use x and y. You can draw a graph on it.
Limit	A limit is what the word "says", a limitation so you know for example within what values to stay.	When you press these [the end points of an interval on the y-axis close to the function] together as much as possible you get a limit.
Derivative	You can describe a derivative as a means to "simplify" equations. It is something you do to get other functions in a graph.	You change the function […] you can get more information from the function, you see the function differently.
Integral	The opposite to derivative. Is used as derivative but in reversed meaning.	You change a function, get different information.
Continuity	It [the function] behaves the same way all the time. There are no "surprises" in the graph.	A continuous function […] changes in a re-occurring pattern all the time. [Linear and sine functions are given as examples]

Table 3. *Kitty's responses to the five tasks before and one year after the course*

Task	Before the course	A year after the course
Function	A function is a constructed series of events.	Numbers and an x to determine. A graph.
Limit	A limit is something you calculate as something tends to for example zero or infinity.	A graph […] closing in on a value but it never gets there.
Derivative	You derive a function and get for example zero values.	Area under a graph. [first but after some thought about integrals changes to] A measure on how fast something accelerates.
Integral	Reversed derivative where you calculate the area under a function on a certain interval.	Area under a graph divided in small rectangles depending on how accurate you are.
Continuity	When there are no gaps in the graph and there is only one x-value per y-value.	If you go from one value to another there can not be any gaps in it.

in a fruitful manner as the image had become a constructive part of Alex's concept image. He was able to explain why the interval on the y-axis ($|f(x) - A| < \varepsilon$) must be the interval that can be chosen arbitrarily small and from that choice find a corresponding interval on the x-axis for which all function values would be contained in the interval on the y-axis. He was not able to present a formal definition of any of the concepts.

Kitty had a conception of functions as a series of events, but she changed it to a view of the objects used when working with functions. On limits, she went from calculating to the limiting process, with the not so unusual misinterpretation that limits are unattainable (e.g. Cornu, 1991; Juter, 2006; Williams, 1991). There was obvious development in Kitty's concept image that remained for the year on derivative and integral. She could not present any formal definition though.

Kitty was confused about some concepts during the interview, but she was often able to sort it out using her concept image when needed. One example is concerning continuity and derivatives when she had answered the question about continuity in table 3:

Kitty: And then there was something about not having any edges.

Interv.: Peaks and so you mean?

Kitty: Yes ... or perhaps it was continuous then too, but there was something about those peaks anyway.

Interv.: Yes.

Kitty: Maybe that you could not take the derivative on those peaks or something like that ... no I might be thinking incorrectly.

The interview went on and the four graphs were presented where Kitty should determine differentiability, integrability, continuity and limits. One had a peak.

Kitty: Integrate, what was that ... it was the area under. [...] But that's right, now I remember [...] If you derive, to determine how the other curve [the derivative] shall look, are you not supposed to draw those lines to see? [She shows a tangent line with her finger]

Interv.: Mm.

Kitty: And that is impossible at the peak there because then you do not know if it, because it is pointy, you do not know what slope it has.

Kitty worked with her existing knowledge and found out the logical and useful properties. She combined her knowledge about constructing tangents, requiring two points to draw the line to be able to determine its slope, with the insight coming from her reasoning about integrability. Her explanation model of an integral as the area under a graph evoked her concept image of differentiable functions where tangents are uniquely defined in each point. The two processes together made Kitty understand why functions are not differentiable at peaks. This way of reasoning was typical for her during the interview.

Networks of concepts

Analysis of the students' links between concepts resulted in a set of categories (summarised in table 6). Tables 4 and 5 show how the students connected different concepts and processes together. Table 4 shows valid relevant links, i.e. links that are correctly justified and true. Invalid links and links that are just briefly mentioned without explanation are presented in table 5. Typical examples are given.

Table 4. *Valid links between concepts for the three students*

Concept	Limit	Derivative	Integral	Continuity
Change		K		
Tangent	I	K		
Sum			I, K	
Limit				I
Difference	A		K	
Function		A, I	A, I	
Area			I, K	
Rule		A, K	A, K	
Interval	A		I, K	
Volume			K	
Quotient				
Approximation	A, K			
Only one *x* for each *y*				
Only one *y* for each *x*				
Derivative			A, K	I
Integral		A, K		
No disruptions				I, K
Impossible to pass beyond				
Maximum value				
Minimum value				
Continuity				
Describes the world		I	I	
Slope		A, I, K	A	
Same change				
Describes functions		A, K	A	
Infinite	I, K			
Infinity	I			
Rate of change		I, K		
Limited				

Note. A (Alex), I (Ian) and K (Kitty).

