A collective enquiry into critical aspects of teaching the concept of angles

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This paper describes an approach to teaching that enhances pupils' learning in mathematics. The model described – *Learning study* – involves teachers and researchers cooperating in an iterative process, gathering data about teaching and pupils' learning, analysing the data, planning and revising their teaching. A particular theoretical framework was used as a guiding principle when designing and analysing learning. The goal was to identify aspects critical for learning the angle concept. It is demonstrated how the teachers were able to identify the critical aspects and change the teaching in a way that promoted pupils' learning. What these critical aspects may entail and what teachers and researchers can learn from a Learning study is discussed.

The 'reflective turn' in the professional development of teachers suggests that teachers can improve their expertise by investigating and reflecting on their own practice. A model for a more systematic reflective process – *Learning study* – where practitioners and researchers work together to improve pupils' learning, is described. Everyone involved try to learn from each other, from the learners and from the study itself. A Learning study starts from the learners' understanding and is focused on the object of learning, i.e. what the pupils are supposed to learn. In the study presented here, the aim was to improve pupils' understanding of the angle concept.

Young children's difficulties with this concept are reported in the literature. For instance, children do not easily recognise an angle that is

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not orientated in the standard horizontal/vertical position. They often perceive angles as corners, have difficulties with angles of 180 and 360 degrees and perceive the angle as the area between the two rays. Several researchers have reported how pupils believe that the length of the arms is related to the size of the angle (Johnsen, 1996; Keiser, 2000; Magina & Hoyles, 1997; Wilson & Adams, 1992). An angle can be defined either as static (wedge, intersection of two lines) or dynamic (turn). It has been discussed whether the angle should be introduced in terms of turning, wedge or both (Keiser, 2000; Mitchelmore, 1998; Wilson & Adams, 1992).

The guiding principles in a Learning study

A *Learning study* (Pang & Marton, 2005), combining development work and a research process, is inspired by the Japanese *Lesson study* (Lewis, 2002; Stigler & Hiebert, 1999; Yoshida, 1999) and *design experiment* (Cobb et al., 2003). In Japan Lesson study is a common model for teachers to improve their teaching. They jointly plan, observe, analyse, and refine actual classroom lessons, particularly for the improvement of Japanese elementary mathematics and science instruction. Lately, lesson study has rapidly emerged in many sites across the United States and other countries as well (Lewis, 2002; Ma, 1999).

A Learning study is similar to a lesson study; the teachers explore and develop their teaching practice, but in a Learning study the variation in pupils' learning plays a more central role. First and foremost, the teachers investigate how different ways of handling the same content afford different learning possibilities and how pupils' learning may be enhanced. As in a design experiment, the process includes systematic and theory-based research but the teachers are involved in the design process. The framework – variation theory (Marton & Booth, 1997; Marton & Tsui, 2004) – does not prescribe a particular procedure, arrangement or way of teaching. It provides some general principles for learning that can serve as a guideline for designing the lessons.

Variation theory deals with the object of learning. In a learning situation there is always something learned. And in a classroom the teacher and the learners interact about *something* (Runesson, 2005). One of the principles of variation theory is that it is necessary to notice or discern certain features of the thing to be learned (Marton et al., 2004). Therefore a Learning study takes its point of departure from that which is problematic for the learners. The teachers ask: "How do the learners understand X? What do they have difficulties with? What does it take to understand X in another way? What is it that the learner must notice or 'get hold of' in order to see it differently?" So, when the teachers start to plan their

teaching, they focus on the relation learner – object of learning, not on the organisation or other arrangements in the classroom.

