Design of a didactic situation – mathematical experiments in linear algebra

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We describe how we used Brousseau's theories of *didactic*[1] *situations* and *didactic engineering* as a framework for the development of an exam project in a first year mathematics course at a life science university. The main learning goals of the project were to (re)discover eigenvalues and eigenvectors partly by studying the asymptotic behaviour of matrix models for population growth and to understand the role eigenvalues play in such models. Moreover, the students would gain experience with mathematical experiments with the use of computers and with drawing conclusions from such experiments.

In this paper we describe parts of the planning of a mathematics course at a life science university. A major point of the paper is to show how our chosen theoretical framework of didactic situations and didactic engineering was used concretely in the design of the course and the development of a particular exam project.

The organisation ofthe paper is as follows. We begin with a relatively brief description of the theories of didactic situations and didactic engineering, and then present some background on the course as well as the mathematics involved in the exam project.In the section *Didactic analysis* we describe how we used the framework of didactic situations and didactic engineering in the development and concrete planning of the course including the exam format and in particular the exam project. We will focus on how the framework helped us in structuring our approach, and from a development point of view this can be regarded as the main section of the paper. We then present the project itself and discuss it with

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reference to the didactic theories. Finally, we sum up our conclusions and discuss some possible improvements, and reflect on the usefulness of the didactic theories in our development project.

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The theories of didactic situations and didactic engineering

As part of our course we wanted to design a number of exam projects as rounded-off learning situations involving experimental work with the use of computers. This way of working seemed to fit nicely with Brousseau's theory of *didactic situations* (see Brousseau, 1997; Sierpinska, 1999). As a concrete way of implementing the theory of didactic situations we used the theory of *didactic engineering*. This theory was originally suggested by Brousseau for this purpose (see Brousseau, 1997), and has been further developed by Artigue (see e.g., Artigue, 1994). In this section we present an outline of these theories.

Didactic situations

According to Brousseau's theory of *didactic situations* (Brousseau, 1997) learning occurs when students adapt themselves to a learning *milieu* (environment). One of the main tasks of the teacher is therefore to construct milieus in such a way that the adaptation leads to the students' acquisition of the target knowledge, which typically is "hidden" from the students and only appears as a result of a successful adaptation. In Brousseau's theory the process of adaptation is often illustrated with "games", where students gradually discover winning strategies. By choosing suitable designs, games can stimulate both of Piaget's processes of assimilation and accommodation. For assimilation the games should be constructed so that the students can fit the "outcome" of the games into their existing cognitive structures, whereas games leading to accommodation must challenge these structures in order to force a change.

Brousseau operates with four different types of didactic situations: action, formulation, validation and institutionalisation (see Brousseau, 1997; Sierpinska, 1999). In most traditional teaching institutionalisation and validation are dominant, possibly with an element of formulation in exercise classes. For the sake of our further discussion we include part of the description of a situation of action from Sierpinska (1999, Lecture 1, p. 4):

> The teacher organizes a milieu for the students to engage with but then completely withdraws from the scene. The milieu for the students is that of a problem so chosen and formulated that (a) the students are willing to adopt it as their own, and are interested in solving it to satisfy their own curiosity or ambition; (b) the students have the means to construct the solution by themselves, either by inventing a new procedure or choosing one among those they know, without, however, the teacher suggesting which one to choose. In this situation, knowledge appears as a means for solving a problem or a class of problems.

The withdrawal of the teacher and the subsequent transfer to the students of the responsibility for the learning situation constitutes Brousseau's notion of *devolution* (Brousseau, 1997). The devolution marks the transition from a didactic to an *adidactic* situation, where the students are working on their own. As part of the devolution the teacher has to decide how to handle "the paradox of devolution"(Brousseau, 1997, p. 41): on the one hand the students often need help from the teacher in order to progress, but on the other hand the more the teacher tells the students what to do the less chance the students have of adapting to the milieu and producing their own learning.

Brousseau's theory of didactic situations was mainly developed for and based on experience from classroom situations in primary schools, but the central parts of the theory can be applied to more general teaching situations. However, because of factors such as the nature of the subject material and the institutional setting, it is to be expected that the theory needs to be adapted for use in other contexts. For instance, the duration of a didactic situation at university level will often be longer than in primary schools.

In a recent addition (Bloch, 2005) to the theory of didactic situations Bloch introduced the concept of *returned situations*. The basic idea is to divide a didactic situation into a *direct game* and a *returned game*. The purpose of the direct game is to familiarise the students with the objects and the strategy of the game in question, whereas the target knowledge of the situation does not play a direct part. Contrary to this the returned game is more elaborate and the game cannot be "won" without using the target knowledge. To some extent the idea of returned situations, and the theory of didactic situations itself, is implicit in certain types of teaching practice, but we find that the conceptualisation of the practice may leadto more efficient and refined uses of the approach. In our conclusion we return to a possible use of returned games in our context.

We finish this section by mentioning Brousseau's notion of (de/re)contextualisation (Brousseau, 1997). General mathematical results (theorems etc.) are typically first discovered in particular cases and examples, and are only later stated in their full generality in a more abstract and decontextualised form. Whereas this increase in the level of abstraction in itself is part of a healthy scientific process, it can be problematic in relation to teaching. Students often begin a learning process by understanding notions and results in exemplified form; thereby following the steps of the researcher, so the teacher first has to recontextualise the knowledge in order to initiate the student learning.

In the theory of didactic situations, games and situations of action are means by which recontextualisation can be achieved. Once the students have grasped such recontextualised versions of the knowledge, they need to (re)decontextualise it in order to obtain an understanding of the generality of the results.

