An anthropological approach to a transitional issue

Analysis of the autonomy required from mathematics students in the French lycee

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This paper intends to contribute to the process of theoretical networking within the mathematics education research community. Some key elements of the Anthropological Theory of Didactics are recalled and used to deal with the issue of French students' transitional difficulties in mathematics between Collège (lower secondary school) and Lycée (upper secondary school). The intention is showing how this theoretical framework, in contrast with a theoretical framework of Advanced Mathematical Thinking, provides tools to analyse the changes between these two institutions and thus supports the following assumption: An increasing autonomy as problem solvers as well as mathematics learners is required from the upper secondary school students. This hypothesis led to a clinical investigation on high school students' homework. This paper addresses the hypothesis by drawing on the case of three high-achieving students.

The general issue this paper deals with is the following: From one grade to the next one, former successful students begin to face important difficulties in mathematics. In France, this first experience of failure involves a significant number of students at two crucial steps of secondary school: grade 10, which is the first year in the so-called *lycée* (upper secondary school), and grade 11 for the students following a scientific course of study [1]. French teachers often attribute these difficulties to an insufficient autonomy to face the upper secondary school requirements. Such transitional difficulties appear in other countries, especially between upper secondary school and university and afterwards within undergraduate courses.

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In the English speaking research community, this issue has largely been approached from a psychological cognitive point of view. Students' difficulties are interpreted in terms of transition to advanced mathematical thinking (AMT) or at least of lacking some thinking skills that are considered as essential mathematical ways of thinking. What is this AMT to which students' cognitive abilities are compared with? The topic has been widely discussed and is still a debated point. I will here differentiate two types of approaches. The first one, largely drawing on AMT ideas is focusing on mental acts during mathematical activities insisting on individuality and contingency: A high-achieving mathematician is someone who is able to cope with the task "surprises". This approach necessarily puts forward the individual and deals with abilities, beliefs, affect, meta-cognition. The second one emphasizes the ideas of invariance and genericity: Every new task has something in common with others and what the mathematics expert knows about those may supply him with useful tools. Within the latter trend, research papers are interested in resources, described as formal and informal knowledge about the content domain. They generally refer to generic mathematical ways of thinking, that is, ways of thinking recursively used within the community of mathematics experts. This is especially the case when the presented elaborations directly aim at curriculum design.

Like other researchers (Grugeon,1995; Praslon, 2000; Bosch, Fonseca & Gascón, 2004 ; Winslow, 2007), I tackle the transitional issue I previously introduced within the anthropological framework developed by Yves Chevallard (1992, 1999, 2006). This anthropological theory of didactics (ATD) focuses on the institutional conditions and constraints of the processes of knowledge creation, teaching and learning in didactic contexts. Within the ATD research field, studies are interested in social objects, so the approach of transitional issues radically differs from the cognitive one that is usual in AMT. That is the first aspect I intend to illustrate in this paper, presenting my own work on the French students' difficulties in the transition between upper secondary grades. The next sections deal with the following questions:

From grade 9 to grade 11 scientific course.

- What changes occur in the way mathematics are taught?
- What changes occur in the mathematical tasks that lead to students' success or failure?
- What has changed in the way three high-achieving scientific grade 11 students work on mathematics at home?

As the Nordic research community may not necessarily be familiar with the ATD, I will at first focus on the key notion of *praxeology*, giving varied mathematical examples (for a detailed introduction see Barbé, Bosch, Espinoza & Gascón, 2005). This will give me the opportunity to explore the connections between praxeologies and notions from the "resources" approach of the AMT field, such as "ways of thinking" (Harel & Sowder, 2005), or "habits of mind" (Cuoco, Goldenberg & Mark, 1997). I will show that, at least for certain issues, the ATD and the cognitive AMT approaches may converge on the same inquiring track.

I will then present the idea of *process of study* or *didactic process* which is a model for the teacher's or students' actions along the process of recreating a mathematical praxeology at the classroom level and at the individual one. This notion is used to describe the evolution of mathematics teaching from lower to upper secondary school in France. The following section uses the tools proposed by A. Robert and collaborators (2002; 2005) to analyse the parallel evolution of the students' mathematical tasks. Based on interview data, I then show how high-achieving students manage to face the new conditions and requirements through an adaptation of their private study process. Finally, some ideas are put forward for the didactic institution to assume better its responsibility to help students succeeding in their study at this level.

The praxeological point of view on mathematics

The praxeological model of human activities and cultural resources

The anthropological theory of didactics considers that every activity an individual is engaged in is at first determined by social constraints and conditions. So every issue is approached from a social point of view. Human societies collectively struggle to overcome the problematic situations their members face, aiming at a certain regularity and efficiency. They create and accumulate strategic resources adapted to the generic aspects of problems. The notion of *praxeological organisation* or *praxeology* is a general model for these cultural resources as well as for human activities.

