Handheld calculators as tools for students' learning of algebra

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What evidence can be found in recent research literature of the potential positive or negative effects of using graphic calculators (GC) and symbolic calculators (CAS) in mathematics education? The focus of this literature review is the use of handheld calculators and their effect on algebra learning, with theoretical backgrounds for the use of this type of technology in classroom practice. Special attention is given to three areas: students' conceptions of literal symbols and of algebraic expressions, fundamental for their ability to work with algebra; functional and modelling approaches, both important for students' view of algebra as a useful tool in problem solving; and approaches within CAS, which put special demands for changes in educational methods. Results of some recent meta-studies, based on a relatively large number of research papers and reports, are also discussed, as well as the importance of students' and teacher's beliefs. Common results are compiled and synthesised for a formulation of some important implications for teaching and pre-service teacher education.

Are calculators important cognitive tools that empower students in their mathematical learning in general and help them to develop a better understanding of algebraic symbols and concepts in particular? Or do they instead function as mathematical "crutches" that prevent students from acquiring proper mathematical skills in arithmetic, algebra or calculus? In a debate article written by two teachers of mathematics at tertiary level, the use of what they labelled "advanced calculators" in mathematics education was highly questioned (Thunberg & Lingefjärd, 2006). Their article was motivated by the fact that CAS calculators for the first time was allowed in parts of the Swedish national tests at upper secondary level. Thunberg and Lingefjärd's claims and hypotheses were in short:

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– Experience shows that the graphing calculators of today are often used in a destructive way in mathematics education. Practice in basic skills does not occur when calculators are available, while the experimental and inquiry-based activities that calculators are said to promote are rare. With CAS calculators these problems are likely to become aggravated.

– Research has failed to show any generally positive effects on mathematical concept formation and understanding with the use of CAS calculators in classroom practice. On the contrary, conceptual understanding is undermined when basic skill training is absent.

They also pointed at general risks with ”the widespread and routine use of calculators at primary and secondary level” (p. 12), such as a decline in manual and mental skills in mathematics for all students, and especially among those who continued their studies at university level.

This reaction is not uncommon among tertiary teachers, and can also be found among some teachers at secondary level (Trouche, 2005a; Thunberg & Lingefjärd, 2006). The mistrust of and reluctance to use calculators is expressed in various ways. They are said to be too crude, to prevent some elementary learning processes, and they do not fit the conception of mathematics which teachers have. But the core of the problem is that ”the integration of complex tools into the classroom requires teachers to undertake deep questioning of their courses, the exercises they have already prepared, and their professional methods” (Trouche, 2005a, p. 19). There are certain teaching styles that are more compatible with the use of calculators than others. Teachers who tend to employ more interactive or inquiry-oriented methodologies during instruction are also more comfortable with ICT tools than teachers who use other teaching approaches.

In my own long experience as a practicing teacher at secondary level I have been directly involved in introducing various types of calculators to students, and as with all new tools, there are initial obstacles and problems. These have all been possible to overcome, but the condition has been my willingness to accommodate the implementation of calculators by changing my own classroom practice. In my daily work with students, I have experienced very little of the serious risks that are claimed above.

My longitudinal study of factors that influence algebraic learning for upper secondary students identified two important factors: *concept development* and *instruction* (Persson, 2005). Graphic calculators were commonly used in all seven classes participating in the study and at least two of the teachers involved tried to explore the possibilities provided by these cognitive tools in classroom practice. Although calculators were not a main issue for the study, they were specifically mentioned in the results. For example, one student who was not a high-achiever uttered that ”I understood mathematics better when I could use the graphic calculator” in the evaluation at the end of his final mathematics course (p. 55). In ”suggestions for further research” calculators were discussed in one section, including proposed questions that are also part of the focus of this paper.

The motivating power for my research and for this article is to contribute to the knowledge of factors for improving classroom practice. Based upon this I propose a set of hypotheses which to a great extent contradict those set by Thunberg and Lingefjärd:

- Calculators are powerful computing and visualizing tools that enable students to try different solving methods and conjectures, without the burden of time-consuming, trivial activities like manual drawing or basic computing and simplification.
- The use of calculators promotes students’ understanding and forming of mathematical concepts, such as algebraic ones, by making it possible to experiment with these concepts and to see them in different representation forms.
- There is no generally observed decline in students’ manual or mental skills.
- Students become more active in mathematical work and show a more positive attitude towards mathematics when they can use calculators.
- Findings in current research support these hypotheses.

An underlying assumption here is that calculators are used ”in the right way” by teachers in their classroom practice. What that means and the importance of teachers’ beliefs is, needless to say, crucial for the outcome of the use of calculators in mathematics education.

The aim of the literature review

There are several motives for this article, like refuting those who claim that calculators are harmful for students’ mathematical learning. Most important, however, is the intention to present an overview of the findings of recent research into calculator use in mathematics education, and the implications for classroom practice that can be formulated. In particular, I wish to make relevant research findings easily accessible to teachers and teacher educators. Therefore, the guiding perspective of the findings I present is the potential usefulness in the readers' own work with school or teacher students.

In a general overview of the literature, it is not possible to provide the readers with a deeper and more profound discussion about the presented issues. Instead the aim is to facilitate the access to information that otherwise could be difficult to reach. My special focus is the influence of calculators on students' algebra learning. However, most of the available research findings do not separate this area from other parts of mathematical education. Therefore, I will include such general results in this study. My research question here is:

> What are the overall findings and implications for teaching in current research on learning mathematics, in particular algebra, with calculators as tools?

First, parts of the theoretical background for the use of calculators and for students' difficulties with algebra in connection with these that can be found in recent literature will be shown. Second, a review of recent research literature in the area, especially focussing on algebraic concepts and skills and on teachers' and students' beliefs, will be given. Third, some recent meta-studies and literature reviews of papers prior to 2002 are presented. Finally, some conclusions from research results will be drawn, along with a set of general implications for teaching.