The schism between the abstract format in which mathematical results are usually communicated and a learning approach based on acquaintance with more concrete cases seems to be a central issue for mathematics education and has been approached from a number of different perspectives. We mention here Freudenthal's conceptof *reinvention* (see Gravemeijer and Terwel (2000) for an introduction to the key ideas in Freudenthal's work) as well as Harel's *concreteness principle* (Harel, 2000, p. 182): "students build their understanding of a concept in a context that is concrete to them".

Didactic engineering

The theory of *didactic situations* is a general theory of (mathematics) teaching and learning which we used to focus our ideas on how we wished the students to work with the projects. For the more concrete planning of the course including the projects we used the theory (or method) of *didactic engineering* (Artigue, 1994) to structure our approach.

Didactic engineering can be viewed as a process that leads to the development of certain teaching products. Schematically this process consists of the following four phases:

(i) The existing course is analysed with emphasis on the parts one wishes to change. This should also lead to a description of the learning goals of the new course.

(ii) The constraints affecting the course are analysed. In particular the main obstacles to teaching and student learning are determined, and ideas of how to overcome these are generated.

(iii) Guided by the goals from (i) and the constraints from (ii), choices are made about the design ofthe course.

(iv) After the course is completed, the results of the process and the course in general are evaluated. The conclusions of this evaluation then serve as input to phase (i) in a new cycle of didactic engineering.

The entire process of didactic engineering takes place on a global level concerning the whole course as well as on a local level concerning specific didactic situations. We will emphasise this point, when we present the didactic engineering carried out in our case.

Artigue (1994) mentions three types of constraints: *epistemological* constraints concerning the mathematical knowledge itself; *cognitive* constraints related to the students and their learning obstacles and finally *didactic* constraints concerning the teaching as well as the more practical and institutional limitations. Artigue also distinguishes between *external* constraints which cannot be avoided and *internal* constraints which are only perceived as constraints; often due to habit.

Background material about the course

In order to explain the context in which our development project took place we give a brief description of the course in this section. We also present some ideas we had before we started the process of didactic engineering, and describe the mathematics involved in the exam project.

Overview of the course

The course is a compulsory mathematics course for about 250 university students in natural resources, food science, biotechnology and agricultural economics in their first year. The mathematical content of the course is divided into 4 modules with *Functions and models*, *Matrices*, *Differential equations* and *Functions of two variables* as the four headings. The time table structure at the university allocates the course a full and a half day per week for a term lasting 9 weeks.

As part of an educational reform, it was decided to include the use of computers (specifically the program "R") in the course; partly to support the learningofmathematics and partly to introduce the students to a few topics from computer science. The computer part takes up 20–25 % of the course. In this paper we have chosen not to focus on the decisions and problems relating to the teaching and integration of computer science in the course.

Over the last couple of years before the present development project, we had revised the mathematical content and the teaching of the course extensively. Briefly, this revision had three main foci: firstly, inspired by the report C*ompetencies and learning of mathematics* (Niss & Jensen, 2002) on mathematical competencies and by Winsløw's notion of *specific competence goals* (see e.g., Grønbæk & Winsløw, 2007) a dimension of "competence" was added to the usual topic-based planning and teaching of the course.

Secondly, applications of mathematics taken from other life science subjects were integrated into the course. Since mathematics is a service subject at our university, the demands on and goals of the course are different from those of a course in pure mathematics. For instance, it is not essential for our students to know the proofs of the theorems, although a certain structural understanding of the mathematical theory is useful, but they need to be able to use and relate to mathematics in other subjects. Also, compared to the deductive approach of most pure mathematics courses we aim to stimulate a more inductive way of thinking in our students.

Thirdly, our presentation of the material was changed to a "from concrete to general" approach, where mathematical notions and results are first introduced through examples and applications, and only afterwards are presented in their traditional mathematical form. In this way we used the relevant examples from applications as a tool for recontextualisation (or reinvention). See Pedersen (2005) for further details about the revision.

The main theme of this paper is the new exam format that we introduced. In the old course there was a 4 hour written exam at the end of the course and from the beginning our clear aim was to change this by including a number of exam projects during the course. In these projects we wanted to include explorative mathematical activities with the use of computers. Furthermore, our intention was to let one of the projects introduce the students to eigenvalues partly through mathematical experiments related to population models. The development of the exam format and of this particular exam project constitutes the core ofthis paper.

The mathematics involved in the exam project

In the old course eigenvalues and eigenvectors were considered to be too advancedmaterial, but we had often wished that we could include the following theorem in the course due to its striking applications to matrix models for population growth. In the rest of the paper we will refer to this theorem as the *main theorem*.

Main theorem

Let $\mathbf{M}$ be a $n \times n$ matrix, $\mathbf{v}_0$ a n-dimensional vector and let $\mathbf{v}_t = \mathbf{M}^t\mathbf{v}_0$ for $t = 1, 2, \ldots$ Under certain conditions (which are usually satisfied) on $\mathbf{M}$ there exists a constant c such that

$$\frac{\mathbf{v}_t}{\lambda_1^t} \to c\mathbf{q}_1 \qquad \text{as } t \to \infty,$$

where λ_1 is the dominant eigenvalue and $\mathbf{q}_1$ a corresponding eigenvector.

In most linear algebra courses, theorems similar to the main theorem are presented as part of a more general spectral theory. Apart from basic matrix algebra this involves notions such as bases (and thus typically linear independence and span), diagonalisation, subspaces and possibly change of bases.