A *praxeology* is composed of two blocks. The *practical block* (or know-how) associates a *type of tasks* T and a *technique* τ. τ is a "way of doing" which is endowed with a certain efficiency for a certain subfield within the set of T tasks. An important point to take into account is that, even if some techniques are algorithmic, i.e. always efficient for the T tasks without any adaptation by the actor, others are not. Nor are they necessarily routine procedures, that is "well-codified but non-algorithmic techniques for solving specific classes of problems" (Schoenfeld, 1985, p. 58). For instance in mathematics, what Schoenfeld calls "problem-solving strategies" are techniques, they are associated to very general types of tasks. We will elaborate more precisely on this point in the following.

The second block is the *knowledge block*. The ATD supposes that

> human practices rarely exist without a *discursive environment*, the aim of which is to describe, explain and justify what is done. [...] This discourse is structured in two levels: the *technology* [θ] (a '*logos*' – discourse – about the '*techne*'), which refers directly to the technique used, and the *theory* [Θ] that constitutes a deeper level of justification of practice. (Barbé et al., 2005, p. 237)

In short, a praxeology is a quadruplet [T, τ, θ, Θ] composed of a type of tasks, a technique, a technology and a theory. "The word 'praxeology' indicates that practice (praxis) and the discourse about practice (logos) always go together" (ibid., p. 237). The word "theory" refers here to organized knowledge fields with an inner developing dynamics. Theories are not directly concerned with practice – etymologically, the term *theoros* refers to a spectator. However they may produce results which give rise to techniques. These results belong to the technology of the considered techniques. But there are also occurrences when no theory exists, at a given historical time, to produce and to explain a technique which was created by communities of practitioners while coping with the tasks of the type. The ATD assumes that in that case, the social elaboration of the technique is generally accompanied, perhaps with some delay, by a technological development. According to Chevallard' s definition, this discourse provides some arguments to legitimize the technique within the community and justify its efficiency and its relevance. In my conception of this theoretical tool, it also expresses the knowledge accumulated by practitioners through their experiences, and it aims at making easier the implementation of the technique. The teaching activity provides us with a lot of such praxeologies with scarce theoretical support, but a developed professional technology. We could use the term *folklore* to name this pragmatic part of the technology, in the etymological meaning of this English word: it is the science (lore) of the folk. As human theoretical effort is developing, part of this *folklore* may be incorporated in some theory which will thus complete the praxeology. A theoretical part θ^{th}, that is pieces of knowledge justified by a theory, enters the technology, but the pragmatic component θ^{p}, depending highly on the practice community and its experience with the type of tasks, remains essential.That is why I propose the following description of a praxeology: [T, τ, $\theta^{p} - \theta^{th}$, Θ] (Castela, 2008).

Mathematical Praxeologies

With the *mathematical praxeologies*, ATD proposes a general model of mathematical activity, strongly connecting practice and theory, problem solving and theoretical elaboration. Praxeologies also describe the resources a mathematical community may build and use while coping with mathematical tasks. This approach puts forward the complexity of knowledge involved in mathematical activity. Hence, it belongs to the research trend that does not reduce to theoretical knowledge the resources considered as useful for problem solving (cf. Schoenfeld, 1985; Lester, 1994, and for a review of papers in English, Carlson & Bloom, 2005; in French, Dorier, Robert, Robinet & Rogalski, 1997, and Castela 2000, 2008). The efficient mathematician is not viewed as being only someone with a in-depth knowledge of mathematical theories, gifted with a great creativity. As evidenced by Schoenfeld's studies (1985), he is also someone possessing more well-connected knowledge, who is, moreover, able to access its resources, to adapt and regulate their use in the solving context. Recently, Carlson and Bloom (2005) and Martignone (2007) have confirmed the importance of well-connected knowledge that appears to influence all phases of the problem-solving process.

So we hypothesize that mathematicians create common praxeologies to face their professional work. The question is the following: How is it possible to find evidence for these praxeologies? The answer depends on the nature of the traces the implementation of the technique leaves behind and on the forms of expression and dissemination of the knowledge block. I will illustrate what I mean with two examples.

The first one is borrowed from Iannone and Nardi's presentation at CERME 7 (2007, p. 2307).

> Based on discussions with mathematicians, both as researchers and university teachers, about the roles of syntactic and semantic knowledge in proof production, this study shows that for the participants both types of knowledge have to interact in sort of "a cyclic process based on drawing on syntactic and semantic knowledge in turn and often simultaneously. [...] Semantic knowledge is of great importance, for example, when an act of choice is involved in proof production.

This work converges with Weber and Alcock's study (2004, p. 232):

> When writing a proof semantically, one can use instantiations of relevant objects to guide the formal inferences that one draws, just as one could use a map to suggest the directions that they should prescribe.