Method and methodological discussion

Sources and search method

Various sources were used to obtain material for this literature review. An electronic search was made using *Educational resources information center* (ERIC), *Academic search elite* and *Springer link*. The focus of the search was graphic calculators (GC) in mathematics education, specifically in connection with algebra learning, computer algebra systems (CAS) with symbolic calculators, and also theoretical perspectives including concepts as *mediating tool*, *cognitive tool*, *instrument*, *register*, *object-process*, and *procept*. The limitation corresponding to "current research" was that these articles and reports would be from the year 2002 or after (although exceptions were made). Some meta-studies and a literature review of older research were found, and these create a platform for more recent findings together with some essential theory-building articles.

Proceedings of the conferences of the *International Group for the Psychology of Mathematics Education* from the years 2004–2006 were searched for papers relating to research on the use of calculators in mathematical activities, and students' and teachers' beliefs and attitudes towards them.

Some important books edited 2002 or later, with chapters discussing calculators and CAS, were also included in this paper, as well as three PhD theses, two of which are from the Nordic countries. A search within ProQuest for dissertations and theses concerning the use of calculators and CAS in algebra learning showed 10 American dissertations since 2002 within this research area. None of these will be reviewed here due to reasons of accessibility.

Criteria for the selection of papers

A set of quality criteria were used for the selection of papers, articles or books to be included in this article. The most important criterion is that they are relevant for my intentions, in that they include findings of importance for classroom development and also give implications for teaching. In addition they must also have undergone a solid review process before publication.

The text in the selected books and papers was read through, in most cases more than once, and interpreted from the perspective of a practitioner. A search was made for findings of relevance and for appropriate citations in order to support and build an argumentation for more general conclusions and implications.

Limitations and critical reflection

A literature search of this kind presents great difficulties in finding all relevant papers and books to include in an article. In fact, it would be almost impossible to do so as not all journals are easily accessible to me. More papers exist that have not been included in the analysis here, but on the other hand, I have found so much material that my conclusions are expected to be valid anyway.

There is a certain risk that I am biased in my search for papers, findings and citations by my generally positive view on the use of calculators as tools for mathematics learning and by my experiences as a practitioner. But I have tried to read with a critical eye, and none of the papers I have found that meet the quality criteria have in fact been excluded. In the article there is a section in which some of the possible obstacles with calculator use are accounted for.

In the debate concerning calculators some (e.g. Thunberg & Lingefjärd, 2006) have pointed out that electronic manufacturers, who obviously have economic interests in selling many units, occasionally fund research. My literature references include one such report (Burrill et al., 2002), which is funded by Texas Instruments. However, the scientific methods are most rigorous and carefully described, and no signs of inappropriate influences can be found as far as I can discover. In a literature search, this type of potential problems must be carefully observed.

Theoretical background for the use of calculators

Within all the studies in the review some theoretical background forms the methods used, the analysis of the data and the interpretation of the results. In this section some of the theories will be presented. They have been chosen partly for the reason that they more or less coincide with the set of theories which form my own view on calculators as tools for learning. I use what Gravemeijer (1994) and others call *theory guided bricolage*, which means that I try to combine and integrate global and local theories that "fit" together well for the specific topic, in this case calculators in mathematics education.

Calculators are in literature often explicitly placed in a context of *socio-cultural learning* through the concepts of *artefacts* and physical and psychological *mediating tools*, first described by Vygotsky (cf. Säljö, 2005). This theory has undergone a progression in different directions. One example of this is Wartofsky's (1979) classification of *primary, secondary and tertiary artefacts*, and another is activity theory (see Nardi, 1996) with the agent-objective-others triad. Other perspectives on learning, like *constructivism* or *interactionism*, have been more implicitly present in some of the literature, but can be detected in a closer analysis.

A tool can develop into a useful *instrument* in a learning process called *instrumental genesis* (Verillon & Rabardel, 1995; Guin & Trouche, 1999; Artigue, 2002; Trouche, 2005a, 2005b), which has two closely interconnected components; *instrumentalization*, directed toward the artefact, and *instrumentation*, directed toward the subject, the student (see figure 1). These processes require time and effort from the user. S(he) must develop skills for recognizing the tasks in which the instrument can be used and must then perform these tasks with the tool. For this, the user must develop *instrumented action schemes* that consist of a *technical part* and a *mental part* (Guin & Trouche, 1999; Drijvers, 2002a, Drijvers & Gravemeijer, 2005).

In a mathematical learning activity one must distinguish between *instrument* (the artefact is used in utilisation schemes) and *instrument*

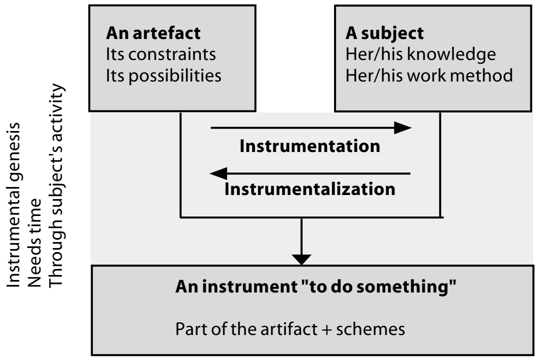

Figure 1. *From artefact to instrument (Trouche, 2005b, p. 144)*

for semiotic mediation (the teacher uses it to develop a specific meaning related to the mathematical content). To learn instrumentation schemes does not in itself induce mathematical meaning and knowledge. Instead the teacher must actively guide the students in a controlled evolution of knowledge, achieved by means of social construction in a class community (Mariotti, 2002).