Yet another theoretical principle says: that which is varied is likely to be discerned (Bowden & Marton, 1998, p. 35) An aspect is more likely to be discerned if it is changed than if it remains invariant. If something varies and something else remains constant, then it is possible that the thing that varies will be noticed. For example, it is more likely that one will understand the 'five-ness' of five if five cups are contrasted with four cups (i.e. the object is the same but number is varied) than if five cups are contrasted with four cars. In the first case, the focus is on the number and it is possible to discern it, whereas in the second case (when number and object are varied at the same time) it is probably not possible to differentiate the object from the number. A teacher can help the learners to discern an aspect by designing and presenting a pattern of variation and invariance. But *how* this pattern should be constituted – what should vary and what should be invariant – cannot be derived from the theory itself. It is specific to each object of learning and to each group of learners. The aim of a Learning study is to investigate 'what it takes' to learn, what the critical aspects for learning are, (i.e. those aspects that should be discerned simultaneously) and to investigate how a particular pattern of variation and invariance can bring these out.

It should be pointed out here that what is meant by variation in this study is not variation in general. The variation/invariance here concerns the object of learning. Variation and invariance are introduced as a means of handling the content, independent of the organisation or arrangement. In a Learning study, the teachers use variation and invariance consciously and systematically to improve learning.

Method and design of the study

A Learning study is a cyclic process; the teachers choose the object of learning and design a pre-test. The pupils are tested and the results are taken as the point of departure when planning the lesson. The lesson is implemented by one of the teachers and video-recorded. The pupils are tested about one day after the lesson. The Learning study group watches the video-recording and analyses the test results. If the learning outcomes do not relate satisfactorily to the teachers' goals, they closely observe the lesson and revise it accordingly. The next teacher implements the new plan in her class (new pupils). This is video-recorded, the pupils are tested, the video-recording observed and a new revision of the plan takes place. This cyclic process continues until all teachers have taught their pupils.

The study took place in three classes of fourth and fifth year pupils (pupils aged 10 and 11 in the same class) in a Swedish compulsory school. The classes were all mixed ability classes. The teachers participated in the study on a voluntary basis. Participation was optional for the pupils. The parent(s) of the participants (100 % of the pupils) had given their written consent. Before starting the Learning study, the teachers attended a one-day seminar about the fundamental ideas of variation theory and how these could be used as a guiding principle for designing and analysing lessons in general.

The teachers (all experienced teachers) and the researcher planned and developed the teaching together, but the way the lessons were designed and enacted was principally based on the teachers' own ideas. Although the significance of the participating researcher in the teacher team cannot be neglected, the study was naturalistic in the sense that the teachers 'owned' the lessons in terms of ideas and how these were enacted.

The data generated in the study were video-recordings of the lessons, audio-recordings for the pre- and post-lesson meetings and results from pre- and post-tests. The pupils were tested one day after the lesson and seven weeks later. The purpose of the delayed test was to examine the long-term effect of the teaching. During these weeks, the pupils were not taught about angles at all.

The data were used to develop the lesson and record how the object of learning was handled and the effect that had on pupils' learning. After each lesson, we wanted to find out what the pupils had actually learned in the lesson. Since some of the items were identical in the pre- and post tests, it was possible to compare the results for each class on every test-item before and after the lesson (intra-group difference). If several pupils had given incorrect answers on a certain item, this was given particular attention. Was it possible to learn that item as presented in that particular recorded lesson? To find the critical differences between the three lessons, we also compared the pre- and post-test results between the classes (intra-group difference). The three classes showed different profiles on the tests. Our analysis aimed to determine how these differences reflected differences in how the object of learning was enacted in each of the lessons, and to describe aspects of the object of learning that were critical for pupils' learning.

In the same way as the theoretical framework served as a guiding principle when designing the lessons, the analysis of the lessons afterwards followed the fundamental principles of the framework (see above). We described the lesson in terms of patterns of variation and invariance, i.e. what aspects were kept constant and what were varied. For instance, if the teacher showed two angles of the same size but with sides of

different lengths, this was a particular pattern of variation/invariance. The analysis followed the hypothesis of Marton & Morris (2002) and Marton & Tsui (2004) that different patterns of variation afford different learning opportunities. They found that the space of variation and invariance identified in a lesson is reflected in pupils' learning outcomes. Experiences from these studies have inspired the introduction of the Learning study model and the way the data in this study were analysed.