These studies reveal a technique used by mathematicians to guide the proof process (type of tasks: Mathematical proof producing). This technique is generally undetectable from the written proof which is the very final product to be broadly disseminated. Nor will it be described in any paper or mathematics book. That does not affect the existence of this praxeology, only the size of professional circles within which it is spread. This kind of praxeology emerges in small research teams and is passed on from an expert to his thesis students, at the moment when it appears useful. The technology is oral, it does not need much formal elaboration. As for the theory, I will consider that none is involved in the semantic proof monitoring but it is not a point I want to discuss here.

For the second example, my argument is based on a classical geometrical situation.

The altitudes of a triangle intersect

Idea of one possible proof

Let ABC be a triangle. The proof is obvious if ABC has a right angle. In the following we suppose that it does not.

Let A' (resp. B') be the intersection point of ABC altitude from the vertex A (resp. B) with (BC) (resp. (AC)).

$(BC)\perp(AA')$, $(CA)\perp(BB')$, (BC) and (CA) are not parallel, so (AA') and (BB') intersect in one point H.

Let us show that H is a point of the altitude from C.

$$\overline{CH}\cdot\overline{AB}=\overline{CH}\cdot(\overline{CB}-\overline{CA})=\overline{CH}\cdot\overline{CB}-\overline{CH}\cdot\overline{CA}$$
$$=(\overline{CA}+\overline{AH})\cdot\overline{CB}-(\overline{CB}+\overline{BH})\cdot\overline{CA}=\overline{CA}\cdot\overline{CB}-\overline{CB}\cdot\overline{CA}$$

since $\overline{AH}\cdot\overline{CB}=\overline{BH}\cdot\overline{CA}=0$

Two praxeologies are involved in this proof.

Praxeology P_1

T_1 Proving that three distinct lines intersect in one point.

τ_1 Introduce the intersection point of two lines and prove that it belongs to the third line.

Θ Affine geometry or Elementary geometry

θ_1^{th} "If two lines are not parallel, they have one and only one common point" (direct corollary of the Euclid parallel postulate).

θ_1^{p} references to other examples (concurrency of segment or angle bisectors, of radical axes ...); in fact, τ_1 is especially relevant when the lines are defined by an equality or any equivalence relation; do not forget to prove that the first two lines are secant.

Praxeology P_2

T_2 Proving a vector relation

τ_2 Split up the vectors into sums of other ones using the existing points.

Θ Affine geometry or elementary vector geometry.

θ_2^{th} Parallelogram relation $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$; vector characterization of some point properties (parallelogram, midpoint, ...)

θ_2^{p} to use efficiently the parallelogram relation, it is necessary to base the decompositions on the aimed relation as well as on hypotheses; if you want to prove an equality, you may start trying to transform one of the equality member, always keeping in mind the second member you want to get; when there are many points or vectors, determine which ones have been independently given (sort of basic points) and use them to express every vector you have to deal with.
If there is a dot product, this technique is still efficient because of the dot product bilinearity; vector decompositions based on orthogonality hypotheses are often interesting.

These praxeologies are rather basic geometrical ones that experienced geometers as well as scientific upper secondary school students in France may use (the interview with *Louison below* gives some evidence for the difficulties a student meets with the second technique). So the technology above gathers, in some artificial way, elements of what could be the practical science of these different communities, but without claiming to be exhaustive. The technique is a routine procedure for a specific type of problems. Its implementation is clearly evidenced in the proof text. It appears in other well-known examples referred to in the technology, so that even if not a single word had be written to present explicitly this technique, we can reasonably consider that it belongs to the mathematics experts' common culture. That is why references to examples are included in the technology though it may be objected that it is not a discourse: these references work as kinds of (web)links, the minimal explicit form of a practical contextualised knowledge. On an individual level any personal experience with the technique may be referred to. At a social level, the fact that the considered examples are well identified, for instance by a name, is important.

To sum up, praxeologies produced and used in the mathematical activity may be roughly classified in two groups:

- Praxeologies regarding mathematics experts' heuristic activities. These techniques are generally theoretically unjustified, they are legitimized by the mathematicians' repeated experience of a reasonable efficiency (example 1).

- Praxeologies emerging in written mathematical works. The techniques are justified by some theoretical elements, at least in the end, as mathematics progress (example 2).

In the following, I will use Chevallard's phrase "Punctual mathematical organisations" (MO) for the second group which is strongly connected to mathematical academic knowledge [2] and "Mathematicians' praxeologies" for the first one.