Cognitive tools are tools that are designed to support cognitive processes, and are instruments that can enhance the power of students in their thinking, problem-solving and learning in different ways. Reznichenko (2007b, p. 6) describes the functions of calculators as cognitive tools. They (1) support cognitive and meta-cognitive processes; (2) share cognitive load by providing support for lower level cognitive skills so that resources are left for higher order cognitive skills; (3) allow learners to engage in cognitive activities that otherwise would be unreachable for them; and (4) allow learners to generate and test hypotheses in the context of problem solving.

Heid's (1997) list of cognitive tools contains *graphic calculators* (GC), *computer algebra systems* (CAS), *micro-worlds* and *dynamic geometry tools* (e.g. Aplusix and Cabri), and technology-based laboratory devices (e.g. CBL). These tools all build on modern digital technology and are summarised as ”ICT tools”.

A group of French mathematics educators have applied instrumentation to the learning of mathematics using calculators, computers and other ICT tools (Artigue, 2002; Guin & Trouche, 1999; Lagrange, 1999). They have all, in different settings, observed and analysed students using graphic and symbolic calculators, and recorded how they developed both instrumental and paper-and-pencil schemes. Ruthven (2002) presents a critical view on the French research and the instrumentation model. He argues that the advantages that calculators in principle bring about will not occur in an ordinary classroom practice and that ”[...] effective, independent use of CAS appears to impose greater technical and conceptual demands on students [...]” (p. 288).

Rivera & Becker (2004) give a sociocultural account of the mediating functions that calculators and social interaction play in students' understanding of polynomial inequalities. They refer to recent investigations that ”provide strong evidence that learning takes place through experiences that are oftentimes mediated by physical or material and symbolic tools and with assistance drawn from other (competent) individuals” (pp. 4–82), which is supported by their own results.

A basic question in the learning of mathematics is which cognitive systems are required to give access to mathematical objects. Duval (2006) has created a framework of *representational registers* that are mobilized in mathematical processes. He distinguishes between two types of transformation of semiotic representations: *treatment* and *conversion*. Treatments occur in the same register (e.g. solving an equation) and conversions involve changes of register (e.g. graphing a function) (see figure 2). His important hypothesis is that ”comprehension in mathematics assumes the coordination of at least two registers of semiotic representation” (p. 115) and also that ”the root of trouble in mathematics learning [is]: the ability to understand and to do by oneself any change of representation register” (p. 122). It is easy to see the role of calculators as

	Discursive representation	**Non-discursive representation**
Multifunctional registers Processes cannot be made into algorithms	*Natural language:* *Orally:* arguments, explanations *Written:* theorems, proofs	*Iconic:* drawing, sketch, pattern *Non-iconic:* geometrical figures constructed with tools
	Transitional auxiliary representations: No rules of combination	
Monofunctional registers Most processes are algorithmic	*In symbolic systems:* Numeric Algebraic Symbolic (formal languages)	Diagrams Cartesian graphs Interpolation, extrapolation

Figure 2. *Classification of registers that can be mobilised in mathematical processes (from Duval, 2006, p. 110).*

facilitators and providers of opportunities working with both treatments and conversions in the classroom practice.

The *process-object duality* is a special source of problems in algebra, discussed by several researchers (e.g. Drijvers, 2003; Graham & Thomas, 2000). It is closely related to the possibility to make conversions between representational forms. A mathematical concept often has two perspectives, as an operational process and as a structural object. For example, the expression $x^2 + 2x - 3$ can be seen (and used) both as a calculation of values in a pattern or for graphing a function, and as an object and representative for quadratic expressions with certain properties, such as the possibility to factorize. The objectification of processes, named *reification* by Sfard and Linchevski (1994), takes time and is hard for many students, and the flexibility to shift between the two perspectives is essential for advanced mathematical thinking (Tall et al., 2000). Tall and colleagues introduced the term *procept* for the combination of the two.

Tall (2008) has later completed the theoretical model in his framework of "the three worlds of mathematics": the *conceptual-embodied world*, based on perception of and reflection on properties of objects in the real world; the *proceptual-symbolic* world that grows out of the embodied world through action and is symbolised as thinkable concepts that function both as processes to do and concepts to think about; and the *axiomatic-formal world*, which reverses the sequence of construction of meaning from knowledge of real world objects to theoretical concepts based on formal definitions. School mathematics usually builds on the first of these "worlds", with the goal that students gradually will be led into the second one. For this, calculators can be important facilitators, if they are strategically used in education. Especially CAS calculators provide interesting possibilities for working with problems within both worlds. The interaction between the two first worlds then form the basis for transition into the third world, which is characterised by formal definitions and theorems with proofs. This is the level of cognitive development that is demanded at tertiary level, but also one that is hard for students to reach. In this, the formation of the proceptual-symbolic world holds a key position.

What can be found in literature reviews and meta-studies?

Some recent literature reviews and meta-studies summarize parts of the research prior to 2002. These studies form the basis for later research in the field, and serve as a platform to build upon in order to make more general conclusions and to formulate implications for teaching.

Reznichenko (2007b) reviews literature about learning with graphing calculators, mainly from 1986 to 2002. Focus is on the use of computers and calculators as tools for mathematics teaching and their effects on students' achievement in algebra and calculus. Results from 44 articles and papers are compiled and summarized. The conclusions Reznichenko draws from these are that electronic technology is an enhancer that puts students in an active role and teachers in a facilitator role. Students perceive problem solving differently when they are free from numerical and algebraic computations to concentrate on problem set up and analysis of solutions (p. 15).