Alex linked difference to limit in the sense that the difference is between the borders in the interval $|f(x) - A| < \varepsilon$ from the limit definition where A is the limit. He also linked limit to interval with a similar explanation implying a conceptual thinking mode (Vinner, 1997). His relational understanding of limits was also apparent in his determination of limits in the four graphs. He was right in all cases. Alex did not appear to understand derivative and continuity relationally, judged from his responses to the four graphs, since he was unable to separate crucial features of the concepts to distinguish them. Table 4 indicates that he showed no valid links to continuity implying that he had a meagre concept image for that part.

Kitty connected change to derivative as she said: "Derivative [...] is a measure of change [...] how the velocity change, kind of, and then you draw it". She first linked derivative to area (table 3), but then she changed her mind:

Kitty: ... then isn't it this area under that graph you calculate by taking the derivative?

Interv.: ok ...

Kitty: ... or is it ... at first we have these squares ... what is two x eee it become these small ... it depends on what values you take between [...] it closes in on the area under the graph

The topic changed to integrals and Kitty's concept image for integrals was evoked

Kitty: Maybe it is this ... yes it is. [...] because there it is between two values, yes that is right, that is integral and there it depends on how accurate you are [...] in the partition you get a good [measure of the] area under.

Kitty's concept images of derivative and integral were paralelly evoked forcing her to decide what the concepts actually meant.

The students' invalid and irrelevant links between the concepts are presented in table 5. Invalid links (bold) are links that reveal a misconception or an erroneous counter perception (the latter marked with ') to other statements of the student at that time. Irrelevant links (italics) are links that students just mentioned without any explanation or with an explanation without substance (the latter marked with *).

Alex showed an example of an erroneous counter perception on continuity as he stated that there is no change in continuous functions. He marked the boxes "only one x per y" and "only one y per x" with the explanation that continuous functions are linear functions, then he marked the box "same change" and gave the example of a sine curve. He repedetly explained continuous functions as having the same slope or same change everywhere. Alex had irrelevant links with explanations without substance on derivative and integral linked to "change": "You change something so it obtains a new structure". He had a similar

Table 5. *Invalid (bold) or irrelevant (italics) links between concepts for the three students*

Concept	Limit	Derivative	Integral	Continuity
Change	**A′**	_A*_, _I_	_A*_	**A′**, _I_
Tangent		_I_	**I**	
Sum	_A_	_A_	_A_	_A_
Limit		_A_	_A_	_A_
Difference	_I*_			
Function		_K_	_K_	**A**, _K*_
Area		**A**		
Rule				
Interval	_I_	_A_, **I**	_A_	_A_
Volume			**A**, _I_	
Quotient				
Approximation	_I_			
Only one _x_ for each _y_		_A*_, **I**	_A*_	**A′**
Only one _y_ for each _x_			**I**	**A′, K**
Derivative				
Integral				**I**
No disruptions		_A*_	_A*_	_A*_
Impossible to pass beyond	**A, K**		**I′**	
Maximum value	_I_	_A*_, _K_	_A*_	_A*_, **A′**
Minimum value	_I_	_A*_, _K_	_A*_	_A*_, **A′**
Continuity				
Describes the world	_A_, _I_			
Slope				**A**
Same change				**A′**
Describes functions		**I**		**A**
Infinite		_A*_	_A*_, **I′**, _K*_	**A**
Infinity		_A*_	_A*_, **I′**	**A**, _K*_
Rate of change		_A*_	_A*_	_A*_
Limited	_A_		**I**, _K*_	

Note. A (Alex), I (Ian) and K (Kitty).

description in responses to earlier tasks (table 2). Alex revealed invalid links between derivative and area, and integral and volume with the following explanations:

Alex: I see area in derivative.

Interv: Why?

Alex: Because it is a square, that is x to the power of two.

Interv: Ok, so you consider it [derivative] to be a square in itself?

Alex: Yes, the same as a square, everything about differentiation is about obtaining some kind of area and that is why it is area for me.

[...]

Alex: I link volume to integral for the same reason I link area to derivative. Volume is to the power of three or ... that is x three.

