# Empirical research in the field of using history in mathematics education

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This paper discusses empirical studies in the proceedings HPM2004 & ESU4. More precisely the paper deals with four of the more clear-cut empirical studies. These are the contributions by B. Smestad, C. Tzanakis & M. Kourkoulos, W.-S. Horng, and Y.-W. Su. These contributions are first presented and then later discussed in the context of whether their purpose of involving the history of mathematics in mathematics education is to promote the learning of mathematics or if it is to bring about aspects of mathematics which are not normally part of the teaching and learning agenda, e.g. cultural or social aspects of mathematics and its history – 'history as a tool' or 'history as a goal'. The papers and their purposes for involving history are then related to a Nordic case, namely the official regulations for the Danish upper secondary mathematics programme for involving the history of mathematics. In the end the need for empirical research studies in the field of using history in mathematics education is discussed as well as further perspectives for the community regarding such studies or the lack of them.

In the spring of 2007 I promised Constantinos Tzanakis, one of the editors of proceedings HPM2004 & ESU4, a small review in a Nordic journal in return for a copy of the revised proceedings. As time went on, however, this small review grew in size and scope and ended up being the here presented paper.

When I received the massive volume (640 pages + 30 pages of preface) I was already familiar with quite a few samples on the use of history in mathematics education, such as (Fauvel, 1990, 1991; Swetz et al., 1995; Jahnke, Knoche & Otte, 1996; Calinger, 1996; Katz, 2000) and last but

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not least (Fauvel & van Maanen, 2000).[1] Another thing which I was also familiar with was some of the critiques of the already existing literature in the field. For instance the one by Gulikers and Blom (1991) who state that:

> Most publications are anecdotic and tell the story of one specific teacher, whereas it is unclear whether and how the (generally positive) experiences can be transferred to other teachers, classes and types of schools. (Gulikers & Blom, 1991, p. 223)

Thus, I decided to look into the empirical studies in the proceedings HPM2004 & ESU4. Out of a total of 78 papers I was able to identify seven papers as being either clear-cut or somewhat empirical studies. Actually this is not such a bad percentage, especially since Siu (2007, p. 269) mentions that at the time of the conference he was only aware of a total of five empirical studies on evaluation of the effectiveness of using history in mathematics education in the English literature within the field.[2] Now, empirical studies need of course not only be concerned with the effectiveness of using history (so maybe there are more empirical studies than the five which Siu mentions on the effectiveness) and the ones I shall discuss in the following do not all concern the effectiveness either. But before I present the chosen papers let me first provide a bit of background information on HPM and ESU.

The history of *The international study group on the relations between the history and pedagogy of mathematics* (HPM) can be traced back to the second ICME of 1972 where it began as a working group. At the third ICME of 1976 HPM was set up as a permanent study group under ICME together with the, today, probably more known *International group for the psychology of mathematics education* (PME). *The European summer university on the history and epistemology in mathematics education* (ESU) is a later initiative taken by the *French mathematics education* community (IREM) in July 1993.[3] The joined conference of HPM2004 and ESU4 was held at Uppsala University from the 12th to the 17th of June 2004 as an ICME10 satellite meeting. In the words of Man-Keung Siu what characterizes both the HPM and ESU is a "medium-sized heterogeneous group, comprising university mathematicians, mathematics educators, school teachers of mathematics and historians of mathematics, [who] come together to learn from each other, to discuss among each other, to argue with each other, and of course also to renew old acquaintances and make new ones in a relaxed and friendly atmosphere".[4] The conference was organized into six main themes, and so are the proceedings: (1) The history of mathematics; (2) Integrating the history of mathematics into the teaching of mathematics; (3) The role of the history of mathematics in teacher's training; (4) The common history of mathematics, science and technology; (5) Mathematics and different cultures; and (6) The philosophy of mathematics. The topics alone indicate the wide scope of both HPM and ESU and the contributions under these topics are as different and diverse as one can imagine. This diversity is, however, in the nature of HPM which can be confirmed by looking at the broad original program of the group as established in 1976 (Fasanelli & Fauvel, 2007, p. xi).