In another review Lagrange et al. (2003) study research and innovation in the field of the integration of ICT in mathematics education. Although ICT represents a broader concept than ”cognitive tools”, it still contains the various types of calculators that are used in the studies that are accounted for in this article. The 79 publications are from 1994 to 1998, and the authors make a meta-study analysis with a multidimensional framework. Their perceived dimensions are: (1) the general approach of ICT in education; (2) the epistemological and semiotic dimension; (3) the cognitive dimension; (4) the institutional dimension with the role of instrumented techniques in conceptualisation of mathematics; (5) the instrumental dimension with the instrumental processes; (6) the situational dimension which deals with the influence of ICT on learning situations; and (7) the teacher dimension which looks at the teacher's beliefs and at the way s(he) organises the classroom activities (pp. 244–245).

Their analysis shows that research in these years went in several directions, with a ”wide range of approaches, from innovation, working on the most recent developments and providing potentially interesting contributions on the use of up-to-date technology, to didactical research that we saw as elaborations from these contributions” (p. 255). But the observation is also that it restricts its analysis to potentialities of ICT itself rather than questions raised by its insertion into ”ordinary” mathematics teaching (p. 256), and the authors also question the real influence of ICT on classroom practices.

Focussing on studies of calculator (GC and CAS) use in algebra and calculus courses, Kulik (2003) reports exceptionally high effect sizes for higher scores on conceptual tests, and also that students' ability to solve computational problems with paper-and-pencil did not suffer from the use of these technologies.

Ellington (2003) made a quantitative meta-analysis of 54 studies, published between 1983 and 2002, on the effects of calculators on students' achievement and attitudes. The criteria for the studies were that they should involve students in mainstream K–12 classrooms, and that the reports of findings should provide data necessary for the calculation of effect sizes. Statistically significant findings were that: (1) operational skills, computational skills, skills necessary to understand mathematical concepts, and problem-solving skills improved for participating students and that: (2) students who used calculators while learning mathematics reported more positive attitudes toward mathematics than their non-calculator counterparts (p.455).

A synthesis of 43 reports, chosen from over 180 research reports with a set of criteria related to the use of handheld technology in secondary mathematics, was made by Burrill et al. (2002). The results provide findings in the areas: comprehension, equity, professional development, usage, approach and mathematical context, and implications for classroom practice are elaborated. Among the notable evidence from the reports is that mathematics and technology must be integrated for the outcomes to be most beneficial, and that students using calculators "are more flexible in their solution strategies, make conjectures and move among algebraic, numeric and graphical approaches, develop calculator-based strategies to manipulate symbolic expressions, and work comfortably with real data" (p. vi).

Algebra learning – what can be found in the reviewed papers?

A main concern for this paper is how calculators affect students' learning of algebra. In a technological environment algebraic symbols and expressions can look different from the conventional one. It is not unlikely that the way the interface (keypad and screen) appears will change the way algebra is perceived and also the way protocols of mathematical activities are written. Some algebraic conventions must be addressed explicitly in instruction. For example, with a CAS calculator *ab* is interpreted by the device as one variable (named with two letters) and not as the product of the two variables, *a* and *b*.

In the following, I will first elaborate on some of the difficulties that algebra presents in mathematics education. Then I have categorized the related literature thematically into three important groups: students' *conceptions of literal symbols and of algebraic expressions*, fundamental for their ability to work with algebra; *functional and modelling approaches*, both by many considered as fundamental for students' view of algebra as a useful tool in problem solving; and *approaches within CAS*, which put special demands for changes in educational methods.

Difficulties with learning algebra

Algebra has always appeared as an especially problematic area of school mathematics. Teachers have presented algebra using various approaches, but in many cases their efforts have ended in poor results. Drijvers (2003, pp. 41–42)) has listed five difficult aspects of the learning of algebra:

- The formal, algorithmic character of the procedures that the student can not relate to informal and meaningful approaches.
- The abstract level at which problems are solved, compared to the concrete situations they arise from, and the lack of meaning that the student attributes to the mathematical objects at the abstract level.
- The need to keep track of the overall problem-solving process while executing the elementary algebraic procedures that are part of it.
- The compact algebraic language with its specific conventions and symbols.
- The object character of algebraic formulas and expressions, where the students often perceive them as processes or actions, and will have problems with the 'lack of closure' obstacle.

In research literature, different approaches to algebra have been presented: the approach by *generalization* pattern and structure; the *problem-solving* approach; the *functional* approach; the *modelling* approach (Bednarz, Kieran & Lee, 1996); and the *language* approach (Drouhard & Teppo, 2004). All of these are highly relevant in the use of calculators, and two of these are particularly obvious when GCs are used.

Literal symbols and algebraic expressions

Understanding of variables is a key to students' learning of algebra. The sometimes great difficulties students encounter when they try to grasp the different aspects of literal symbols have been thoroughly described in research literature. Graham and Thomas (2000) introduced literal symbols to students, age 13–14, using a module of work based on graphing calculators. These modules provided an environment where students could experience letters in the role of "hidden numbers" and as "generalised numbers" in simple algebraic expressions. One example of an activity in the study is shown in figure 3. For each algebraic expression the students first discussed what the possible result could be, and then entered

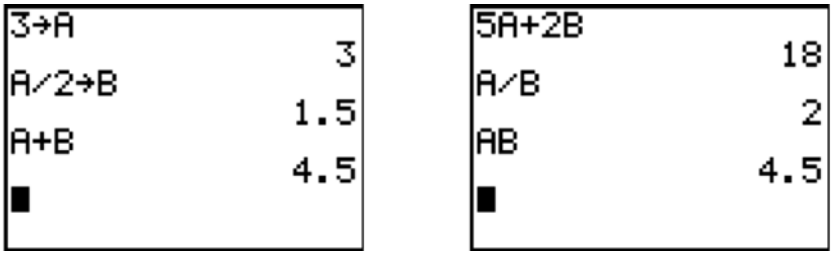

Figure 3. *Letters as "hidden numbers".*

it in the calculator to verify their conjecture. The conclusion was that the use of a GC had significantly improved students' understanding of the way literal symbols are used in elementary algebra and that students became versatile in their perspective of these. Moreover, "the gains were particularly noticeable, in terms of relative advancement, for the weakest students" (p. 279).