We will now interpret the main theorem in the context of population matrix models. If a population is divided into n age groups and $\mathbf{v}_t$ denotes the population vector at (discrete) time t and if $\mathbf{M}$ is the so-called *Leslie matrix* for the population, then we have $\mathbf{v}_{t+1} = \mathbf{M}\mathbf{v}_t$ and thus $\mathbf{v}_t = \mathbf{M}^t\mathbf{v}_0$. When λ_1 and $\mathbf{q}_1$ are known, the asymptotic behaviour of the population vector $\mathbf{v}_t$ can thus be described by

$$\mathbf{v}_t \approx c\lambda_1^t\mathbf{q}_1.$$

Moreover, with $\mathbf{v}_t = (v_{1t}, v_{2t}, \ldots, v_{nt})$ it follows from the main theorem that

$$\frac{v_{k,t+1}}{v_{kt}} \simeq \lambda_1 \qquad \text{and} \qquad \frac{\mathbf{v}_t}{v_{nt}} \simeq \mathbf{q}_1 \tag{$*$}$$

for large values of t. In the long run the growth rate, that is $v_{k,t+1}/v_{kt}$, of *each* age group therefore approaches the dominant eigenvalue λ_1 and the distribution, that is $\mathbf{v}_t/v_{nt}$, between the age groups becomes asymptotically stable and given by the eigenvector $\mathbf{q}_1$. In this way the main theorem also allows us to determine approximations to λ_1 and $\mathbf{q}_1$ by first calculating a suitable number of the iterates $\mathbf{v}_t$.

Didactic analysis

In this section we describe how we used the theories of didactic situations and didactic engineering to structure and guide the development of the course and in particular the exam project about eigenvalues in population models. We will attempt to show what roles the various didactic notions played in our case. In the closing section we will reflect on the usefulness of these theories based on our experiences from this development project.

In our presentation of the didactic analysis we use the structure of didactic engineering described earlier except that the evaluation of the process is described in our conclusion. We will describe what in our case were some of the more important parts of the various phases of the didactic engineering. Since we are particularly interested in the didactic situation about eigenvalues in population models, we not only mention our global analyses and decisions about the whole course, but also the local ones about this particular didactic situation. In our presentation, each phase is therefore divided into a global and a local part.

Some of the decisions about the course were actually taken before we started the process of didactic engineering. For instance, because of the earlier revision of the course, we decided that the mathematical content could more or less be carried over to the new course. Also, we had already, at least implicitly, decided to include project work in the exam format, so the process of didactic engineering would focus on *how* this change could be carried out. Other decisions made in advance will be implicit in the following presentation.

Phase (i): analysis of the course

In this presentation of our analysis we will focus on the exam format of the course. The global analysis mainly concerned the overall format of the exam, whereas the local analysis lead to the learning goals of the didactic situation about eigenvalues in population models.

Global analysis of the course

As indicated earlier we wanted to include project work in the exam in order to stimulate student activity throughout the course and at the same time test a wider range of competencies. Since the mathematical content of the course was divided into 4 modules our idea was to have some kind of exam project in each module.

Apart from being part of the examination we wanted the exam projects to be rounded-off learning situations involving explorative activities with the use of computers. In the concrete design of these projects we were inspired by the emphasis Brousseau puts on devolution of the situation to the students through suitable designs.

Also, one of our main ideas was to let the students discover a mathematical object in a more active way than usual without first having the full theory behind it presented. We felt that the four different types of didactic situations described by Brousseau could help us to conceptualise this idea. For instance, situations of action can be hard to organise and are rather time consuming, so it is probably realistic only to include a small number of such situations in a course. However, we felt that the presence of computers gave us a chance to create such situations in our course by letting the games take the form of mathematical experiments.

By mathematical experiments we mean explorative mathematical activities often using a material milieu (computers, cardboard etc.) that extends beyond pen and paper. The aim of such experiments will typically be to introduce mathematical notions or ways of working before a traditional introduction to these is given. The explorative approach is in some ways similar to the use of research-like situations in the classroom, for instance, work on open problems, in more advanced mathematics courses.

Local analysis of the didactic situation about eigenvalues

We hoped that we could include the main theorem and its applications in the exam project about matrices. As part of this project we therefore wanted to construct a didactic situation about eigenvalues in population models with the following learning (or specific competence) goals: the students had to be able to

(a) understand the definitions of eigenvalues and eigenvectors and know how to determine them,

(b) understand the modelling leading to matrix models $\mathbf{v}_{t+1} = \mathbf{M}\mathbf{v}_t$ for population growth,

(c) calculate some of the iterates $\mathbf{v}_t$ with the use of computers and thereby gain experience with mathematical experiments and with drawing conclusions from such experiments, and

(d) describe the asymptotic behaviour of the iterates $\mathbf{v}_t$ in terms of the dominant eigenvalue and the corresponding eigenvector and conversely use (∗) to determine approximations to the dominant eigenvalue and the corresponding eigenvector from the iterates.

In (c) and (d) the outline of a mathematical experiment can be seen: firstly, the iterates $\mathbf{v}_t$ are calculated for sufficiently many values of t. This part requires no knowledge of eigenvalue and eigenvectors. Secondly, the iterates and (∗) are used to calculate approximations to the dominant eigenvalue and the corresponding eigenvector. In our conclusion we will discuss this approach in the context of Bloch's returned situations. This use of experiments meant that the didactic milieu the students had to adapt to would consist of the experimental environment and the results obtained from it as well as the very basic theoretical introduction to eigenvalues and eigenvectors that we intended to give the students at the beginning of the project.

Phase (ii): analysis of constraints

As mentioned earlier, some decisions about the course were taken before the didactic engineering took place.In the context of constraints these decisions can be seen as a kind of external constraints; at least if the decisions would not be changed.

Generally, our global analysis of constraints resulted mainly in didactic constraints, whereas the local analysis of constraints resulted mainly in epistemological and cognitive constraints. To some extent this division reflects the nature of the three types of constraints, since, for instance, epistemological constraints relate to concrete mathematical issues, whereas didactic constraints often affect the whole course.