Video-recordings, transcripts and test results were analysed in parallel and in shifts. The inter- and intra-group differences in test results provided information for studying instances in the lesson that were assumed to be critical for the learning outcomes. The differences in how these instances were structured were analysed in detail to catch patterns of variation/invariance in the lessons. To avoid being too concentrated on features of the lesson that can easily be directly related to the test results, we disregarded the test results in the first phase of the analysis and just described the different successions of the patterns of variation. Not until these descriptions were completed, did we attempt to relate them to differences in the learning outcomes.

Results

In the following, I first report how the teachers investigated and developed their teaching, then I analyse the lessons more deeply and account for how the critical aspects identified by the teachers were related to differences in the three lessons, and how these aspects were analysed through the researcher's enquiry.

Analysing and refining the lesson

On the basis of results on the pre-test (table 1), the Learning study group agreed upon a common object of learning: to recognise the 180° angle, to differentiate an angle from a non-angle, and that the size of an angle is independent of the lengths of its sides. The most striking differences in the pupils' learning outcomes between the classes were found in connection with the last item. The findings are reported in this article.

Being informed about the post-test results after the first lesson, the teachers observed the video-recorded lesson. They were not satisfied with the results; they thought the pupils' had not learned what the teachers had intended them to learn (see table 1).

After lesson 1 only 33 % of the pupils could disassociate the lengths of the sides of the angle (i.e the arm lengths) from the size of the angle (see figure 1, item 1). Although this is some progress compared to the pre-test

Table 1. *Results in pre- and post-tests in the three classes.*

Item	Class/ Lesson 1 (n=18)		Class/ Lesson 2 (n=20)		Class/ Lesson 3 (n=23)	
	Pre-test	Post-test	Pre-test	Post-test	Pre-test	Post-test
1 Can disassociate the arm lengths from the size	6	33	40	95	26	78
2 Position of the marking unimportant to the size	-	67	-	95	-	78
3 Can see that the angles in two similar triangles are the same size	-	11	-	40	-	43

Note. Percentage of the pupils giving correct answer on the different items.

before the lesson, the teachers were not satisfied with the results. Items 2 and 3 (see table 1 and figure 1) on the post-test were only given after the lesson. After being taught, 67 % realised that the position of the marking is unimportant to the size (item 2), which is far better than the result on item 1. Only 11 % realised that the size of the angles are the same in two congruent triangles.

When analysing the audio-recording of the discussion, I found that the teachers paid much attention to the pupils' understanding of the content. They tried to relate what was learned, i.e. the pupils' understanding (as depicted by the test), to what was brought out in the lesson. For example, they came to the conclusion that the pupils see the sides as a constituent part of the angle, and this way of understanding is reasonable and logical from the point of view of how the definition of an angle was given in the lesson. So, they decided to come up with another definition in the next lesson and bring out the idea of angle as *a turning* more clearly. However, they did not discuss more explicitly how this should be done. They just called attention to the fact that 'turning' must be elicited more clearly in lesson 2 in order to promote better learning.

After the second lesson in the series, the teachers were more satisfied with the learning outcomes. After lesson 2, almost all the pupils (95 %) in the class gave correct answers to questions 1 and 2, and 40 % knew that the angles are equal in two congruent triangles (see table 1). The teachers decided to use the same lesson plan for the third lesson. After lesson 3, the scores were almost as high as after lesson 2.

A deeper enquiry into the identified critical aspects

Avoiding falling into the trap of seeing the teaching-learning process as a one-to-one correspondence, we identified differences in the lessons that might be significant for failure and success on the test. Three different items were designed to test if the pupils realised that the length of the sides is not related to the size of the angle (figure 1). Item 1 appeared both in the pre- and the post-test, while the other two items were tested after the lesson only.