Alex's links between derivative and integral through area and volume resembles the relation of derivative and primitive function in the particular case of x^3 and x^2. The transfer was done based on procedural (Weber, 2004) processes from, in this case, irrelevant parts of the concept image. Several of Alex's links were weak in the sense that the explanations were without relevant substance. His links from derivative, integral and continuity to rate of change were all explained from his view of the concepts as functions:

Alex: I see rate of change in all of them, derivative, integral and continuity ... since I see them as curves it is some kind of rate of change [...] it [the rate of change] can be very small or very large depending on what they measure.

His explanation discloses shallow links to graphs in general rather than the specific concepts at hand.

Kitty showed an invalid link on continuity to "only one y for each x" as she thought that was a sufficient condition for a function to be continuous. At the beginning of the course she stated that continuous functions had "only one x-value per y-value" (table 3) indicating an uncertain link.

The students showed several valid connections to different words from derivatives and integrals, but not so many from limits and continuity (table 4). Table 5 indicates that the students perceived representations of concepts as well integrated in their concept images since there are so many links between them, links of which most are practically useless or even damaging to the conceptual development. Alex, who had few or no links to concepts in table 4, had most links in table 5. Kitty's links illustrate an opposite pattern to Alex's in tables 4 and 5. She had a flexible concept image and seemed to understand most connections. She only showed two misconceptions.

Discussion and conclusions

Students' conceptions

The students' pre-knowledge was variously depicted in their descriptions when they came to the course as tables 1 to 3 show. The descriptions were in some cases intuitive, some had focus on methods for problem solving and some were erroneous or not written at all. A number of pre-conceptions endured the course and a year, for example Alex's unfortunate perceptions of derivative and integral as means to change functions. A drawback of pre-conceptions is when they are wrong and remain, despite teaching and work within a course stating the opposite of the pre-conceptions (Smith, diSessa & Rochelle, 1993). A concept image that has been established for some time is not easily changed since it also demands discontinuous change (Merenluoto & Lehtinen, 2004a) affecting the nearby parts of the concept image. Other reasons to retain familiar structures are the comfort and security of the known that may not be readily surrendered. Kitty's concept image of integrals was partly the same after the year, but a development of continuous growth (Merenluoto & Lehtinen, 2004a) had occurred (table 3). This way of learning is steady since no change of prior knowledge is required; there is only a phase of adding new knowledge strongly linked to the former.

Mental representations naturally connect to images, self constructed or otherwise, supporting understanding. Both Kitty and Alex mentioned graphs. Kitty mixed up derivatives with integrals as she stated that the derivative is the area under the graph. When she, shortly after, was describing integrals she was able to make sense of her pictures of "areas under graphs" and she went back to rethink derivatives. Alex's description of limits after the year included an image resembling pictures used at lectures in the course. He had used the picture to strengthen his concept image in a, for him, useful manner. Pictures can however, as afore mentioned (Aspinwall, Shaw & Presmeg, 1997), cause confusion rather than insight. The same picture as Alex used give many students the impression that limits actually are limits determined by the endpoints of the intervals from the absolute values in the limit definition mentioned before (Juter, 2006, 2009), i.e. an upper and a lower bound. Students learn differently and hence use whatever input they get according to their purposes.

Students' links between concepts

Kitty's ability to find errors in her concept image and correct them by means of logic became apparent several times during the interview implying that she learned intentionally (Vosniadou, 2007). She did not use formal notations but her natural approach (Pinto & Tall, 2001) allowed her to deduct new or forgotten concepts from her prior knowledge.

Alex had many links, variously validated, in his concept image enabling him to confidently talk about relations between concepts. Ability to use concepts strengthens students' confidence further as long as they believe in their abilities, as in Alex's case. He did not show any traces of axiomatic-formal treatment of the concepts (Tall, 2004, 2008) and was often unable to discern correct parts from wrong parts. The interview situation may have caused him to try to make connections he did not really have, in a pseudo-conceptual manner (Vinner, 1997), which could be a reason for his connections of area to derivative and volume to integral. He could, however, lucidly explain limits in connection to the essential process of letting intervals tend to zero. He understood the meaning of the quantifiers in the definition without formally using the definition or any symbols, i.e. he had a conceptual-embodied understanding of limits (Tall, 2004, 2008) which was well established in his concept image, despite that he was not a mathematically strong student otherwise. This contradicts previous results (e.g. Hähkiöniemi, 2006; Juter, 2006a) where students' conceptions often were inadequately developed for them to have a relational understanding of limits, particularly average or below average students.