Global analysis of constraints

Didactic constraints

In a way, one of the most motivating constraints for us were the full days. Traditional teaching with only lectures and exercise classes did not seem optimal when the students had to be sensibly engaged for a full day. We therefore had to consider in what other manner we could make the use of the full days effective and which possibilities this time table structure gave rise to. Our earlier decision of including the use of computers in the course actually loosened this constraint by allowing us to vary between mathematics and computer activities.

In Brousseau's original theory the teacher withdraws after having "devolved" the didactic situation. Since Brousseau's didactic situations took place in primary schools, our guess is that they were relatively short. We intended our projects to last a full day, and therefore found that the *adidactic potential* – that is the opportunities for independent student work – of the situations would be improved, if, as a phase in the didactic game, we could offer some kind of support to the students instead of completely withdrawing for the duration of the project. Whereas situations of action typically are adidactic and institutionalisation typically didactic, situations of formulation and validation can be both. Our idea was to provide support to the students when they were in situations of formulation and validation in such a way that the support acted as a link between didactic and adidactic situations. Since the projects would be part of the exam, this assistance should be more indirect than at normal exercise classes, but should at the same time be sufficient for the students to progress in their problem solving process.

Local analysis of constraints

Epistemological constraints

Our main wish was to let the course include the more advanced material contained in the main theorem. As indicated in our discussion of the mathematical background this is normally presented as part of a larger mathematical structure known as spectral theory. This theory involves many notions which are abstract in the sense that they are hard to visualise even in dimensions 2 and 3. Even if we had the time to cover all these notions, the level of abstraction would be too high for a course such as ours. We therefore had to analyse whether we could present the main theorem to the students in a way which on the one hand lead to a sufficient understanding of the theorem and on the other hand was not too abstract.

The actual statement of the main theorem is fairly concrete and accessible. Its use basically just requires determination of the dominant eigenvalue and a corresponding eigenvector, and it has illustrative applications relevant to our course. Although eigenvalues and eigenvectors are also abstract notions, they are often easier to handle from an operational point of view than many of the other abstract notions involved in spectral theory. The definitions are reasonably accessible and the procedure used to determinate the eigenvalues and eigenvectors of a 2 x 2 matrix is straightforward. For larger matrices we would either use the computer (the program "R") to determine eigenvalues and eigenvectors or use the main theorem "backwards" by using the iterates $\mathbf{v}_t = \mathbf{M}^t\mathbf{v}_0$ to determine approximations to the dominant eigenvalue and a corresponding eigenvector. We therefore felt it possible to carve out a suitable coherent part of the theory which together with mathematical experiments could form a didactic situation about eigenvalues in population models focused on the learning goals described in phase (i).

The role of the experiments in this context can be seen in relation to the different modes of thinking described in (Sierpinska, 2000). Geometric notions are often used in the teaching of linear algebra to give the students an intuitive feel for some of the more abstract notions. In our course we put rather little emphasis on the *synthetic-geometric* (Sierpinska, 2000) mode of thinking and, in a sense, we replace it by an experimental mode before moving on to the *analytic-arithmetic* mode.

Because we would only present a small part of the spectral theory, we would not be able to prove or even justify the main theorem, so the institutionalisation of this knowledge would in some sense be less rigorous than normal. However, the focus of our course is on the applications of the mathematical theory rather than its inner structure, and we felt that this justified the inclusion of the more advanced material.

Cognitive constraints

We wanted the students to work with concrete matrix models, typically for population growth, which as descriptive models can be understood with very basic matrix skills. However, the analysis we had in mind of the models requires the use of eigenvalues and eigenvectors, which form part of a deeper and more abstract theory. The cognitive problems related to the students learning and understanding of eigenvalues and eigenvectors is of course closely linked to the *Epistemological constraints* described above. Our students generally see the need for mathematics as part of their studies and have a direct learning attitude, but at the same time they often find it hard to handle more abstract notions. This does not mean that they should be spared from such abstract notions, but generally the course does not attempt to reach a high level of abstraction. We intended to accommodate this by presenting only a very simplified, but hopefully coherent, part of the spectral theory for matrices, and we did not expect the students to become acquainted with the more abstract notions involved.

Nevertheless, even with this simplified approach we expected to encounter certain cognitive problems, for instance, related to *semiotic registers* (see Duval, 2006). To illustrate this notion we mention that the two descriptions of a straight line either by the equation $y=ax+b$ or by its graph take place in the algebraic respectively graphic register. In our case, in order to understand the definition $\mathbf{Mv}=\lambda\mathbf{v}$ of eigenvalues and eigenvectors one essentially has to work in the geometric register, whereas the determination of these consists of first solving the equation $\det(\mathbf{M}-\lambda\mathbf{E})=0$ for λ and then $\mathbf{Mv}=\lambda\mathbf{v}$ for $\mathbf{v}$ in the algebraic register. *Conversion*, the act of changing from a representation of a mathematical object in one register to a representation in a different register, for instance, between the equation $y=ax+b$ and the graph of a straight line, is typically cognitively difficult, so the students may be tempted not to move between these registers, but instead work only in the algebraic register. Thereby they would be able to determine the eigenvalues and eigenvectors, but not develop a clearer perception of the notions. In addition to this comes the complication that the eigenvectors only are determined up to a constant (at best), whereas the students are only familiar with equations with unique (or a finite number of) solutions.

Furthermore, our basic idea about introducing eigenvalues and eigenvectors through an experimental approach may in itself create a cognitive hurdle. The students first had to perform experiments using the computer, where they would calculate some of the iterates $\mathbf{v}_t$, and afterwards had to interpret the conclusions obtained from the experiments in terms of eigenvalues and eigenvectors by using the main theorem about the asymptotic behaviour of iterates. Whereas this approach could give the students an illustrative introduction to eigenvalues and eigenvectors as well as the role they play in population models, it, and similar situations in the other projects, also requires the students themselves to make transitions between action, formulation and validation, which they are probably not used to. This can also be compared to the problems students have moving between different modes of thinking (see the discussion above under *Epistemological constraints* and Sierpinska, 2000).