Functional and modelling approaches

Bardini, Pierce and Stacey (2004) analysed the consequences of having a functional-modelling approach to the teaching of algebra with the use of GC, with special attention on the impact on students' use of symbols. Among the conclusions are that this technology afforded the students support for their exploration of real world problems and that they quickly came to make sensible use of symbols and understand functions.

An interesting question is how students manage to produce graphs that enable them to solve tasks. With special software that captures the students' keystrokes Berry and Graham (2005) could follow and analyse students' solving strategies and the efficiency of the use of the technology. Beside the purpose of investigating the instrumentalization of tasks, this software also offered students the possibility to reflect on their own learning of mathematics concepts and skills by replaying the key-strokes alongside their written solutions. However, what the key-strokes do not show is how students interact in the solving process. In a teaching sequence with grade 11 students, the students were asked to produce graphs corresponding to the relationship between time and distance of a cylinder moving up and down an inclined plane, using a GC (Radford, Demers, Guzman & Cerulli, 2003). Among the results of the study were that a complex relationship between diverse semiotic resources such as gestures, graphs, words and artefacts became interwoven during the mathematical activity, and allowed the students to make sense of the time-space graphic expressions.

A central question is how the use of calculators affects results on tests, especially national tests. Heller, Curtis, Jaffe and Verboncoeur (2005) investigated the relationship between instructional use of GC and students' achievement in an algebra course at secondary level. Their results from the study including 458 high-school students show that the more calculators were used and integrated in instruction, the higher end-of-course test scores were achieved, including in the parts of the test where calculators were not allowed.

In his PhD thesis, Bergqvist (2001, p. II:9) describes a series of studies where students at upper secondary level explored polynomial expressions and functions with GC. They were also presented with conjectures like:

– A linear function and a quadratic function always intersect in two points.

– If the graph to a quadratic function crosses the x-axis in two points there is a point between the intersections where the tangent to the graph is horizontal.

– The graph of a third degree polynomial always crosses the x-axis.

The students were asked to decide if the conjectures were true or false and to verify their conclusions. Among the results it can be noted that many students were reluctant to use the calculators for verification, and believed that they had to work in the ”normal” way (i.e. pen-and-paper) to get the teacher's approval.

How then are calculators to be used at tertiary level? In some courses, especially in ”pure” mathematics, they appear to be banned, but a shift towards a more widespread use can be seen. Hennessy, Fung and Scanlon (2001) used GCs in an undergraduate course at the Open University, including functions. The purpose was to investigate perceptions of the GC and the features which facilitate graphing and linking between representations. Among the results of the study were the following:

– Portable graphing technologies present a unique opportunity to help mathematics students (at secondary and university level) develop concepts and skills in a traditionally difficult curriculum area. (p. 282)

– Experimentation and cumulative experience with certain critical features of the technologies encouraged engagement with the calculator using an exploratory approach. (p. 283)

– The findings have contributed some information concerning how – and which (physical and perceived) features of – graphic calculators can mediate collaborative problem solving. (p. 283)

In a study of less successful algebra students using graphing software, also in a functional approach, Yerushalmy (2006) showed that these students obtained a broader view of mathematics. The students could confirm conjectures and complete difficult operations that had been unreachable for them before they had access to graphing software. However, they also delayed using symbolic formalism and most of their solution attempts focused on numeric and graphic representations. For this, Yerushalmy proposes working more in a CAS environment:

> We should explore the possibility of integrating software with symbolic manipulation capabilities in the work of beginning algebra students, especially for those who need it to become fluent in algebra, even at the price of giving up mastery of manipulations. (p. 385)

Approaches within computer algebra systems

The development of symbolic calculators, together with software for symbolic computation, opens many new possibilities for working with algebra, but it also creates a new set of problems. What kind of changes must be made in educational practice, which tasks must students be able to do mentally and by paper-and-pencil, and what is algebraic knowledge, really? In addition, which particular problems do students face in the instrumental genesis with this tool?

Drijvers (2002a) investigated, in a CAS environment, the instrumentation triad *screen-paper-mind*. One of his conclusions was that ”to establish the complex relationship among machine technique, mental conception, and paper-and pencil work, students must perceive the congruence among them” (p. 32). He also provided implications for teaching such as the importance of ”paying explicit attention to the development of instrumentation schemes”, ”taking care of a simultaneous development of both the technical an the mental aspect of the instrumentation schemes” and the fact that ”many difficulties that students encounter while working with a computer algebra device can be viewed as instrumentation obstacles” (p. 31).

Drijvers' PhD thesis (2003) concerns students' understanding of the concept of parameter in equations, systems of equations and functions, again with the instrumentation schemes as a background. One example is (see figure 4):

9.4 Consider the equations $x^2+y^2=25$ and $x \cdot y=10$.

a. Isolate y in both equations and have the graphs drawn.

b. Solve the system of equations.

c. What is the biggest value that can be substituted instead of 10 so that the system still has a solution? (p. 142)

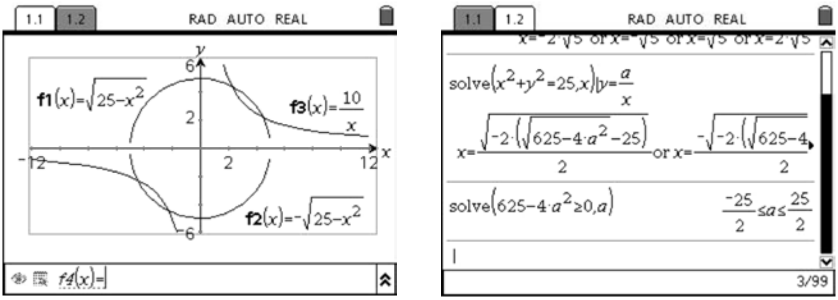

Figure 4. *A possible solution to question 9.4 in Drijvers' PhD thesis on a Texas TI-Nspire.*