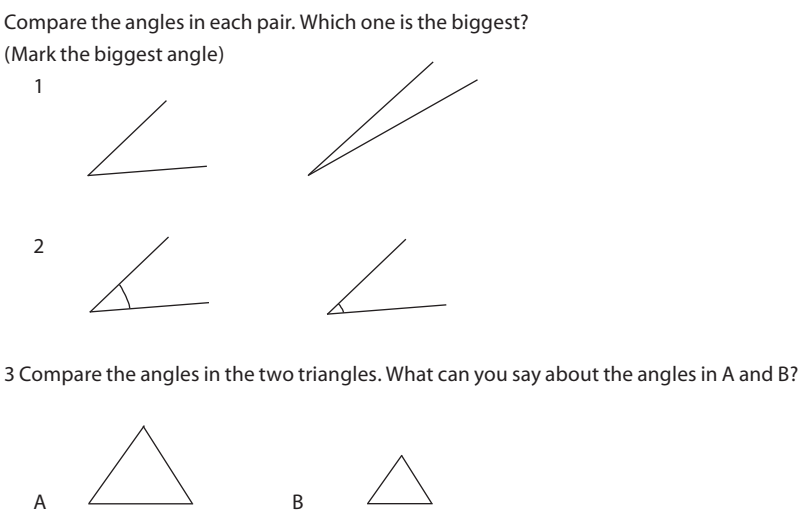

Figure 1. *Examples from the pre- and post-test (c.f. table 1)*

The outcomes on item 1 in the pre-test differed in the three classes (see table 1). Class 2 performed best. 40 % of the pupils answered correctly on this item before the lesson, compared to 26 % and 6 % respectively in classes 3 and 1. After the lesson, class 2 still performed best, almost all pupils answering correctly on item 1 (and, in fact, also on item 2). However, the biggest increase in correct answers was found in class 3; from about a quarter of the class giving correct answers on the pre-test to 3/4 on the post-test.

On items 2 and 3 (these examples were not explicitly taught in the lesson) classes 2 and 3 performed better than their counterparts in class 1. There was a particularly marked difference between class 1 and the other two classes with regard to item 3.

Following the teachers' anticipation that how 'turning' was brought out in lesson 1 was significant and their revision of the lesson plan, I concentrated my analysis on the different ways 'turning' was explained and the pupils' experience of this.

Lesson 1

In lesson 1 the focus was very much on defining an angle. The teacher defined an angle as "two arms meeting at one point". She used this definition as a reference when comparing two clock-faces showing the same time but with hands of different lengths (i.e. the rays of the angle formed an angle). To the question: "What is an angle?" one of the pupils replied: "Two arms meeting at a point", and the teacher summarized:

Excerpt 1 [L1].

T: Good. Two arms of an angle meet at a point, but the arms don't need to be the same length. There may be two angles here [points to the pupils' report papers] .... that are exactly the same size but whose arms aren't the same length.

The next topic thus became 'rotation of angle'. Again the 'clocks' were used to illustrate this. That an angle could be orientated differently was then connected with the definition of angle previously given; two rays meeting at one point.

Excerpt 2 [L1]

1 T: Now we have something else we know about angles. They are two rays that meet at one point. What was it we knew about the arms? Pernilla? They meet at a point, what else did we know?

2 Pernilla: They are called like that, arms ... They don't need to be the same length.

3 T: They don't need to be the same length. And they don't need to lie ... the angles don't need to lie in the same plane but can lie in all sorts of different directions.

In this manner, the teacher added some more information to the definition previously given. Next, the teacher brought out the idea of turning. She pointed to a clock showing five past twelve. She turned one of the clock's hands until it was 'half past twelve', and asked: "What happens to the angle?" The pupils answered in chorus: "It gets bigger". However, the teacher did not explicitly say that it is by turning the arms that the size of the angle is changed. Neither did she relate the turning of the arms to the angles formed by the hands on the clocks, which they had compared earlier in the lesson. Instead she continued to turn the arms of the angle until the angle was almost 180° and said: "What will happen to the angle?

Will it disappear?" In this way the turning of the clock's hands served mainly as a transition for presenting one of the other goals for the lesson: the 180°angle. By this transition and, by the question the teacher asked, the teacher drew the pupils' attention to the difference between an angle and a 'non-angle' (i.e. whether a 180°angle is an angle or not). One could anticipate that this could change the focus of the learners' awareness – from experiencing 'turning' to noticing 'different angles'.