Finally, we hoped to avoid the worst problems associated with the technical level of calculations by only dealing with eigenvalues and eigenvectors of 2x2 matrices by hand and using the computer for larger matrices.

Phase (iii): decisions about the course

We will now describe the decisions and choices concerning the course in general as well as the didactic situation about eigenvalues in population models that we made based on the learning goals and constraints. The project itself will be discussed in the next section.

Global choices about the course

Two of our main aims, making good use of the full days and choosing a suitable new exam format, we attempted to combine by using 4 out of the 9 full days on exam projects. Each of these days started with a brief lecture as introduction to the project and throughout the rest of the day the students worked in groups on the project. On the remaining full days we attempted to achieve some variation by using half the day on mathematics and the other half on computer science.

As mentioned under *Didactic constraints* the long duration of the exam projects meant that we could not simply ”devolve” the project to the students and then withdraw. At the same time it was important that the setting of the project was different from that of the normal exercise classes. The choice we made was a centrally located ”help desk”, where the students could get hints and suggestions but not answers to the questions in the project. This set-up also forced the students to consider whether their need for help was great enough for them to seek help at the help desk; contrary to the typical situation in exercise classes. The use of the helpdesk is in a strict sense not included in Brousseau’s original theory, but we felt that it was a necessary adaptation of the theory for it to be used in our context. Also, the students' visits to the help desk can be regarded as guided situations of formulation and validation, and finally the help desk can be seen as an attempt to relate to "the paradox of devolution".

The contents of the projects had to satisfy two criteria: firstly, we wanted most of the material in the course to be covered by the projects, which for instance, meant that the didactic situation about eigenvalues in population models could not take up an entire project. Secondly, the projects had to involve integration of mathematics and computer science.

Since the projects were considered to be exam papers, the students could not get their marked projects back. After the students themselves had worked through the stages of action, formulation and validation, we considered it important to finish the project with an element of institutionalisation. Together with the mark for the project, each group were therefore given a list of their major mistakes with suggestions for improvement. Moreover, correct answers to the questions were made available.

We decided on an overall exam format consisting of two parts which had to be passed separately and with the total grade for the course calculated as the average of the grades for the two parts. One part consisted of the 4 projects of which the 3 best counted for the grade. While the course responsibles were thinking of the exam projects more as learning situations than examinations, the students obviously were more focused on their grades for the projects, but this difference in objectives did not seem to cause problems and we decided not to discuss it in any detail with the students. The second part of the exam was a 2 hour multiple-choice test at the end of the course. This test served two purposes: firstly, it tested basic skills and typically dealt with only one mathematical notion at a time, where the projects tended to involve more complex situations. Secondly, the projects were done as group work and we had no way of checking whether the group members contributed equally to the projects. From this perspective the test ensured the reliability of the exam format by preventing students who had not done their share of the group work from passing the overall exam. See Grønbæk and Winsløw (2007) for another example of an exam format which combines project work with a written exam testing basic skills.

We made serious efforts to communicate the relevant information about the exam format and the projects to the students. In this connection we mention Brousseau's notion of a *didactic contract* (see Brousseau, 1997; Sierpinska, 1999), that is, the often implicit rules of the didactic milieu. We attempted to make part of the contract *explicit* by emphasising to the students that the projects took place in a different learning environment from what the students were used to since they to a higher extent had to work out the solutions on their own. Also, with respect to the help desk we asked the students not to think of the projects as exercise classes with less help than normal, but rather as exam situations with access to some assistance. The help desk also served to clarify some of the more implicit parts of the didactic contract, such as, if necessary, explaining the purposes of the various parts of the projects.

Local choices about the didactic situation about eigenvalues
The idea of working with eigenvalues in population models through an experimental approach was conceived before the process of didactic engineering started. After our analysis of constraints and the global choices mentioned above, we felt further convinced of the feasibility of the idea. The next step consisted in making local choices about a didactic situation centred around this idea; ”to ’engineer’ situations aimed at the construction of a certain knowledge”(Sierpinska, 1999, p. 3). Some of these local choices are described below, whereas others are implicit in the project itself.

We decided not to mention eigenvalues and eigenvectors before the project day, which therefore began with a one hour lecture on this topic. In our conclusion we will mention an idea for a different set-up where the project day does not start with institutionalisation of the theory of eigenvalues. We first gave the definition of eigenvalues and eigenvectors and showed how these can be determined for 2 x 2 matrices by hand. However, emphasis was on the main theorem: both how conclusions about the ”population” $\mathbf{v}_t$ can be obtained from the dominant eigenvalue and a corresponding eigenvector, and conversely how the dominant eigenvalue and a corresponding eigenvector can be determined from the iterates $\mathbf{v}_t$ using (∗). In this way we made didactic use of the specific mathematical result (∗): because of (∗), eigenvalues lend themselves to discovery through experiments, so the use of computers allows the students to work with mathematical content that would otherwise have been out of their reach.

We felt that the lecture gave a rounded-off introduction to eigenvalues and eigenvectors and made no attempts to introduce the students to related abstract notions such as diagonalisation, or to show them how the main theorem is part of the general spectral theory for matrices. From a design point of view the didactic situation actually consists not only of the project itself, but also of the introductory lecture with devolution marking the boundary between the two.

Finally, since we wanted the project to test widely within the module, we decided to include exercises in the project that tested basic pen and paper matrix skills. Typically these exercises would involve handling of parameters; for instance, determination of parameters from given information.