Drijvers emphasises the importance of the didactical contract (Brousseau, 1997) between teacher and students for the use of calculators in the classroom and of discussing problem-solving strategies. He also concludes that "the possibility to solve equations in the computer algebra environment with respect to any unknown improved the students' flexibility concerning the roles of the literal symbols" (p. 324), and that "two issues seemed to influence the development of the higher level understanding of the concept of parameter: the use of realistic problem situations and the insight into the meaning and structure of expressions and formulas" (p. 324). Results from the same study are further elaborated in a chapter describing how instrumented action schemes for solving parameterized equations and substituting expressions can be used to assess students' progress in developing mental conceptions of algebra (Drijvers & Gravemeijer, 2004).

Kieran and Drijvers (2006) explore the relationship between thinking and technique, and report results from two teaching experiments about the differences between equivalence, equality and equation, and on generalizing and proving within factoring of the expression $x^n - 1$ for different values of n. The study is part of a larger project, with results also discussed in Kieran and Saldanha (2005). They adopted an anthropological view, summarized by *task-technique-theory* (TTT), and the main finding was evidence for the relation theory-technique, and that these emerge in mutual interaction. CAS was used as a didactical tool for working with underlying theoretical ideas in algebra that are rarely discussed in mathematics classes. However, ”the epistemic value of CAS techniques by themselves may depend both on the nature of the task and the limits of students’ existing learning” (Kieran & Drijvers, 2006, p. 258).

There can be a problem in the way that students present algebraic syntax in written records when they use CAS (Ball & Stacey, 2005). Students might use different syntax and calculator methods to solve even basic problems. What can be accepted in writing, and how will students know what is specific to their CAS and what is standard mathematical notation? Ball & Stacey show that teachers’ discussions with their students about proper mathematical syntax in comparison with CAS language were a key factor in helping them to develop good practices.

Obstacles with the use of calculators

Research has revealed obstacles and possible negative effects of using calculators of various types (e.g. Drijvers, 2000, 2002b). Many of these are of a *technological* and *cognitive* nature. The calculator is a ”black box”, and its working modes are not transparent to either the students or, in many cases, the teachers. A graphic calculator only works numerically and this often creates problems with interpretation of the results it gives: numbers, graphs etc. There can be a tendency for students to accept only numerical answers and to over-interpret all of the decimal places that are presented on the display.

With CAS calculators the black box character is even more apparent. In a study of how symbolic calculators were introduced in an upper secondary class, Drijvers (2000, p. 205) points out some important problems of both of a technical and mathematical nature, especially concerning algebraic understanding:

- The difference between the algebraic representations provided by the CAS and those students expect and conceive as ’simple’.
- The difference between numerical and algebraic calculations and the implicit way the CAS deals with the difference.
- The limitations of the CAS and the difficulty in providing algebraic strategies to help the CAS to overcome these limitations.
- The inability to decide when and how computer algebra can be useful.
- The flexible conception of variables and parameters that using a CAS requires.

Students managed to solve optimisation problems in a meaningful way and showed understanding of relevant concepts and strategies for solving problems. In a later article (Drijvers, 2002b), more obstacles were added, such as the difficult transfer between CAS technique and paper-and-pencil and the problems with interpreting the CAS output.

However, this list of obstacles should not be interpreted as evidence that CAS is inappropriate for mathematics education. On the contrary, "... obstacles are opportunities for learning that can be exploited in interaction with individual students and in classroom situations" (ibid., p. 228). Many of them are in fact existing cognitive obstacles in mathematics that are more obvious in a CAS environment, and new technology can be utilised for a better conceptual development.

> Seemingly technical difficulties often have a conceptual background, and the relation between technical and conceptual aspects makes the instrumental genesis a complex process. (ibid., p.195)
>
> [...] obstacles offer opportunities for learning, which can be capitalized on by reflecting on their conceptual aspects and the relation with the corresponding paper-and-pencil technique. Of course, the teacher plays an important role in turning obstacles into opportunities. (Drijvers & Gravemeijer, 2005, p.195)

Along with the technological obstacles, Balling (2003) in her PhD thesis also refers to national and institutional obstacles like *curricula*, *national exams* and *tests*, (lack of) *instruction material*, *time* and *economy*. Her study investigates the potentials in introducing graphic calculators as a cognitive tool in upper secondary school instruction, and the obstacles there are for this introduction. Her findings show that the most important obstacles can be found within students' and teachers' beliefs about technology, mathematics and mathematics instruction and learning.

Teachers' and students' beliefs about the use of calculators

In his theory of *didactical situations*, Brousseau (1997) defines the *didactic contract* as the mutual recognition of the roles that teacher and students play in the classroom. The outcome of teaching depends highly on the beliefs of what should be "going on" during instruction and the results of different working methods. Calculators (especially CAS) have the capacity to fundamentally change the way mathematics is taught, and therefore they could represent both a progress and a threat for teachers as well as students. This is being discussed by several authors (e.g. Bergqvist, 2001; Balling, 2003; Drijvers, 2003) as an essential perspective in the use of technology in mathematics education.