As mentioned I was able to identify seven papers involving empirical studies in the proceedings. Out of these I have selected four, which I believe to have a more clear-cut empirical focus, for a further discussion. Two of these appear within the second main theme of the proceedings, the first is an evaluation by the Norwegian B. Smestad of an international video study, the other is by the Greeks C. Tzanakis and M. Kourkoulos and includes a classroom observation. The two others of the four appear within the third main theme. These are by the Taiwanese W.-S. Horng and Y.-W. Su and concern mathematics teachers' professional development in terms of applying history in their teaching.

As the reader may have noticed no main research question has been given for this paper, besides the focus of it being on empirical studies. The lack of a general research question is due to the review nature of the paper. The idea is to present the four selected HPM2004 & ESU4 contributions on their own premises, which is done in the following three sections, and then give a discussion and comparison of them. However, in order to structure and order the discussion and comparison an instrument for analyzing the four papers is presented and applied. This instrument is a categorization of the purposes of involving history of mathematics in mathematics education into being either concerned with using 'history as a tool' or using 'history as a goal'. Thus, the instrument serves as a basis for the discussion. The situation of the Danish upper secondary mathematics programme is then related to the discussion of both the categories and the four papers as well as the need for empirical studies in the field in general. In the end, conclusive remarks and further perspectives regarding empirical studies within the field are made and presented.

## Use of history of mathematics in the TIMSS video study

In 1997 history of mathematics was included in the Norwegian national curriculum from 1st to 10th grade. Smestad (2002) reported on the involvement of the history of mathematics in the new generation of textbooks due to this public school reform. The results were that "the treatment of history of mathematics was problematic, and that textbook writers struggled to include history of mathematics in a meaningful way" (Smestad, 2007, p. 278). Another Norwegian study suggests that the history of mathematics despite being part of the national curriculum still does not play a significant role in the Norwegian classrooms (Alseth, Breiteg & Brekke, 2003). Due to this situation Smestad decided to look into the TIMSS (Trends in International Mathematics and Science Study) video study of 1999 to compare the Norwegian situation with that of seven other countries.[5] The TIMSS 1999 includes 638 lessons of mathematics teaching in 8th grade classrooms and the lessons where history of mathematics appears are labelled 'historical background'. Smestad found that only 21 out of the 638 lessons have a treatment of the history of mathematics in some way. The treatment of history in these amounts to 69 minutes, and removing two longer treatments apparently leaves us with 19 shorter treatments of a total duration of 18 minutes. Smestad found that about half of the examples of the treatments of history deal with the theorem of Pythagoras. The rest include, amongst others, Thales, Plato, Venn, Descartes, and Euler. Apart from one extreme example (a lesson long discussion of several historical proofs of the theorem of Pythagoras) these treatments are anecdotal, in the sense of giving only names and biographical information.

Smestad proposes a tool for analyzing the involvement of history in the classrooms, a tool providing "a division of knowledge into five categories: facts, skills/concepts, strategies, attitudes, and others" (Smestad, 2007, p. 280). The biographical treatment of the theorem of Pythagoras belongs, of course, to the category of facts. An example from the lessons developing the students' mathematical skills and grasp of concepts is that of Egyptian multiplication (multiplication by successive doubling). Examples of history of mathematics used for developing students' strategies for solving problems do not appear in the lessons. According to Smestad it seems that the far majority of the treatments of the history of mathematics in the classrooms concern the improvement of pupils' attitude towards mathematics. As an example of this Smestad says:

> One way of influencing students' attitudes towards mathematics, is to explain the role of mathematics in society. This can of course be done by focusing on the situation today, but it can also be done with reference to the history of mathematics.
>
> (Smestad, 2007, p. 281)

In the TIMSS material there are only two examples on this, one dealing with magic numbers and another with art. Regarding the fifth category, 'others', Smestad mentions that history of mathematics provides opportunities for crosscurricular work. However, there are no examples of this in the material. On the other hand, the material does include a few examples of history of mathematics serving as a means to increase the respect of other cultures, e.g. about Egypt's pyramids and the Babylonians.

Smestad concludes that his impression is that ”there are a few teachers who include history of mathematics as part of their teaching, but it seems that most teachers only make historical connections ’in passing’, if at all” (Smestad, 2007, p. 282). On top of this the history of mathematics is often lectured and hence student participation seems at a minimum in the treatment of the history. And in such lectures the history of mathematics usually limits itself to biographical information often with little anchoring in the mathematics taught.