Presentation and discussion of the project

In the project we attempted to implement the ideas and decisions concerning the didactic situation about eigenvalues in population models. In this section we present the parts of the project that involved experiments and eigenvalues and which are therefore most relevant in the context of this paper. In total this was 40–50 % of the project. We will discuss the learning purposes of the various parts of the project and how they were inspired by and linked to the the theory of didactic situations. The entire project can be found on www.dina.kvl.dk/~vils/exam-project.pdf. We refer to the previous section for a brief description of the introductory lecture.

Exercise 1

This was the part of the project that dealt most with experiments and eigenvalues. A population of worms is divided into 3 age groups: cocoons, juvenile worms and adult worms. At time t (measured in months) we denote by C_t, J_t and A_t respectively the number of cocoons, juvenile worms and adult worms and we let

$$\mathbf{v}_t = \begin{pmatrix} C_t \\ J_t \\ A_t \end{pmatrix}$$

be the population vector. It is assumed that the development of the population follows the matrix model

$$\mathbf{v}_t = \mathbf{M}^t \mathbf{v}_0 \qquad \text{with} \qquad \mathbf{M} = \begin{pmatrix} a & 0.51 & 1.8 \\ 0.22 & 0.68 & 0 \\ 0 & 0.32 & 0.60 \end{pmatrix},$$

where $a \geq 0$ is a parameter (This model was inspired by a PhD-project (Svendsen et al., 2005) at our university). The students were furthermore given a data set containing the number of worms in each of the three age groups found in an experiment over a 10 month period. Excerpts from the project:

(b) Let $a = 0.4$. As it can be seen from the data set, there were at the start 100 cocoons and no juvenile or adult worms.

Use R to calculate the vectors $\mathbf{v}_t$ for $t = 1, ..., 10$. Plot the calculated values of C_t, J_t and A_t together with the measured values.

Afterwards calculate $\mathbf{v}_t$ for $t = 1, ..., 10$ for different values of a in such a way that you gradually get better agreement between the calculated and measured values of C_t, J_t and A_t. Try to give the best value of a with 2 decimals.

Hand in graphs corresponding to at least 3 different choices of the parameter a (including the choice $a = 0.4$). Briefly argue why you from graph to graph change the parameter a as you do.

(e) For a population of worms living under different conditions than those in the experiment it has been established that $a = 0.4$.

Use the calculations from (b) of $\mathbf{v}_t$ for $t = 1, ..., 10$ for $a = 0.4$ to determine approximate values (with 3 decimals) of the dominant eigenvalue λ_1 and a corresponding eigenvector. Explain the calculations leading to the approximate values.

Which percentwise distribution between cocoons, juvenile worms and adult worms appears in the long run?

(f) For the worm population in (e)the parameter a can be changed by altering the living conditions. One wishes to change the value of the parameter a in such a way that the dominant eigenvalue is 1.234.

Use R to calculate the vectors $\mathbf{v}_t$ for $t = 1, ..., 10$ for different values of a. For each value of a use these calculations to determine an approximate value of the dominant eigenvalue λ_1. The values of a must be chosen in such a way that the dominant eigenvalue gradually approaches the desired value of 1.234. Try to give the best value of a with 2 decimals.

The calculations of $\mathbf{v}_t$ for $t = 1, ..., 10$ and the subsequent determination of λ_1 must be done for at least 3 different choices of the parameter a (including the choice $a = 0.4$ from (e)). Briefly argue why you change the parameter a as you do.

The purpose of part (b) was to give the students an idea of how the development of the population depend on the parameter and to show how an experimental approach can be used to find optimal values of parameters. The students would play a didactic game, where they alternated between situations of action: calculating iterates and plotting; and formulation: comparing with data and deciding how the parameter *a* had to be changed. No mathematical theory as such was involved, but the game required the students to have the necessary computer prerequisites to calculate and plot the iterates and plot the data. Before the project we made an effort to make sure this was the case.

The purpose of part (e) was for the students to learn how to calculate the dominant eigenvalue and a corresponding eigenvector from the iterates $\mathbf{v}_t$, so in a sense this part was the core of the didactic situation. The action part had been done in part (b). In (e) the students first had to use the main theorem and more specifically (∗) to realise that the ratios C_{t+1}/C_t, J_{t+1}/J_t and A_{t+1}/A_t had to be calculated. Following these calculations, they had to validate that the ratios approach the same value for all 3 age groups and conclude that this limit is the dominant eigenvalue. Similarly for the corresponding eigenvector, which afterwards also had to be interpreted to give the long term distribution between the age groups.

In the theory of didactic situations, the target knowledge is usually hidden from the students, who "discover" new mathematical concepts and results through adaptation to suitably designed didactic situations. In our case this approach was weakened, since eigenvalues and eigenvectors were introduced before the project and therefore only were "rediscovered" by the students in the mathematical experiment consisting of (b) and (e). In our conclusions we will discuss a set-up which might allow a more real "discovery" of eigenvalues and eigenvectors.

Finally, the approaches in (b) and (e) were combined in (f) which required the students to work exploratively at two different levels: the game of calculating iterates and using these to determine approximations to the dominant eigenvalue for given values of *a* was placed inside an overall game, where the value of *a* had to be adjusted in order to obtain a certain value of the dominant eigenvalue.

Exercise 2

A herd of cows is divided into "good cows" (with high yield of milk) and "bad cows" (with low yield of milk). Based on information about the development of the herd from one year to the next one, the students first had to formulate the affine matrix model

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} 0.8 & 0.1 \\ 0.2 & 0.5 \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix} + \begin{pmatrix} 12 \\ -8 \end{pmatrix},$$

where x_t and y_t are the number of good and bad cows respectively in year t. The exercise mainly dealt with pen-and-paper matrix skills and the only question which is relevant for us here is the following:

(c) Determine the equilibrium for the herd. Use theorem ... [which states that λ is an eigenvalue of $\mathbf{M}$ if and only if $\det(\mathbf{M}-\lambda\mathbf{E})=0$] to determine the eigenvalues of the matrix in the model. What can we deduce from these answers about the numbers of good and bad cows in the long run? How do these numbers depend on the start herd, that is the numbers of good and bad cows at time $t=0$?