Praxeologies, ways of thinking and habits of mind

The ATD is interested in mathematical resources as social production. Praxeologies are considered as objects existing independently of individuals, especially revealed through visible gestures with material tools or signs and at least partly justified and described by discourses. But as seen before, these public manifestations are not so evident for the *Mathematicians' praxeologies* which live in a somewhat esoteric way within the small communities of mathematics experts. Hence establishing the existence of a given praxeology may need some work.

Most of AMT-based studies start on the contrary from an individual cognitive point of view. Let us quote Harel & Sowder (2005) to represent this approach:

> In our usage, the phrase way of understanding, conveys the reasoning one applies in a local, particular mathematical situation. The phrase way of thinking, on the other hand, refers to what governs one's ways of understanding, and thus expresses reasoning that is not specific to one particular situation but to a multiple of situations.

> A personal' s ways of thinking involve at least three interrelated categories: beliefs, problem-solving approaches, and proof schemes. (Harel & Sowder, 2005, p. 31)

Thus the phrase "ways of thinking" refers to the mental resources an individual produces to face generic aspects of problematic situations. Yet the examples the authors give in the referred paper may be considered as widely shared knowledge within the mathematics experts' community.

Let us now consider the expression "habits of mind" used by Cuoco, Goldenberg and Mark (1997). Apparently the notion sounds very much as a psychological cognitive notion. But they write:

> We believe that every course or academic experience in high school should be used as an opportunity to help students develop what we have come to call good general habits of mind.
>
> Good thinking must apparently be relearned in a variety of domains [...] high school graduates should be accustomed to using real mathematical *methods*. They should be able to use the research techniques that have been so productive in modern mathematics [...]. We are after mental habits that allow students to develop a repertoire of general heuristics and approaches that can be applied in many different situations. (p. 377)

So these habits of mind clearly appear to be widespread techniques, broadly experienced in the research communities. Again this paper refers to social knowledge that we may describe within the ATD as praxeologies, *Punctual mathematical organisations* and *Mathematicians' praxeologies*.

How is it that the researchers I have just mentioned could not avoid leaving their initial individual psychological approach for a social point of view? How is it that esoteric praxeologies should appear in broader circles? The reason here is the common curriculum design project, as referred to in the title of Cuoco et al. (1997). The fact is that to be implemented as a curriculum objective, an individual cognitive skill or resource needs to be a socially shared skill or resource, i.e. to be a praxeology. This praxeology needs its technology to be developed: To support firmly the relevance of teaching this praxeology, to introduce teachers who are not necessarily mathematics experts to the considered technique, to give them possible situations to train students, and to elaborate a discourse to speak about this technique with classes, not only in a one to one conversation between peers. It comes from this necessity that the psychological cognitive approach of AMT may meet the ATD approach on some points in practice.

From french grade 9 to grades 11–12 scientific course: what changes in mathematics teaching?

Let us now begin with the transitional issue this paper will tackle from the anthropological point of view. We will, at first, consider that students' difficulties may result from changes in the way mathematics is taught. According to the praxeological approach, teaching mathematics is giving the students opportunities to recreate for themselves mathematical praxeologies, especially the praxeologies directly connected with concepts and theorems, that is, mathematical organisations (MO). I will focus the comparison between the considered institutions on the MO recreation process.

Re-creation of a mathematical organisation in a didactic context

In the ATD, the process of recreation of a mathematical organisation is modelled by the notion of *process of study* or *didactic process*. This process is organized into six distinct intertwined moments: the moment of the *first encounter*, the *exploratory* moment, the *technological-theoretical* moment, the *technical* moment, the *institutionalisation* moment and the *evaluation* moment.

> The second moment concerns the exploration of the type of tasks T_i and elaboration of a technique τ_1 relative to this type of tasks. [...] The third moment of the study consists of the constitution of the technological-theoretical environment [...] relative to τ_i. In a general way, this moment is closely interrelated to each of the other moments. [...] The fourth moment concerns the technical work, which has at the same time to improve the technique making it more powerful and reliable [...] and develop the mastery of its use.
>
> (Chevallard, 1999, pp. 250–255, English translation in Barbé et al., 2005, pp. 238–239)

I will use this model to evidence a major change from lower secondary school to grade 10 and 11 in mathematics teaching.

Lower secondary school mathematics teaching: A well developed process of study

From grade 6 to grade 9, mathematics teaching rhythm is moderate, a limited amount of theoretical objects and correlated MOs are introduced. Hence, teachers have time enough to organize the different moments of study. In particular they give their students the opportunity to handle with a rich sample of variants of a given type of tasks. The common work on the students' productions is a moment when a collective *folklore* may be elaborated. In short, the teacher creates good conditions for the MO appropriation by the students within the math class. This point clearly appears in Felix's study on grade 9 students' homework (Felix, 2002). Interviewed on the way they prepare periodic assessment in mathematics, two high-achieving students claim that, when it comes to exercises, they only read the solutions given by the teacher to make sure that they have understood the solution. They are sure that they need no more learning. They add that assessment tasks are always similar to the previously studied exercises. Felix concluded that for these students, the learning process essentially happens during classroom time. So I will retain the following: Although in France the explicit mathematics syllabus is expressed in terms of theoretical objects and most of the related MOs remain implicit during the lower secondary school years, the didactic process of recreation of these MOs is institutionally organized by teachers.