## A classroom observation on the use of history of statistics

Tzanakis and Kourkoulos' paper begins with a short historical account for the development of statistics with a special emphasis on the concept of variance. Based on this account the authors draw three didactical implications: (i) First, the development of statistics has followed two complementary routes:

(1) The desire and need to manage, control and elaborate on data of various kinds, related to social and/or physical problems.

(2) The study of chance problems, in an effort to grasp the meaning of randomness and consequently, to conceive basic probabilistic notions. (Tzanakis & Kourkoulos, 2007, pp. 285–286)

Hence, the didactical implication is to base teaching on these two routes, that is to (1) ”collect, manage and elaborate on empirical data” and (2) ”to discuss and work theoretically on probabilistic problems and concepts and compare the results with experiments” (Tzanakis & Kourkoulos, 2007, p. 289). (ii) Second, since the development of statistics is deeply connected to physics and physical problems this connection should be explored in the mathematics classroom. The authors mention that physical models may be used to ”introduce, make plausible, or interpret statistical notions and relations” on one hand, and on the other statistics may also provide new insight knowledge on (already known) pieces of physics. An example of the first might be to interpret variance as (mean, kinetic or potential) energy, thus improving ”students' intuitive understanding, by linking variance to a physical concept, quite familiar to them” (Tzanakis & Kourkoulos, 2007, p. 293). (iii) Third, history of statistics is a nice example of how research in mathematics is based on both solving problems and posing new problems. Therefore statistics may be an area in which ”guided research work” is a suitable approach to teaching, i.e. that students are given some initial questions and problems and are invited to elaborate on these and perhaps formulate new ones, all under the guidance of a teacher. Such a setting may, according to the authors, also reveal more clearly the "possible analogies between students' conceptions and learning, and those of mathematicians in history" (Tzanakis & Kourkoulos, 2007, p. 289).

Based on this Tzanakis and Kourkoulos carried out a classroom experiment by introducing statistics in the manner described above to a class of prospective primary school teachers working in small groups of three to five. Tzanakis and Kourkoulos were able to identify several similarities between the historical development and students' guided research. For instance, the students were unwilling to accept variance as the 'natural' measure of dispersion over the more intuitive mean absolute deviation in much the same way as this was the case in history:

> Being more similar to mathematicians' research activity, students' guided research work probably made possible to observe rough similarities between the historical development and students' learning and difficulties. This provides some new input on the old, but still unsettled, issue of the parallelism between historical and ontogenetic development of mathematical knowledge [...].
>
> (Tzanakis & Kourkoulos, 2007, p. 294)

## Two studies on teachers' professional HPM development

The two Taiwanese empirical studies by W.-S. Horng and Y.-W. Su are part of a larger project on teacher's professional development in terms of HPM. The studies tell the personal story of two, different teachers; Horng tells the story of Yu and Su relates her presentation to the story of the teacher called $T_1$. Notice that the term 'HPM' in the papers by Horng and Su is used as a concept rather than just the name of the independent ICME study group, a concept describing the teachers' development due to the use of history.

Su looks at the HPM development of three high school mathematics teachers (plus herself) who have joined the research project. These teachers meet once a week for three hours to study and discuss the history of mathematics in both primary as well as secondary sources and to produce worksheets for later use in class. Su's empirical data for evaluating the possible HPM development of the teachers include these worksheets, reflection articles written by the teachers, classroom videos, interviews with the teachers, and audiotape accounts of study group meetings. Su is able to identify three different phases of professional development of the teachers:

> In Phase One, they made it clear that the purpose of HPM is to teach mathematics rather than teach the history of mathematics. Nevertheless, they came to realize what the history of mathematics is about. When entering the second phase, they understood that the logical aspect of mathematical knowledge, the historical aspect of mathematical knowledge and the aspect of student's cognition can be interconnected. [...] In the last phase, the colleagues find that they all have enhanced their professional expertise in terms of the HPM. (Su, 2007, p. 370)

Su describes the three phases be telling the story of $T_1$ who has been a teacher for eighteen years but has no prior experiences with the history of mathematics. In phase one $T_1$ hopes that history of mathematics may be a way to reach the students who are not particularly fond of mathematics. The initial idea of $T_1$, however, is that history may only work as a sort of 'decoration' to make the framework of mathematics richer. In the second phase $T_1$ is able to use the prepared worksheets in a more constructive manner, namely by using these as a point of departure for further use of history in class. Su concludes that $T_1$ in phase two "has realized that students themselves are the producers of knowledge, not only just a consumer of knowledge given by teachers" and that $T_1$ gradually has "changed from the idea of making students interested to the connection with the recognition of students" (Su, 2007, p. 377). Also, $T_1$ becomes critical towards the textbook material and starts to produce historical supplements. In phase three the teacher begins to apply the two other teachers' worksheets and is able to analyze, criticize, and discuss these, as well as his own, with the other teachers. Due to the positive results of the research project Su concludes that the "HPM approach can help the participant's professional development in an efficient way" and that it can be another way to do the in-service training of teachers (Su, 2007, p. 379).