The lack of valid connections between limits and continuity is clear, and consistent with Merenluoto and Lehtinen's (2004b) results. The pattern of the invalid or irrelevant links is different with several links between the concepts. Change and sum, for instance, are not linked to limits or continuity in table 4 (valid links) but they are in table 5, meaning that these students have invalid or irrelevant explanation models of the concepts influencing their acquisition of new knowledge (Vosniadou, 2007). Understanding concepts is not the same as being able to formally express them. Students also need to have a strong and rich foundation tightly linked to the formal expressions to make them meaningful, which has been proven to be difficult (Hähkiöniemi, 2006; Juter, 2006; Viholainen, 2006). Kitty had a functional foundation to formalise and she showed evidence to be on her way to reach the formal world in her deductions. Alex showed less such evidence, except for limits. Students' lack of insight in their own mathematics conceptions and their often vague concept images are serious problems, particularly for students training to be teachers in mathematics as they need to be able to transfer their own knowledge to understandable presentations for their students and to understand their students' reasoning.

Classification of links

The classification from the analysis of the students' links between concepts resulted, as afore mentioned, in the set of categories summarised in table 6.

Table 6. *Definitions of links between concepts in the classification.*

Type of link	Definition	Tables and markings
Valid link	True relevant link revealing a core feature of the concept.	4
Invalid link, misconception	Untrue link due to a misconception of the concept.	**5**
Invalid link, counter perception	Untrue statement contradicting prior statements of the student.	**5'**
Irrelevant link, no reason	No actual motivation for the link is provided.	*5*
Irrelevant link, no substance	Peripheral true link without substance relevant for the concept.	*5**

Note. The categories are used for students' links in tables 4 and 5 marked as indicated in the last column.

The categorisation is a tool to distinguish different types of links in students' concept images and is general enough to pertain to other mathematical concepts than those in the study reported here. It is important to investigate the different types of associations students have between concepts. A vast number of valid links means that a concept is well understood, but students with several invalid or irrelevant links may, based on these deceptive grounds, feel as if they understand the concept on account of the large number of links. Invalid links require rearranging of the concept image to become valid, whereas irrelevant links may be more easily turned to valid ones through added details and awareness. The categories could be used in analysis courses for future secondary teachers to help them become aware of the state of their own conceptions and hence to gain a deeper understanding of their, and their future students', processes of learning mathematics.

Further research

The three worlds of mathematics (Tall, 2004, 2008) provide a useful broad frame including various mathematics thinking modes. The added theories in this article (Merenluoto & Lehtinen, 2004a; Vinner, 1997; Vosniadou, 1994, 2007) enable refined descriptions of students' development within the three worlds with respect to their awareness and intentions. Further research on students' identities and their conceptual development can endow with more detailed answers to questions about why some students knowingly settle with partly underdeveloped concept images, or why others remain unaware of their abilities despite indications of inconsistensies. The categorisation of links presented in table 6 can be further elaborated, particularly the valid links part, to give valuable information on students' conceptions.

The three students, Alex, Ian and Kitty, have been interviewed again two years after the course end as part of a study comprising four student groups with an emphasis on their teacher identity. One aim is to find correlations between their professional attitudes to matematics and learning, and their own mathematical development. The divided perspective of teacher identity and mathematical development is meant to give insight to intentions and abilities of our future teachers.

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Sammanfattning

Blivande gymnasielärares förkunskaper och begreppsliga utveckling i analys undersöktes i anslutning till deras första analyskurs. De fick en enkät i början av kursen och en intervju genomfördes ett år efter kursens slut för att spegla utvecklingen i deras begreppsbilder. Kopplingar mellan olika begreppsrepresentationer undersöktes och kategoriserades i en klassificeringsmodell som utvecklades efter analysen av studenternas hantering av begreppen. Den teoretiska ramen är kognitiv men har inslag av kontextuella teorier som komplement. Två studenter har valts ut för djupare beskrivningar i artikeln. Studenterna visade många kopplingar mellan begreppen, men också en oförmåga att avgöra om de är meningsfulla. Studenter upplever då att de har förmåga att koppla samman begrepp, som i sin tur ger ett obefogat starkt självförtroende. Delar av båda studenternas begreppsbilder var så pass utvecklade att studenterna kunde beskriva avgörande processer, men ingen av dem hade en formell begreppsbild.