In this lesson it was pointed out that two equal angles do not necessarily have to have the same arm length, but that the size can be changed by turning the arms. That the size is independent of the length of the arms was presented twice (firstly comparing the same angle with different arm lengths, and secondly by turning the arms). However these *were not consecutive instances* but were interrupted by the presentation of 'orientation'. The sequence of lesson 1 is illustrated in figure 2.

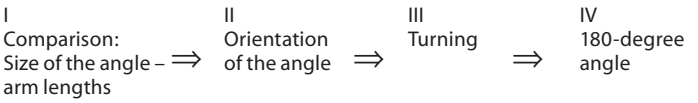

Figure 2. *The sequence of 'comparison', 'turning' and 'orientation' in lesson 1*

Lessons 2 and 3

In lessons 2 and 3 the teachers gave another 'definition' of angle to the class; "there is a point and two straight arms coming from this point", a change they had decided when revising the lesson plan. Two clock-faces showing the same time but with hands of different lengths (i.e. the rays of the angle formed) were compared. In lesson 2 the teacher instructed the pupils to set the smaller clock face at four o'clock and the bigger at two o'clock (i.e. the bigger clock having the smallest angle and vice versa). These were displayed on the pupils' desks so they could see them *simultaneously*. At the same time the teacher used a big clock to demonstrate the turning. She very clearly turned the hands of the clock by moving them back and forth. In the dialogue she alternated between focussing on changing the hands of the bigger clock and pointing to the two clocks showing different times on the pupils' desks. This is illustrated in the following:

Excerpt 3 [L2]

1 T: If I have this clock now. My clock shows two, just like your clocks do. If I want to make mine show four, how should I do it? Eva?

2 Eva: Just move it two steps.

3 T: Now what I wonder is this; if you look at my clock, which angle is biggest? When my clock shows two or when my clock shows four?

4 Sonja: When it is four o'clock.

5 T: Why?

6 Sonja Don't know

7 T: Do you know what I did to make the angle bigger? Indeed, I turned one of the arms of the angle. I moved one of the arms of the angle. If I want to make it smaller, if I want to make the angle smaller, how do I do it then?

[Then she asked]

8 T: Your big clocks show two and the small clocks show four. Which angle is biggest? The angle on the little clock or the angle on the big clock?

9 Esther: The angle on the small clock. The smallest.

[The teacher explained how the size changes]

10 T: Look here. Here's twelve o'clock. When I move one of the arms in one or other direction, then the angle becomes bigger because the angle is what comes from the point and the turn you make. The angle, then, is the turn here. Okay? So, it doesn't make any difference how big the clocks are, how big the hands are, even though mine is this big, it's how much I turn the hands that determines how big an angle I get.

Here she contrasted the two angles, that is, she focused on the difference (line 3 and 8) and the change in the size of the angle (line 7 and 10) *at the same time*. It was possible to see different angles with different arm lengths *simultaneously* with the change in the position of the hands on the teacher's clock (displayed on the two clocks on the desks in front of them, and the clock displayed by the teacher at the front). The teacher brought out the turning of the arms of the angle distinctly when she turned both the hands of the clock and said: "Does the angle get bigger? Yes, because the angle is what comes from the point and from the turning." This demonstration of the turning followed directly after the comparison of angles (see figure 3). So, the sequence of the lesson is different from that in lesson 1.

In lesson 3 the pupils, besides comparing the two clock-faces showing the same time, but with different lengths of the hands, compared several examples of pairs of angles (similar size – different arm lengths, different size – different arm lengths). The teacher said: "The figure is bigger, yes, but the angle ... it makes no difference how long the arms of the angle are. The angle is just as big all the same". In order to convince the pupils, the teacher took one clock showing five past twelve and, turning one of the hands, she said:

T: The more we open it, or turn it rather, the bigger the angle becomes. So the more we turn it, can you look here, the bigger the angle becomes.

So, apart from telling the pupils that the angle size is independent of the arm lengths, the teacher also demonstrated this by a systematic comparison of pairs of angles *and* by turning one of the arms of the angle.