Horng[6], as opposed to Su, describes a teacher, Yu, who already as a master's student had an interest in history of mathematics. Yu began as a teacher in 1998 and was then part of an earlier research project of Horng's on "Ancient Mathematical Texts introduced into Classroom" lasting from 1998–2001. She then joined the new project on "Mathematics Teacher's Profession Development and the HPM" in 2002. As part of the research project Yu had to become acquainted with different topics of the history of mathematics and to prepare worksheets related to these. Yu did the worksheets "On Circle" (1999), "On Pascal's Triangle" (2000), "On Ptolemy" (2001), "On Conic Sections" (2003), and gave a talk to her colleagues of the high school called "On Geometric Aspects of $\sqrt{2}$" (2003) (Horng, 2007, p. 3). The worksheets, together with a series of interviews, also serve as Horng's empirical data of Yu's HPM development. According to Horng Yu's professional HPM development seems to have occurred in two phases (or periods); one stretching from 1998 to 2002 and another from 2002 to 2004. Horng says:

> For Period One, Yu's way of integrating history of mathematics into teaching is to provide her students cultural aspects of the topic in order to motivate studying of mathematics. For example, in developing her worksheet, 'On Circle', Yu's idea is to explore the cultural and humanistic aspects of mathematical knowledge in history, which are related to the concepts of the circle. [...] In this period, Yu's approach in designing her worksheets apparently is teaching-oriented. That is, she had in mind an ideal of what should be given to her students no matter what their learning conditions are. She was of course very enthusiastic about the history of mathematics and its relevance to teaching. Yet, at this stage she did not seem to care about students' reaction largely because she was tempted to teach history of mathematics rather than mathematics per se. (Horng, 2007, p. 5)

In the second phase (period two) Yu becomes more critical to the textbook's contents and presentation. Also Yu comes to regard the approach from period one as 'too superficial' and she switches her approach from a 'teaching-oriented' to a 'learning-oriented' approach for integrating history of mathematics. Horng[7] concludes the following about Yu:

> [She] recognizes that although HPM always has a role to play in teaching mathematics, teachers should regard student's learning as first priority. After all, the commitment of HPM is to help teaching mathematics efficiently. (Horng, 2007, p. 7)

## Two purposes of involving history: discussing the papers

Saying that the commitment of HPM, as Horng does, is to help teaching mathematics more efficiently is, I believe, not the entire truth. As can be seen from the article by Fasanelli and Fauvel (2007, p. xi) the HPM Study Group of 1976 has eight "principal aims" and only one of these, number 4, speaks of relating "the teaching of mathematics and the history of mathematics teaching to the development of mathematics in ways which assists the improvement of instruction and the development of curricula." The remaining ones speak, amongst others, about promoting international contacts and awareness of relevance, producing materials, facilitating access to materials, deepening the understanding of mathematics' evolution, and as number 8, promoting "awareness of the history of mathematics as a significant part of development of culture" (Fasanelli & Fauvel, 2007, p. xi).

## Categorizing the use of history as a tool or as a goal

Out of the eight principal aims only two, number 4 and 8, are, or can be seen as, actual arguments for involving history in mathematics education. And they are in fact examples of the two different types of arguments or purposes, which you in general may argue that the use of history of mathematics can serve: (1) as a *tool* in the sense of assisting the actual learning of mathematics (mathematical concepts, theories and so forth) and (2) as a *goal* by, for instance, bringing about a dimension of 'meta-mathematics' in mathematics education (Jankvist, 2007, pp. 72–76). By meta-mathematics I am thinking of posing and suggesting answers to questions about the 'meta-issues' of mathematics, e.g. how mathematics evolves over time, what forces and mechanisms cause the evolution of mathematics, how mathematics interacts with society and culture, whether or not mathematics can become obsolete, and so forth (Niss, 2001, p. 10). So where the second purpose of using history in mathematics education is to teach the students something about the 'meta-issues' of mathematics – perhaps you can even say that it is a matter of *general education* – the first purpose is concerned with teaching the students something about the inner issues, or 'in-issues', of mathematics.