Mathematics teaching in grades 11–12 scientific course of study: A mere starting of the didactic process

The mathematics syllabus for the scientific course of study introduces a great number of concepts and theorems, each of them central to several MOs which still do not appear as explicit objectives. The teaching rhythm strikingly increases. Consequently, the teacher no longer has enough time to develop the didactic process for the new MOs. Except for the basic fundamental ones, he hardly enters the fourth moment, which reduces the opportunity for the class community to elaborate the technological environment, especially its *folklore* component. For instance, when working on the barycentre associativity, the teacher shows that it may be used to prove that three lines intersect in a common point and that three points are on the same line, but the he will not vary the exercises involving these techniques. Hence, students lack the opportunity to really become aware of the subtleties of the technique. To sum up, the teaching system focuses on theoretical knowledge and leaves it to the students to develop the process of study for the new MOs. Yet assessing tests in mathematics consist mostly of exercises or problems for which students are expected to write complete solutions (cf. the section below for some examples of such exercises and a detailed analysis of what they require from students). So students' success in grade 10 and still more in the scientific course at grade 11 and 12 highly depends on their autonomy as "MO developers". As seen previously, it was not the case at the lower secondary school level.

What changes in the school mathematical problems?

Let us now tackle the mathematical activities required from students, intending to examine what aspects of these tasks may produce a higher degree of difficulty. I will not elaborate further on the obvious fact that new theoretical objects and new techniques are involved, and that they are taught in the previously described conditions. My intention in this section is to focus on difficulties regarding familiar MOs which students previously successfully used. If I worked in another framework (see e.g. Rasmussen et al., 2005; Zazkis & Applebaum, 2007), I would say that I am interested in evidencing advances in mathematical thinking. Within the ATD, I will present tools for analysis to evidence advances in mathematical problems.

Analysing mathematical problems in a given school context: examples

Problems or exercises? This point cannot be eluded. Most research papers which deal with problem solving agree with Schoenfeld' s definition:

> A problem is only a problem (as mathematicians use the word) if you don't know how to go about solving it. A problem that has no 'surprises' in store, and can be solved comfortably by routine or familiar procedures (no matter how difficult!) is an exercise.
>
> (Schoenfeld, 1983, p.41)

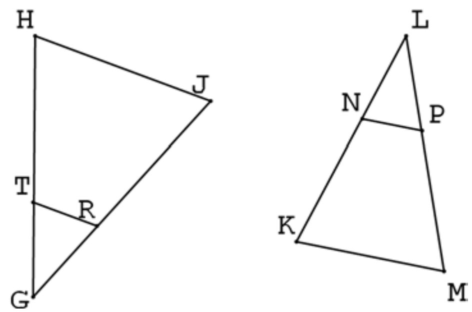
This dichotomy is much too rough to be efficient in our study which needs a more fine-graned scale to differentiate the tasks given to students. Even if a technique is familiar, the conditions of its use may change a lot from one task to another, thus requiring a variety of activities from the student. To follow, three examples which come from French textbooks will illustrate this claim and let us see what kind of tools are used to analyse evolutions. The words "exercise" and "problem" will be used without any particular discrimination in this paper.

What the following exercises have in common is that they use what in France is referred to as the *Théorème de Thalès*[3]. They appear at different moments of the curriculum, from grade 8 to grade 10.

The exercise in example 1 appears in the chapter where the Thales theorem is taught for the first time. The derived technique to calculate a missing length is not yet familiar. The students are required to use it in different conditions: variations affect triangle orientation in the sheet, point names and given lengths. In particular, the second case introduces the necessity of an intermediary step (calculating LK).

In grade 8, the students' pragmatic *folklore* will probably include some elements regarding the presence of two triangles, whose correspondent

Example 1. *Triangles and parallel lines*



For both joined figures, calculate the requested length.
Left figure: (TR) // (HJ),
HJ = 9, TR = 4, GJ = 9, calculate GR.
Right figure: (NP) // (KM),
LN = 5, NK = 7, NP = 4, calculate KM.

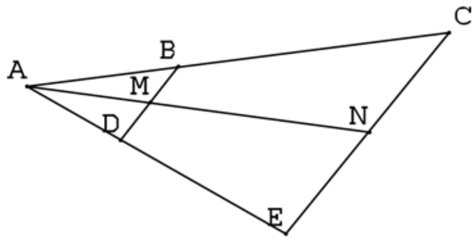
(Hatier 4e (2002), Collection triangle mathématiques. Grade 8, chapter 12, p. 195)

sides are associated in the ratios. Such an observation could prevent errors such as considering $\frac{LN}{NK}$ in the second figure because these two lengths are given.