The sequence in lessons 2 and 3 is similar, in that 'comparison' and 'turning' were consecutive episodes in the lessons, as is shown in figure 3. in lessons 2 and 3 the episodes 'comparison' and 'turning' were followed by the demonstration of 'orientation of the angle', and 'the 180° angle' respectively.

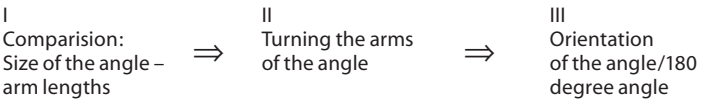

Figure 3. *The sequence of 'comparison', 'turning' and 'orientation in lessons 2 and 3*

The significance of difference of sequence and the space of variation

Can the better performance after lessons 2 and 3 be understood in the light of these differences? The lessons were similar in many respects; their duration was about 50 minutes, the same teaching material was used, and the same activities took place in all three lessons. It is true that classes 2 and 3 performed better on the pre-test, which might have meant that there had better preconditions for learning. On the other hand, one could argue that the possibility of improving the post-test result was greater in class 1.

The very close analysis of the lessons demonstrates a change of sequence in the lessons and of the space of variation created. These changes seem to correspond to changes in the learning outcomes. Is it possible that the difference in sequence and variation can account for certain differences in learning outcomes? Relating learning to teaching is complex (Nuthall, 2004). There are, of course, many factors that affect learning outcomes and learning can hardly be predicted. This is not the aim of a Learning study, but rather to understand and describe possibilities for learning. Or put differently, to understand how learning is promoted by certain aspects of the learning situation. This is the point of departure when focusing on the differences between the lessons as regards how aspects of the angle concept were sequenced and brought out as patterns of variation/invariance.

In lessons 2 and 3 the event in the lesson when 'turning' was shown and talked about followed directly after the 'comparison' event (see figures 2 and 3) They were consecutive instances in the lesson. That the size of the angles can be changed by turning the rays and not by extending

them became an answer to the questions: "What is different/similar?" and "Which is bigger/are they the same?" in those lessons. In that way, the pupils were given an opportunity to pay attention to *differences in arm lengths and change in the positions of the arms* at the same time. You can say that the demonstration of 'turning' resolved the problem of what determines the size of the angle.

In lesson 1 there was another sequence. The presentation of 'orientation of angle' followed upon the comparison of angles of the same size but with different arm lengths. You can say that the question about whether or not the length of the arms is related to the size was interrupted by the question about whether the orientation is related to the size. In my view, the result of this sequence was that the pupils' attention was drawn to another form of 'turning', that is the turning of the angle in the plane (the same angle was rotated in different positions) not to turning of one or both arms.

In all the lessons the teacher clearly stated that the length of the arms is not related to the size of the angle. However, this statement was embedded in a pattern of variation/invariance of the aspects 'size of the angle', 'length of the arms' and 'orientation'. So, apart from the difference in sequence, the patterns of variation/invariance created in the three lessons were different. Four different patterns of variation were identified:

A Angles of the same size but with arms of different lengths.

B Angles of the same size and with arms of the same length but oriented differently.

C Angles of different sizes but with arms of the same length and with the same orientation.

D Angles of different sizes with arms of different lengths but oriented similarly.

These are summarized in table 2.

Table 2. *Patterns of identified variation (v) and invariance (i)*

Pattern of variation/ invariance	Angle size	Length of arms	Orientation
A	*i*	*v*	*i*
B	*i*	*i*	*v*
C	*v*	*i*	*i*
D	*v*	*v*	*i*

All patterns but pattern D were present in the three lessons. A situation when the size *and* arm lengths varied *at the same time* was never demonstrated to the pupils in lesson 1. So, compared to lesson 1, the pupils were exposed to a more elaborated pattern of variation/invariance in lessons 2 and 3. Moreover, the succession of the patterns was different, as is illustrated in figure 4.