Any categorization may, of course, be productive in some respects and unproductive in others. For instance, when designing teaching material, including student worksheets, a designer and/or teacher may be interested in using the history of mathematics as both a tool to present in-issues and as a goal in terms of bringing about related meta-issues. So as a guide for designing teaching material the tool-goal categorization may not be the most productive one. In the ICMI Study (Fauvel & van Maanen, 2000) Tzanakis and Arcavi (2000, pp. 203–207) provide, in this respect, probably a more productive classification of arguments/purposes of using history in mathematics education. Based on an extensive literature review they discuss seventeen different arguments for involving history and classify these within five main areas. However, it is my claim that any one of these seventeen arguments may be placed in either the category of tool or the category of goal. Some arguments, though, may be interpreted in different ways. This may result in one interpretation of a given argument being placed into the category of tool while another interpretation of the same argument will be placed into the category of goal. For instance, one argument says: "Students may learn that mistakes, heuristic arguments, uncertainties, doubts, intuitive arguments, blind alleys, and alternative approaches to problems are not only legitimate but also an integral part of mathematics in the making" (Tzanakis & Arcavi, 2000, p. 205 (b1)). Now if the point of this argument is to illustrate to the students something about mathematics in the making then it belongs to the category of goal. But if the point is to show, for instance, the students that they shouldn't be afraid to make mistakes in the process of learning since also great mathematicians in history have been known to do that, then the argument goes into the category of tool. Another argument which is a bit tricky in categorizing is the one saying, that history may enrich the didactical repertoire and serve as a resource for the teacher (Tzanakis & Arcavi, 2000, p. 204 (a2), 206 (c4)).[8] Here we need to ask the question: What is the purpose of enriching the teacher's repertoire and for what reason does the teacher need this resource? Does the teacher need it to enhance the students' learning of mathematical in-issues? Then the argument must be placed in the category of tool. Or does the teacher need it to discuss certain meta-issues of mathematics in class? Then the argument goes into the category of goal. Let me provide one last example of an argument, one that is not mentioned by Tzanakis and Arcavi (2000), and which I have been told a couple of times does not fit the categories of tool and goal. The argument may be stated as: "If you teach mathematics, you must also teach history of mathematics, for the history of the subject is part of the subject" (Heiede, 1992, p. 152). Now, even though this may be an argument, it is not an argument about the use of history of mathematics. It does not say what involving history in mathematics education is good for. In this respect the argument is inane and, hence, it doesn't make sense trying to categorize it into being either tool or goal. However, if the argument is supplied with a remark about 'history giving a more authentic picture of what mathematics is' then the involvement of history has a purpose, one regarding meta-issues, and may be placed in the category of goal.

In the remaining parts of this section I shall use the categorization of the two purposes of tool and goal as a basis for the discussion of the four selected papers, as well as an analyzing tool in general.[9] And for this purpose I find the categorization to be quite useful.

## Discussion of Tzanakis and Kourkoulos' paper

Tzanakis and Kourkoulos (2007) deal (almost) entirely with the first purpose, i.e. using history as a tool to enhance learning of the in-issues. In fact, it seems that an underlying motivation for the study is to discuss similarities between the historical evolution of mathematics and the way in which students learn. In the abstract the authors say that the paper comments on "similarities of question, objections and difficulties that appeared historically and aspects of which seems to have reappeared in the classroom" (Tzanakis & Kourkoulos, 2007, p. 284).