In a study of high school students' experiences of learning mathematics with GC, Reznichenko (2007a) showed that GC's affect their enjoyment of mathematics, mathematical skills and conceptual understanding in a positive direction. Students viewed the GC as an all-around tool in mathematics (particularly in problem-solving), and their ability to learn mathematics was enhanced with the calculator. Importantly, students claimed that they enjoyed mathematics more, e.g. because the GC reduced the time consuming nature of certain mathematical tasks. These findings are in coherence with the meta-studies reported above (e.g. Burrill et al., 2003; Ellington, 2003; Reznichenco, 2007b). A number of the studies analyzed in this article found that students using calculators are generally more motivated and have a more positive attitude towards mathematics than those who seldom or never use them.

Students use different styles of working as they solve problems with technology such as calculators, but ”ways of promoting the use of the GC are needed if the students are to gain the full benefit of having them as an aid to their studies” (Berry, Graham & Smith, 2006, p. 302). There is also a progression from students focussing on the use of buttons and menus for entering direct single procedures into the calculator, to a stage when they make decisions about when and how they use the calculator in longer or more difficult problem solving (Thomas & Hong, 2004). Pierce and Stacey (2004) show that, even if the students were well able to master the technical aspects of a CAS while learning mathematics, there is a complex interaction between cognitive and affective factors. Both the teachers' and the students' attitudes have an effect, and the planning of teaching requires awareness of individual differences.

Teachers' knowledge of the didactical tool is of crucial importance for the integration of calculators into their pedagogical approach. In a study of secondary teachers, Thomas and Hong (2005) found essentially two groups, with a third progressing between the two. In the first group are teachers who are still coming to grips with basic operational aspects of the technology, such as key presses and menu operations. In the second group are teachers who are competent in basic instrumentation and can focus on other important aspects, such as linking representations like algebra, tables and graphs, and who use other features of the calculators. It is a difficult task for a single teacher to make progress on his/her own, and there are great needs for support in various ways and for recurrent in-service training (Ball, 2004). This could be one of the reasons for the fact that teachers generally are not very involved in the process of calculator appropriation (Trouche, 2005a).

A prime factor for teachers engagement in integrating technology into their instruction is whether it is included in the national or local curriculum or not, and if it therefore is allowed or even demanded in the national tests and examinations (Scheuneman et al., 2002; Graham, Hedlam, Honey, Sharp & Smith, 2003; Tan & Forgasz, 2006). On the other hand, teachers' personal beliefs about what mathematics is and how mathematics teaching is performed can either promote the use of calculators or make obstacles for them (Brown et al., 2007). This is especially true for CAS, which has the problem of becoming legitimized within the school culture (Kendal & Stacey, 2002). Again, there is a strong dependence of the way mathematics as a subject is seen:

> – these tools do not fit the conception of mathematics which teachers have. One teacher explains it in this way: ”calculators deny the mathematical reflex”. Reducing mathematics to an experimental practice restricts the place of formal proof. (Trouche, 2005a, p. 19)

So it is really a matter of what philosophy of mathematics the individual teacher has. This must be recognised and understood, at the same time as the demands of society for progression in mathematics education must be considered.

Synthesis of findings

The main question in this article is what the overall findings and implications in current research are for teaching and learning mathematics, in particular algebra, with calculators as cognitive tools. It is a delicate and most difficult task to make a synthesis of the results from a large number of research reports. In some parts they can also be pointing in diverging directions.

One such example is the question of whether lower achieving students benefit from the use of calculators or not. More than one researcher finds that these students show improvement in both understanding and skills, at least in terms of relative advancement, compared to higher achieving students (e.g. Graham & Thomas, 2000; Hennessy et al., 2001, Yerushalmy, 2006). But other researchers (e.g. Ruthven, 2002; Balling, 2003) instead point out that there is a risk that calculators increase the gap between these two categories of students, mainly because the higher achievers have better possibilities to use the calculator as an advanced cognitive tool.

Nevertheless, there are some important findings that are recurrent and generally converge in summaries and conclusions. They will first be presented in a short list, in which *examples* of supporting references are given:

Students who use calculators usually:

- become more active in solving mathematical tasks (Burrill et al., 2002; Reznichenko, 2007b),
- see problem-solving in a new way when they are freed from routine calculations; both numerical and algebraic (Reznichenko, 2007b; Hennessy et al., 2001),
- are more flexible with strategies for problem-solving and with representational forms (Bergqvist, 2001; Berry et al., 2006; Reznichenko 2007b; Hennessy et al., 2001),
- improve their ability to understand and use mathematical concepts (Bergqvist, 2001; Drijvers, 2003; Graham & Thomas, 2000; Hennessy et al., 2001; Kieran & Drijvers, 2006; Kulik, 2003; Reznichenko, 2007a),
- develop a clearer and deeper conceptual understanding of algebraic syntax, expressions and functions (Ball, 2004; Drijvers, 2003; Kieran & Saldanha, 2005; Reznichenko, 2007b),
- make significant improvements in problem-solving skills as well as computing and operational skills (Burrill et al., 2002; Ellington, 2003; Yerushalmy, 2006),
- take advantage of them as a common communication medium on a higher level than what is possible with paper-and-pencil in their cooperative work (Balling, 2003; Hennessy et al., 2001; Radford et al., 2003; Rivera & Becker, 2004),
- show no decline in paper-and-pencil or mental skills. On the contrary, in most cases they also make improvements in these skills (Ellington, 2003; Kulik, 2003; Graham & Thomas, 2000), and
- show a more positive attitude towards mathematics and are more motivated than those students who are not using calculators (Bardini et al., 2004; Ellington, 2003; Hennessy et al., 2001; Reznichenko, 2007a).

Some characteristics of cases when the use of calculators had no noticeable influence on students' knowledge, or could even be negative:

- Students were not explicitly instructed in the utilisation of the calculators, and did not understand their limitations (Burrill et al., 2002; Ruthven, 2002; Trouche, 2005a).