This type of tasks appears frequently all along grades 8 and 9, so that when they leave the lower secondary school, most students recognize on their own that the Thales theorem may be relevant from the type of drawing we have above. At this level, this knowledge belongs to the *Thales MO* related to the type of tasks "Calculate the length of a segment".

The second example appears in a chapter which intends to work again on the whole geometric knowledge taught previously. Hence, when they face a task, students cannot guess from the chapter context which theorem to use. But, as pointed out before, the drawing may be here considered as a good fit for the Thales theorem. However we will not consider that the procedure goes on here as a simple routine, especially because of the following analysis. When calculating AM, this procedure leads to the equality $7{,}5AM = 3(AM + 3{,}6)$. Students have to recognize a linear equation to finish this question. Solving the equation $7{,}5x = 3(x + 3{,}6)$ is mere routine in grade 10. But here, the usual symbol x for the unknown is missing; the general class context refers to geometry and not algebra. Hence students are completely in charge of identifying the type of mathematical question involved and mobilising the relevant technique. I will consider that facing this responsibility will be supported by an evolution of the corresponding MO: what is at stake is the perception of the type

Example 2. *Configurations of the plane*



We know that AB = 3; BC = 4,5; MN = 3,6
BM = 1,5; AD = 2,5
The straight lines (BD) and (CE) are parallel.
Calculate AE, AM and CN.

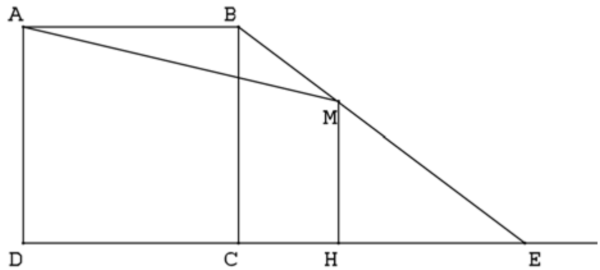
(Hachette Seconde (2000), Collection Déclic Maths. Grade 10, chapter 9, p. 247)

of tasks itself, that shall be characterised by the research of an unknown quantity and not by the presence of the symbol *x*, which is usually the very point lower secondary school students keep in mind.

The third exercise belongs to a chapter which deals with functions, i.e. a rather new subject for students. In this *precalculus* context, the first questions are geometric ones and require calculating some lengths depending on the variable *x*. Several techniques have been taught relying on Pythagoras and Thales theorems or using trigonometry. Here, unlike what we found in the second example, there is no strong indication that one of these procedures might be more relevant than the others – the drawing is complex, the Thales configuration is not especially visible, and the text does not refer to parallel lines or to a right-angled triangle. Hence the solver needs to mobilise the different procedures he knows and check by himself if one or another is efficient. Then he has to adapt it to the *x* context.

A more complex organisation of knowledge than the one technique *Punctual mathematical organisations* considered in the second section of this article would help the student to face the initiatives required by the orienting and planning phases of this solving process (Carlson & Bloom, 2005). Dealing with the general type of tasks *Calculating a length*, this MO connects several one technique punctual MOs, eventually developing their technology to describe their efficiency conditions: The Thales technique requires knowing that some straight lines are parallel, some points on the same line; and the Pythagoras theorem and trigonometric techniques need a right-angled triangle.

Example 3. *Functions: generalities.*



ABCD is a square with AB = 3. E is a point situated on the half-line [D,C] so that DE = 7. Let M be a point on [B,E] so that EM = *x* and let H be M orthogonal projection on (DE).

This problem aims at studying the area of the trapezium ADHM.

1. Calculate BE.

2.a Express the distances MH et EH as functions of x, and then the distance DH.

2.b Infer from these results an expression of the distance DH according to *x*.

3. Express the area of ADHM as a function of *x* . [...]

(Magnard Seconde (2004), Collection Abscisse. Grade 10, chapter 9, p. 292)

From lower secondary school to grade 11: advances in mathematical problems

The analyses presented here originate in Robert's works (see for example Robert & Rogalski, 2002). According to her proposals, the level of difficulty of an exercise or problem concerning the use of a given technique is assessed through two questions. First, is the related MO in some way present in the exercise wording? Secondly, is the technique efficient in its familiar simplest form or does it need some adaptation? Of course, the analysis must take into account the task context. For instance, we will consider that, in grade 8, the Thales procedure is not used in its common form in the first example, at least in the right case; while in grade 10, this exercise would be a routine one.