Whereas the approach in exercise 1 to a high extent was based on actions, the purpose of exercise 2 was to let the students work in a more institutionalised way, by using eigenvalues to show stability of an equilibrium. More precisely, the students had to combine the theorem that states that the population vector $\binom{x_t}{y_t}$ converges to the equilibrium as $t \to \infty$ if both eigenvalues are numerically less than 1 with an interpretation of "the numbers of good and bad cows in the long run" $\lim_{t\to\infty} x_t$ and $\lim_{t\to\infty} y_t$ to conclude that the herd approaches its equilibrium in the long run.

Exercise 4

Let

$$\mathbf{M} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}.$$

(a) Use R to calculate the powers $\mathbf{M}^t$ for $t=1, ..., 10$. Use these calculations to guess an expression for $\mathbf{M}^t$, that is an expression for each of the 4 elements in the matrix $\mathbf{M}^t$ for an arbitrary value of t.

(b) [Difficult question] Verify the expression for $\mathbf{M}^t$ that you guessed in (a) by showing that if it is correct for t, then it is also correct for $t+1$.

This exercise did not involve eigenvalues, but showed a different kind of use of mathematical experiments, and contained a whole range of types of didactic situations: action (the mathematical experiment), where the powers $\mathbf{M}^t$ were calculated; formulation, where the students had to make a qualified guess ($\mathbf{M}^t = \begin{pmatrix} 1 & 2^t-1 \\ 0 & 2^t \end{pmatrix}$) based on the results of the experiment; and finally validation, where the students should attempt to verify their guess. Generally the students hardly had any knowledge of proof by induction, but our idea was that the experimental approach would support an inductive way of thinking.

Conclusions and ideas for improvement

We will now present some of the conclusions we drew from the first run of the course and critically evaluate the design of the didactic situation about eigenvalues in population models. We also present some changes which we implemented in the second run as well as some ideas for a further development of the course. This section can also be seen as phase (iv) in the process of didactic engineering.

Global conclusions about the exam format and the use of computers

The new exam format was in most ways a major success. On the project days there was an exceptional high level of student activity and the students seemed very comfortable with this way of working. The set-up with the help desk worked well in that the students were content not to be left entirely on their own, while at the same time accepting that the assistance was more indirect than usual.

No systematic studies were done in connection with the reform and we have no concrete measurements of increased learning outcome as a result of the exam projects, but in the opinion of the students themselves, the teaching assistants (who marked the projects) and the course responsibles the students learned a lot from their work on the projects. For the project involving eigenvalues we will analyse the learning outcome in somewhat more detail below.

However, there was one area where the reform lead to a definite and measurable improvement. Since there was a project every two weeks, the students did not have the option of pretending to themselves that they could catch up with the material later on, but were instead motivated to work at a steady pace throughout the course. In the old course about 15 % of the students did not attend the final exam; either because they had dropped out completely or because they considered themselves too far behind to stand a chance of passing the exam. In the new course basically all the students who handed in the first project after two weeks also attended the final multiple-choice test, so it seems that the exam format also helped to prevent the students from dropping out by making them feel more attached to the course.

On a less positive note, various comments from the students seemed to show that they did not fully grasp the idea that the final multiple-choice exam and the projects test different competencies and only together give a picture of the abilities of the individual student, so in the future we have to communicate this more clearly to the students. This may also reflect that although a competence based course description can be a very useful tool in planning and teaching the course, it may be more difficult to communicate it to the students.

Local conclusions about of the didactic situation about eigenvalues

The students were generally able to answer the questions concerning eigenvalues and eigenvectors as successfully as they answered other questions in the project, so in this respect it did not seem to have a negative impact that the topic was only introduced on the same day. However, a few students commented that they found the introduction of new material on the project day unfortunate, so we may have to make it clearer that the projects are not only part of the exam, but are also learning situations.

In exercise 1 the main idea of ”rediscovering” eigenvalues and eigenvectors through mathematical experiments worked well and the students seemed to develop a reasonable understanding of the role eigenvalues play in population models. In particular the fact that the main theorem was presented without any links to the underlying theory of diagonalisation etc. did not seem to impair the students ability to use the theorem in the way we intended. However, not all students were comfortable drawing conclusion from mathematical experiments and for instance, found it confusing that the ratios C_{t+1}/C_t etc. give approximations to, but not the exact value of, the dominant eigenvalue.

Also, some students were dissatisfied that they had to determine the dominant eigenvalue from experiments and were not simply allowed to use the built-in function in ”R” that gives the eigenvalues and eigenvectors of a given matrix. One way to improve on this would be to be more aware of ”the teacher’s epistemology”: the teacher not only expects the students to answer the questions, but to answer them according to certain standards, which are implicitly but most often not explicitly approved by the teacher (see Brousseau, 1997; Sierpinska, 1999). Hence we should change the design to avoid such unwanted ambiguities about how questions are meant to be answered, or to put it differently, we must communicate the didactic contract more clearly by making the purposes of the learning situation more explicit to the students. In the second run of the course we partly tried to do this by having not only questions similar to the quoted parts of exercise 1, but also a question where the students first had to use ”R” to find the eigenvalues and eigenvectors of a matrix and then use the main theorem the other way round to draw various conclusions about the development of the population v_t in the long run.