Lesson 1	A ⇒ B ⇒ C
Lesson 2 and 3	A ⇒ D ⇒ C ⇒ B

Figure 4. *The succession of the patterns of variation and invariance in the lessons*

The same patterns of variation and invariance appeared in lessons 2 and 3 and their succession was identical, but we found a difference regarding *how* this variation was brought about. In lesson 2, the pupils had the opportunity to notice the teacher turning the two hands of the clock *and* to look at the two different angles with different arm lengths *at the same time*. The teacher thus turned the clock hands several times and alternated between paying attention to the turning of the hands of her big clock and the angle position of the hands of two clock-faces on the pupils' desks. In this way there was a (repeated) shift between change and difference. The teacher pointed to the angle as a static phenomenon (wedge, intersection of two lines) on the clock faces on the pupils' desks *and* compared this to the dynamic change in the angle size on the teachers' clock. This demonstrated the dynamic and the static character of the angle at the same time. In lesson 3, on the other hand, the variation in angle size was created when several examples of pairs of angles were compared. This set of pairs was composed in a way that probably afforded an opportunity for the learners to see that the length of the arms was not related to the angle size.

If we take a closer look at this simultaneous pattern of variation, we will find a particular *combination of variation and invariance* in lessons 2 and 3. The pupils were asked to set the biggest clock at two o'clock and the smallest at four o'clock and compare the angles. So, the bigger angle had the shortest arms and the smallest one the longest arms. This combination of variation/invariance really challenges the idea that the length of the arms has something to do with the size of the angle. You can say that the two aspects coincide. If you think, like the majority of pupils did before the lesson, that 'bigger' refers to the size of 'the area' marked off by the sides of the angle, and thus, the longer the arms the bigger the 'area', your idea is challenged by this example.

Table 3. *Results in pre- and post-tests 1 and 2 in the three classes*

Item	Class/ Lesson 1 (*n*=18)			Class/ Lesson 2 (*n*=20)			Class/ Lesson 3 (*n*=23)		
	Pre-test	Post-test 1	2	Pre-test	Post-test 1	2	Pre-test	Post-test 1	2
1 Can disassociate the arm lengths from the size	6	33	44	40	95	95	26	78	78
2 Position of the marking unimportant to the size	-	67	67	-	95	95	-	78	78
3 Can see that the angles in two similar triangles are the same size	-	11	0	-	40	25	-	43	22

Note. Percentage of the pupils giving correct answer on the different items.

Long-term effects of learning

We asked the teachers not to teach about angles until after the second post-test seven weeks later. The results of the pre-test and the two post-tests are presented in table 3.

The overall impression is that the knowledge the pupils acquired during the lessons is retrieved over a period of time. On item 2 the proportion of pupils giving the correct answer is exactly the same in post-tests 1 and 2 in all the classes. This also goes for item 1 and classes 2 and 3. The recall of item 3 (seeing that the angles in two similar triangles are the same), however, deteriorated in all the classes. This task is the most complex one and furthest away from what actually took place during the lesson. Despite that, about one quarter of the pupils in classes 2 and 3 answered correctly even after seven weeks. What surprised us was that class 1 performed better on item 1 on the second post-test seven weeks later. However, it must be noted that the increase from 33 % to 44 % correct answers represents two pupils in the class. If this is a result of the pupils either talking to each other, to the teacher or their parents, or if it an effect of the teaching, we can only speculate about. Despite that, what is worth considering is the effect one single lesson seems to have on learning.

Learning from a Learning study

A Learning study aims at mutual learning and learning at three levels: pupils', teachers' and the researcher's learning.

Changes in the lesson design and in the manifested lesson were based on the teachers' own conclusions about pupils' understanding and on

their reflections and analysis of the lessons. My interpretation is that the teachers managed to identify aspects necessary for learning. When investigating the first lesson and how the object of learning was related to the post-lesson test results, the teachers realised that the necessary conditions were not present. They explained the pupils' comments in the lesson and their performance in the tests after the lesson as an effect of the absence of the idea of turning and of how the concept was defined in the lesson. They concluded that it was necessary to change the lesson and, accordingly, adjustments were made in the following two lessons. So, it is suggested that the teachers in this study learned how to improve teaching in a way that enhanced pupils' learning