We can now give at least a partial answer to our second question: What changes in mathematics tasks that lead to students' success or failure? The evolution we met through the three examples analysed before is paradigmatic of what happens from grade 9 to grade 10 and still more from grade 10 to the scientific course of study in grade 11 where the rhythm of introduction of new objects becomes greater. Knowledge taught in the previous years is considered familiar. These resources are involved in exercises where they need to be coordinated between them (example 2) and more and more often with completely new objects and techniques (example 3). In this latter case, it is often left to the student to perceive the relevance of some familiar technique. This responsibility becomes especially demanding when several techniques have been taught for the same type of questions. This case becomes more frequent in the upper grades, as more mathematical knowledge has been studied.

To sum up, the problems in grades 10 and 11 require that students take more and more initiatives on their own using familiar resources. Within the anthropological epistemology of mathematics, this higher degree of autonomy requires that previous MOs evolve. The characterization of the type of tasks may be renewed, sometimes restricted (e.g. trying to describe the type of tasks especially well adapted for a given technique), sometimes enlarged (e.g. considering the tasks ”Solving an equation” and ”Proving an inequality” as referring to one type of tasks). The technology is developed to take into account the experienced adaptations. Moreover MOs relative to the same type of tasks T shall be organized into a new superstructure connected to T in order to support a broad summing up of the relevant resources when a T task appears in a problem. But in the conditions we have described in the third section, it is very difficult for teachers to spend some time in class going back to objects that have been taught in previous grades. These objects generally remain implicit on the didactic scene. Hence, in grade 10 and still more in grades 11 and 12 in the Scientific course, students must not only develop the new MOs introduced by the teacher in connection with the new theoretical knowledge, but they should also feel the interest to go back over old familiar MOs and achieve the renewing process on their own.

About scientific high-achieving students' homework

From the previous analysis, I infer that success in mathematics highly depends on the students' homework, especially in the scientific course of study. Some previously successful students ignore this new self-teaching charge or fail to face it; therefore they encounter increasing difficulties in mathematics. Others manage to adapt their homework to their new responsibilities. In order to investigate grade 11 scientific students' work, I have interviewed the students of one class who were willing to contribute to a research on homework in mathematics. I met ten students – about one third of the class – between January and May. Only two were boys, which may be due to my own gender and to the fact that perhaps boys are more reluctant to talk about themselves in a face to face discussion. The interview was semi-structured with a focus on moments of stimulated recollection. The general question was the following: Tell me what you have done to prepare yourself for the latest test in mathematics.

Among the students, three girls, Louison, Paula and Juliette, were especially high-achieving – with respectively averages of 15, 17.5 and 17 out of 20. I will present here the salient points I have drawn from their interviews.

Going back over what has been organized by the teacher

Louison and Paula spontaneously brought up the accelerated learning pace from grade 10 to grade 11. At the beginning of the year, they felt stressed because they could not understand everything that was going on in class and this was new for them. For the first weeks, Louison had to work at home on these shortcomings, before the following mathematics lesson; but later on, this problem disappeared. Paula encountered such difficulties all year long. She copes with them only when working for the tests but she had to totally change this preparing homework. As for Juliette, such adaptation occurred in grade 10. Louison claims having always worked as we will see now.

Louison and Juliette begin studying the theoretical part of their notes, intending to understand the proofs and to memorize the results. Paula does not, she considers that this learning is achieved through her working on exercises. As for this aspect of preparation, the three girls have the same method. At first, they solve again almost every exercise studied with the teacher. While the successful grade 9 students interviewed by Félix (2002) have no doubt on their learning during the class, these students have experienced the necessity to make sure that they are really able to find the solution. Doing so, they give themselves the opportunity to tackle the original task prescribed by the teacher. This one often requires more initiatives than the one effectively worked on in class, where, as evidenced by Robert and Rogalski (2005), the teacher gives hints to speed up the solution process. If they do not succeed, they study the teacher's solution and try again to solve the exercise. Louison and Juliette especially go back over their errors. Paula does not.

In summary, something that has not been anticipated from the previous analysis appears here. Due to the accelerated pace, the learning process regarding the activities initiated during the class has to be completed at home by a deliberate return to exercises. Among the interviewed students, only those with particular difficulties (with an average grade under 8 out of 20) appear not to do this type of work.

Developing the new MOs or going back over previously taught ones

The three girls generally do not solve new exercises because they would not have any way to check their production validity. Only Paula practices somewhat more, using exercises from the textbook, even without solutions, in the case of basic techniques she feels she may control by herself. In fact, it appears that they do not consider it necessary to extend the technical work (the third moment of the didactic process) beyond the limits established by the teacher.

But Paula and Juliette systematically complete their written work on solutions by a verbal phase in which they describe the solution. In doing so, they begin to decontextualize some generic elements of the solution and to elaborate a personal technology. They are clearly aware that each exercise intends to introduce them to a given type of tasks with an associated technique.