The argument that history, due to similarities in evolution and in learning, should serve as the guide for teaching mathematics seems to be a somewhat frequently debated one. F. Furinghetti and L. Radford, for instance, have devoted an entire chapter to this argument in the soon to be published *Second handbook of international research in mathematics education*.[10] The argument is derived from the statement that "ontogenesis recapitulates phylogenesis", which is attributed to Haeckel (1906, pp. 2–3) who applied it to biology where it was known as the biogenetical law. The statement, however, found its way into psychology and became known as the psychogenetical law, saying that the mind of the individual has to go through the same development as the mind of mankind. And for about a century now, this angle of attack has been present in the field of using history in mathematics education, either in its hardcore version or in more soft-core ones, saying that history merely should serve as a guide to the teaching and learning of mathematics.[11] An example of this is the genetic method of Toeplitz (1927). Another example of the soft-core version is Freudenthal's guided reinvention of mathematics (Freudenthal, 1991). Now, whether or not the approach of "guided research work" as used by Tzanakis and Kourkoulos is a variation of Freudenthal's guided reinvention is not to say based on the paper alone. However, there seems to be quite a few similarities and if, in fact, the applied approach is the same as that of Freudenhtal it makes you wonder why it doesn't bear the same name.

Whether or not the psychogenetical law holds in the case of mathematics is not to tell. But Tzanakis and Kourkoulos' paper indicates that to some extent it may be the case of the history and learning of the concept of variance. And as a matter of fact, the research by Tzanakis and Kourkoulos is not the only empirical study claiming some truth of the argument. Also Harper (1987) describes an empirical study which indicates a possible parallelism between evolution of algebra and school children's conceptual development. Sfard (1995) used the study of Harper, and her own empirical data, together with her earlier developed theoretical framework for investigating mathematical conceptions, i.e. the idea of mathematical objects and processes (Sfard, 1991), to show that:

> ... the reification [turning processes into objects] that is needed for a deep understanding of a concept (say complex number) cannot

> be expected before some familiarity with secondary processes (e.g. operations on complex numbers) has been attained.
>
> (Sfard, 1995, p. 35)

This lead to the somewhat controversial hypothesis that: ”Sometimes the teacher and the students must put up with the necessity of practicing techniques even before they are fully understood” (Sfard, 1995, p. 35), a statement in the wake of which followed an intense debate in JMB.

## Discussion of Smestad's paper

Smestad (2007), with his tool of five categories, finds that the far majority of the uses of history in the TIMSS concern improvements of the students' attitudes towards (or beliefs of) mathematics. The results of Smestad suggest that history is rarely used as a tool to enhance the actual learning of mathematics (excluding the use of it as a motivating factor to promote learning). Far more often history serves the purpose of illuminating meta-issues of mathematics in order to change students' beliefs of what mathematics is, for instance, in terms of mathematics' historical origins and evolution, the role it plays in society, etc. But, as Smestad also points out, the treatment of meta-issues are often done ”in passing”, and this may suggest that the meta-issues are not necessarily always sturdily anchored in the related in-issues. Another couple of interesting remarks which Smestad (2007, p. 281) makes are that the role of mathematics in the development of technology is not touched upon in the TIMSS video study and neither are the motivations of the passed mathematicians. This is a shame, I think, since both of these angles of approach to the history of mathematics may especially challenge students' beliefs of mathematics, its evolution and place in society.

Smestad's five categories mainly deal with the *why* of using history in mathematics education, however no argument is given for choosing exactly these five. In the light of this it does seem peculiar that Smestad does not at all relate his five categories to the ones given by Tzanakis and Arcavi (2000, p. 203), especially since he refers to the book in his conclusion. Had this been done, Smestad might have been able to discover another criticism which his five categories may also be subject to, namely that they seem to play down the idea of history of mathematics as a form of general education. Tzanakis and Arcavi deal with this issue of using history in mathematics education, without calling it general education though, in a much more elaborated fashion that just referring to it as 'attitudes' and 'others'.

Smestad's study, interesting as it is, seems to leave some questions unanswered. For instance, why are there so few features involving history of mathematics in the TIMSS study? And why do they appear so randomized in the material? Even though Smestad (2007, p. 279) mentions that he due to the relatively few TIMSS samples with "historical background" refrains from discussing particular countries and instead views the material as a whole, one wonders how many of the answers could have been anticipated with a different approach. For instance, how many of the countries in TIMSS have history as part of their mathematics curriculum? And if some of them do, in what manner must history then be included? What purpose does it serve? Could it have been anticipated that strategies for solving mathematical problems would not be touched upon in connection with history?