A more radical idea for a solution to the above problem would be to move the lecture on eigenvalues to later in the day and divide the didactic situation about eigenvalues into two separate didactic situations: in the first one the students would not have any theoretical knowledge of eigenvalues or eigenvectors. They would be asked to perform iterations of a matrix model as in exercise 1 (b), and then be guided to discover the patterns in the development of the population; more precisely that ratios such as C_{t+1}/C_t and C_t/A_t stabilise as $t \to \infty$. Afterwards the students would have to interpret their conclusions as the long-term growth rate and percentwise distribution of the population. This would be more in the spirit of the theory of didactic situations, since it would allow the students to discover at least some aspects of the new mathematical notions of eigenvalues and eigenvectors before these were formally introduced. The first didactic situation would be followed by a lecture on eigenvalues similar to the present one, but with the students' discoveries from the first didactic situation as the starting point. Finally, in the second didactic situation the students would have to use this basic theory about eigenvalues to analyse their results from the first one; possibly combined with more advanced questions related to eigenvalues.

This organisation resembles Bloch's returned situations, but with some differences. In Bloch's case the situation consists of two separate games each containing explorative activities, but where only the second game requires the target knowledge. In our case the second situation would not necessarily contain experimental work, but would instead focus on interpreting the first situation in the light of the target knowledge that was not previously available to the students. We have not yet analysed the implications such a division of the situation would have for the practical organisation of the project day.

We finish our conclusions about exercise 1 by mentioning that some of the manual tasks involved in calculating the iterates and using these to obtain approximations to the dominant eigenvalue seemed too repetitive. Although some repetition of the procedure may be useful, it will probably be an improvement if such repetitions are chosen so that they lead the students to draw conclusions about a wider range of cases; for instance, by including populations that die out, that is with $|\lambda_1| < 1$.

The idea in exercise 2 of using the eigenvalues to study the stability of the equilibrium and thereby the long-term behaviour of the model worked well except that a number of students did not mention that the eigenvalues had to be numerically less than 1 in order to draw the desired conclusions. This exercise could also have been used within a more traditional presentation of eigenvalues, but it is our impression that our partly experimental approach had given the students a more intuitive understanding of the long term behaviour of the iterates, which they benefited from when doing this question. Also, we think it is beneficial for the total learning outcome of the project to combine questions involving an explorative approach such as exercise 1 with questions of a more theoretical nature such as exercise 2.

Finally, as we hoped the approach in exercise 4 seemed to stimulate the students' inductive thinking. In the context of our course we doubt that the exercise would have worked well without the use of computers to calculate the powers of **M**, so, as in exercise 1, the experiments allowed us to include material, which would otherwise have been out of reach. In part (b) the degree to which the students were able to formalise their answers in the validation varied significantly: from answers like "it is clear from (a)" to fully correct proofs using mathematical induction.

Reflections on the usefulness of the didactic theories

We end the paper with a brief evaluation of how the theories of didactic situations and didactic engineering worked as development tools in our case.

The decision to include some type of project work in the course was taken early on in the process. Because of this, our initial motivation for using the theory of didactic situations was its focus on rounded-off learning situations, devolution and the distinction between different types of situations. In this respect we felt that the theory served a purpose by making us think consciously about the design of the situations, and thereby focusing our approach. As an example of this, we mention our analysis of the epistemological constraints related to the didactic situation about eigenvalues in population models. We found that the material would be too difficult for the students if we did not cut it down to the absolute minimum necessary to understand and apply the main theorem. Without this analysis we might have either included too advanced material in order to explain the background for the result, or rejected the idea of including the main theorem thinking that it would be out of reach of the students. Furthermore, regarding the mathematical experiments as didactic games lead us to a conscious use of different types of situations, as can for instance, be seen in exercise 1 (b) and exercise 4. In this way the theory helped to make clearer to us the role the experiments played in the projects.

As mentioned earlier, in some directions we felt a need to adapt the theory of didactic situations to our context. Our inclusion of a help-desk is an example of such an adaptation, and can in our opinion be seen as an addition to the theory which does not violate the central ideas of the original theory. However, as discussed in some detail in the conclusion, the introductory lecture before the project presents a more serious departure from the theory. This was a case where practical considerations ended up out-weighing the theory.

The method of didactic engineering and in particular its emphasis on analysis of constraints proved useful to us as a tool to structure our development process. In our experience such processes tend to jump directly from the goals to the decisions and could often benefit from a more thorough investigation of the challenges and feasibility of the development project.

The interplay between the two theories was in our case that didactic engineering helped us to make the framework of didactic situations more operational. In particular, we wish to emphasise that one of the strengths of the process of didactic engineering, in our opinion, is that not only is it useful in the overall planning of the course, but it can also be used on a local level, where epistemologically based discussions of concrete issues can lead to the development or revision of local teaching products.

Overall, it is our impression that the framework of didactic situation and didactic engineering was well suited for the type of development project described in this paper. Generally we found that the largest benefit was that the didactic notions helped us to conceptualise our development ideas for ourselves.

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Notes

1 In this paper *didactic* is used instead of *didactical*. Both adjectives are correct but didactic is simpler and seems to be more frequently used outside the field of research in mathematics education.

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Sammendrag

Vi beskriver hvordan vi benyttede Brousseaus teorier om *didaktiske situationer* og *didaktisk ingeniørarbejde* som en ramme for udviklingen af et eksamensprojekt i et førsteårs matematikkursus på et biovidenskabeligt universitet. De vigtigste læringsmål i projektet var at (gen)opdage egenværdier og egenvektorer bl. a. ved at studere den asymptotiske opførsel af matrixmodeller for populationsvækst samt at forstå den rolle egenværdier spiller i sådanne modeller. Derudover skulle de studerende opnå erfaring med matematiske eksperimenter med brug af computer og med at drage konklusioner fra sådanne eksperimenter.