Louison generally stops working when she can solve every exercise. She is confident in her ability to adapt what she knows to the specificities of the assessment test. More over, like all the students, she expects that the test will not be much different from the class exercises. However, it may happen that an exercise appears to be especially difficult. In that case, she struggles to draw from the solution elements of the teacher's efficiency. She gives a very convincing sample dealing with the monitoring of the parallelogram relation with vectors. On this occasion, she goes back to a grade 10 MO:

L: Sometimes I have difficulties knowing what to do, which vectors to use, which ones to add ... So, I remember it, I mean I had begun to ... there were lots of calculations ... and I was totally lost. So I studied how the teacher did and he did it directly, so I looked at how he did it and after I try to do it alone.

Int: Okay, can you explain to me how you have picked out what he did, and how it was smarter than what you had done?

L: Well I started with vectors that were not really linked to the question [...] the parallelogram relation is really convenient but it can be used with lots of vectors and we can write a full page of calculations whereas it takes just two lines for the teacher. If we don't use the clever vectors, we can't manage it.

I: We can try to go into details ... in your reflection, did you find the reason why it was so tricky?

L: Yes, I do. Well in fact, what I did ... I used to begin quite randomly and I tried to get to the end, while the teacher ... he began with the conclusion and tried to come back to the ... so it is inevitably faster [...].

I: Did the teacher explain all that? or did you figure all that out on your own?

L: Oh no, he didn't, I found that alone, I don't ... well the teacher, he does his stuff, he writes the solution and after he does something else, he doesn't take time to ...

Thus, these high-achieving students take upon themselves, through their homework, a certain development of the *technological-theoretical* moment. The other students with middle or weaker results I interviewed never refer to this working mode, which, in the limit of this clinical study, appears to favour success in mathematics. This confirms the outcomes of a previous investigation regarding university students (Castela, 2004).

Conclusions and perspectives

In this paper I propose a double diagnostic to explain the difficulties which former successful French students encounter in mathematics in grade 10 or later, in grades 11–12 of the Scientific course of study. First, the mathematical problems require the solver to take more and more initiatives. To face this demand, the student's familiar resources should evolve. Second, at the same time, the teaching system partially leaves it up to the students to re-create for themselves the Mathematical Organisations at stake in the syllabus. Hence, an important autonomy as a learner appears to be demanded from the students. This generally requires some evolutions of the homework that many students are probably unable to imagine on their own. Therefore I consider it necessary at this point of my investigation to think of experimental proposals to help students to adapt their style of working. Teachers should provide the students with means to extend the MOs developing, especially some more exercises with solutions in order to sustain the technical work and the technological elaboration. Some training to the study of solutions should be organized, aiming at the detection of techniques, for new MOs as well as for previously taught ones. Such work would bring to light the importance of praxeologies. We may hope that students would transfer this working form to the developing of more general mathematical organisations. At the same time, conditions should be institutionally created to favour collective study in small groups, so that the self-teaching responsibilities may be socially faced.

Regarding possible connexions between the ATD approach and some studies of the AMT field, I will emphasize the following point. As I am interested, from an anthropological point of view, in the resources involved in mathematical activities, I cope with praxeologies mathematics experts do not widely share outside of small professional communities. These more or less esoteric praxeologies are very closed to objects like ways of thinking or habits of minds which, though introduced in the

AMT field from a psychological cognitive point of view, refer to generic aspects of the mathematics expert's competency. The intention to design a curriculum aiming at improving students' achievement in mathematics regularly leads researchers to move away from a strictly psychological approach. Social dimensions are required to legitimate curriculum choices. I hope I have contributed to show that the ATD provides a relevant framework to face such necessity as well as efficient tools to describe the components of the curriculum.

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Notes

1 During the last two years in the upper secondary school students have to choose a specific course of study. This choice is not totally free. For the science course of study, it highly depends on the student's results in science in grade 10 (Seconde): those who follow this course of study (Première, Terminale Scientifiques) were generally rather successful in mathematics.

2 In this paper, we will only consider the most elementary level of Mathematical Organisations, the *punctual* ones which are based on a unique type of tasks *T*. Punctual MOs referring to several types of tasks are then integrated in more complex organisations: a local OM is composed of punctual OMs sharing the same technology; local OMs with common theory are integrated in a regional MO (see Barbé et al., 2005, p. 237–238).

3 What is in France considered as the *Thales theorem* is the following one: "Let d and d' be two straight lines with O in common. A and B are two points on d, A' and B' two points on d'. If the straight lines (AB) and (A'B') are parallel, then OA/OB = OA'/OB' = AA'/BB' ". Students do not learn any more general result, for instance that OA/AB = OA'/A'B'.

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