By closely examining what the differences between the lessons involve and imply for pupil's opportunities to learn and discern the critical aspects, the researcher was able to learn more about the critical aspects identified by the teachers. Two major differences were found. First, the teachers changed the definition given and how this was used. In lesson 1 the definition was used as a criterion for classifying angles, whereas in lessons 2 and 3 the pupils' own understanding was drawn upon, for instance, when classifying angles/non-angles. Second, the patterns of variation and sequence brought out in lesson 1 were different from those in lessons 2 and 3. There seemed to be a systematic relationship between the difference in sequence and variation constituted and the difference in pupils' learning outcomes. It seems likely that the particular pattern of sequence and variation/invariance in lessons 2 and 3 made it possible for the learners to experience differences between angle size and arm length and changes in size at the same time. In the researcher's view, this could account for differences in learning outcomes.

To what extent can the result indicate to an audience something going beyond the particular study and the group of teachers in question? What can the results of this study reveal about the teaching and understanding of the angle concept in a more general sense? Although teamwork is common among Swedish teachers, they very seldom work co-operatively around a specific topic like they did in this study. This and other Learning studies (e.g. Holmqvist, 2006; Lo et al., 2005) have demonstrated the potential of this subject-oriented model for school developmental work in other contexts. So, given sufficient conditions, it is most likely that it would be possible for groups of teachers to work like this on a regular working basis and without the support of a researcher.

Follow-up studies are needed before one can say that the patterns of sequence and variation described in this study have potential value for other groups of learners as well. However, I find it interesting to note that, in this study, learning outcomes were almost the same after lessons 2 and

3 although the teachers and the pupils were different. The results for item 1 in post-test 1 (i.e. can disassociate the arm length from the size) in classes 2 and 3 were similar; the majority of the pupils in these classes gave the correct answer, whereas in class 1 only one third of the pupils answered correctly. This could be compared to the similarities between lessons 2 and 3 regarding how the idea of turning was handled. In these lessons, the sequence and variation of aspects related to the angle concept were presented in a similar way, although not identical in detail. In both of these lessons, 'turning' of the arms of the angle (pattern C and D, see table 2) was presented against the background of a comparison of angles of the same size but with different arm lengths (pattern A). It is likely that this particular and common feature of lessons 2 and 3 is reflected in the learning outcomes in the post-tests in these classes.

Some final reflections on the study

There are some other issues that need to be considered concerning this study. For instance, it could be questioned whether the material used is an appropriate means of teaching the angle concept, especially from the point of view that angles mostly appear in other contexts than the clock's face (Magina & Hoyles, 1997). However, the aim of this study was to give the teachers as much free scope as possible as regards choosing material, planning and carrying out the lessons and the use of the clock as a teaching aid was a decision taken by the teachers themselves.

In a study like this, the tests and how they are used become critical. The pre-test had two functions: a means of 'screening' the learners' pre-knowledge and of comparing pupils' learning before and after the lesson. It is somewhat problematic to combine those interests. It is not until the study is analysed and the critical aspects of the object of learning are identified, that you know what the test should contain. There is, of course, always a risk of missing something in the lessons that has an impact on learning. For instance, the pupils' attention levels, their previous learning experience or the teacher's involvement may all be underestimated. So, there are several variables that cannot be controlled in a study like this. We did not make any attempts to control all these variables but, to enhance learning by identifying critical aspects of the object of learning, and in that respect the Learning study seems to have been effective.

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Sammanfattning

I denna artikel beskrivs en modell för hur lärare kan samarbeta för att förbättra elevernas lärande. Modellen – Learning study – grundar sig på forskning om lärande och har en teoretisk grund. Det är en modell för samarbete mellan lärare i lärarlag, där lärare tillsammans utvecklar en gemensam kompetens kring frågor som: Hur kan man på bästa sätt lära ut något som är svårt? Vad gör skillnad mellan olika möjligheter att lära? I den aktuella studien undervisade man om vinklar och avsikten var att undersöka vad som var nödvändigt för att eleverna skulle lära sig begreppet. I artikeln beskrivs hur lärarna lyckades komma underfund med vad detta var samt hur de lyckades förändra sin undervisning så att fler elever lärde sig.