- Calculators were not used with consequence and continuity in classroom work, but were only brought in occasionally (Burrill et al., 2002; Ellington, 2003; Heller et al.,2005).
- The teacher had no adequate training for using calculators in their teaching, neither in a practical nor in a didactical perspective (Ball, 2004; Balling, 2003; Drijvers & Gravemeijer, 2005; Heid & Edwards, 2001; Kendal & Stacey, 2002; Lagrange et al., 2003; Thomas & Hong, 2005; Trouche, 2005a;).
- Instruction had not been adjusted for the use of calculators, but had been essentially the same as when the students only worked with paper and pencil (Artigue, 2005; Ball & Stacey, 2005; Balling, 2003; Drijvers, 2003; Heid & Edwards, 2001; Thomas & Hong, 2005; Brown et al., 2007; Trouche, 2005a;).

These common outcomes of recent as well as earlier findings can be interpreted as general standpoints of today's research. They also make it possible to formulate some guiding principles and implications for teachers as well as teacher educators for developing strategies in this area. For the last two bullet points I have intentionally given several references, and these refer both to studies where some negative sides of calculator use have been recorded, and to studies in which teacher training and instructional design have been a prerequisite. The fact that they are frequently recurring over many years points again at the significance of the teachers' role in implementation and integration of calculators in mathematics education, and the urgent need for both in-service and pre-service teacher development.

Concluding discussion

The writing of this article was partly triggered by a debate article with claims both of the negative influence of calculators on mathematics learning and of what research has, or has not, shown. One of my intentions with this literature review has been to demonstrate the misjudgement in both respects, and that my own hypotheses instead can be well defended. The results of a vast number of studies, especially those within the bullet points in the previous section point in one direction:

> Meta-studies of earlier research as well as findings in recent research do in fact support my hypothesis that calculators can be powerful cognitive tools for the enhancing of students' skills and understanding of mathematics, and in particular algebra. Calculators are not the cause

> of allegedly decreasing manual or mental skills. Instead they put students in a more active role and support motivation as well as a general positive attitude towards mathematics.

The use of technological tools in mathematics instruction is however not unproblematic, and a number of problems and questions concerning the widespread use of GC and CAS still need attention from researchers. One example is students' conceptions of mathematical symbols and syntax and use of new algorithms. Another is teachers' beliefs, knowledge of and skills in using calculators in mathematics instruction. As has been underlined by many researchers this is one crucial factor for a beneficial outcome.

Heid and Edwards (2001) discuss important challenges in the use of calculators, e.g. that teachers must rethink curricular and didactical aspects of mathematics learning in general, and particularly the nature and purpose of school algebra. They underline that "the new face of algebra is multirepresentational instead of primarily symbolic, centered on applications instead of solely on theory, and focused on symbolic reasoning instead of primarily on symbolic manipulation" (p. 129). But we aim for more than developing an effective instrumented practice:

> The educational legitimacy of tools for mathematical work has thus both epistemic and pragmatic sources: tools must be helpful for producing results but their use must also support and promote mathematical learning and understanding. (Artigue, 2005, p. 232)

CAS technology has opened a wide field of new possibilities, in fact too wide seen from many teachers' point of view. In-service-training as well as ongoing support is needed to enable teachers to use calculators in an effective way, and for them to take the step from "button-pressing" to users and to orchestrate instrumentation.

> As regards teaching algebra using computer algebra, the results suggest that it is important to anticipate on computer algebra use, to be explicit about the changing didactical contract, to orchestrate individual and collective instrumentation, to have students compare CAS techniques with paper-and-pencil techniques and to have students reflect on the way CAS works. (Drijvers, 2003, p. 330)

Artigue (2005) underlines the necessary evolution of the didactic contract as regards instrumented techniques, according to the advancement of mathematical and instrumental knowledge. She proposes developmental projects with didactic engineering as one way to achieve this.

The evolution of handheld technology is an ongoing process. In the newest generation of calculators, dynamic geometry is not only included, but also integrated with the graphing tools and computer algebra. For the first time, dynamic graphs are available on calculators, which create even further possibilities in algebra and calculus and we might see even more dramatic changes concerning handheld technology. The mobile phone has successfully taken over several functions that earlier demanded special devices, like digital camera or GPS. The newest generation has even the capacity to replace computers in many respects. Most students are quite experienced in taking advantage of the mobile phone's possibilities, and in a few years the "handheld calculator" could very well be software that we download to our phone. Development projects in that direction have already been launched. One example is Mobile Learning Environment (2008), which focuses on learning abilities of mobile games, aimed at mathematics and science, for different platforms. In these games students use a virtual calculator to make necessary computations and to graph functions that appear in the context. Introduction of such technology would of course call for still new research.

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Sammanfattning

Vilka belägg kan man finna i nyare forskningslitteratur för potentiellt positiva eller negativa effekter av att utnyttja grafräknare (GC) och symbolhanterande räknare (CAS) i matematikundervisningen? Fokus för denna litteraturgenomgång är användningen av räknare och deras effekt på algebralärande, inklusive teoretisk bakgrund för användningen av denna typ av teknologi i klassrumsarbetet. Särskild uppmärksamhet ägnas tre områden: elevers uppfattningar om bokstavssymboler och algebraiska uttryck, fundamentala för deras förmåga att arbeta med algebra; funktions- och modelleringsansatser, båda viktiga för elevers syn på algebra som ett användbart verktyg i problemlösning; samt ansatser med CAS, som ställer särskilda krav på förändringar i undervisningsmetoderna. Resultat från några nyare metastudier, baserade på ett relativt stort antal forskningsrapporter, diskuteras såväl som betydelsen av elevers och lärares uppfattningar om räknare. En sammanställning och syntes av vanligt förekommande resultat görs för en formulering av några viktiga implikationer för undervisning och lärarutbildning.