## Discussion of Su's paper

Su in her paper, like Smestad in his, discusses elements of both purposes (tool and goal) for involving history in mathematics education. For instance, she says:

> By adopting the approach of HPM, now the colleagues [the three teachers] can use the history sources to help students' mathematics learning both in cognitive and cultural aspects. (Su, 2007, p. 370)

Both purposes are also present in the story of $T_1$. As explained earlier, in phase one $T_1$ believes that history may serve as a way of motivating the more inactive students in class by sort of "decorating" the curriculum-tied in-issues of the mathematics teaching. But in phase two $T_1$, in one of Su's quotes, talks about history of mathematics more in terms of general education, when he says that in using history "the passing down of math culture could be introduced to students for their enlightenment" (Su, 2007, p. 376). Su comments herself, actually, on the argument of motivation for using history which $T_1$ talks about in phase one. Su says:

> In addition to confirming that the using of history of mathematics is like lubricant which makes the material more active and attractive to students, history of mathematics also provides more literal and social aspects of the knowledge, thereby shortening the distance between students and math. Math is not always boring anymore.
>
> (Su, 2007, p. 374)

Now, it is not an empirically proven fact that history always motivates students. In fact, this is very likely to differ from student to student which is why this argument possibly may only be applied on a subjective and personal level. This is exactly the weakness of the argument; that it assumes an historical approach to really be of interest to a given student. According to Schubring (1988) it then becomes a prerequisite condition that historical questions, do in fact, have value within our culture. And a question might be whether or not this is the case for today's students in the same way it was for students in earlier times (Schubring, 1988, p. 138; Mosvold, 2002, p. 6). Generally speaking, using history with the purpose of motivating students is problematic both in terms of justification, as the discussion above illustrates, but also in terms of documentation. Su's empirical data mainly concerns the teachers, not the students, and a conclusion as the one she draws in the quote above therefore seems much too hasty.

It should be mentioned that Su, in passing, proposes "six manners" in which the teachers use history of mathematics in the classroom: "isolation, addition, introduction, execution, integration, and decision-making" (Su, 2007, p. 370). However, Su provides no explanation of any of these six manners, nor does she seem to apply them in her study. Now, this is a shame since her manners seems to complement Smestad's five categories on why to involve history, since Su's six manners seems to deal with the *how* of doing this. Of course, both the why and how of using history is discussed by Tzanakis and Arcavi (2000).[12] But where Tzanakis and Arcavi discuss the issue of implementing history into classrooms more in general, i.e. at all levels of education and in textbooks, student projects, etc., the interesting thing in the categorizations by Su and Smestad is that they deal with actual implementations and the way in which teachers handle the involvement of history here.

A positive aspect, in my view, of the study of Su in terms of the two purposes is that the majority of the time 'history as a tool' and 'history as a goal' seems to co-exist on something like equal terms – something which is not the case in study by Horng.

## Discussion of Horng's paper

Horng (2007) describes Yu's initial idea of involving history much like Su described $T_1$'s initial idea, namely as thinking of history as having only the purpose of motivating the students. For this reason, according to Horng, Yu takes a 'teaching-oriented' approach towards the involvement of history in class. Horng states that Yu "brought HPM into her classroom as an 'added-on' material" and further points out that it wasn't surprising that "she would ignore the cognitive meaning the HPM can fruitfully provoke" (Horng, 2007, p. 7). In the second period of Yu's professional HPM development she changes her approach of involving history to a 'learning-oriented' approach where she, again much like Su's teacher $T_1$, comes to realize that the point of departure must be the students and thus tries to make a connection of "historical materials with students' prior knowledge and current topical knowledge" (Horng, 2007, p. 7). According to Horng it seems that Yu in the second phase now is able to transform her role from a teacher to a historian and vice versa (Horng, 2007, p. 8). Even though the teachers Yu and $T_1$ go through similar changes from phase one to phase two, now being more aware of the students' topical knowledge, the outcome of these changes in terms of HPM seems to differ a bit. Where $T_1$ seems quite aware of both purposes (tool and goal) of involving history Yu's changes amounts to a focussing on 'history as a tool' only.

The above actually touches on a debate regarding what should be the base for students learning of mathematics: the historical development of mathematics or the students themselves. According to Schubring (1978, pp. 189–197) people like Walther Lietzmann, Wolfgang Klafki and Hans Freudenthal have criticized the first approach, i.e. the historical-genetic principle. They were of the opinion that history should not make up the base for the education, but the students themselves should be the starting point. Both the pedagogic and the didactic perspective should be the students' own experiences from the environment surrounding them every day (Mosvold, 2001, p. 32). One way of fulfilling this might be to discuss the role of mathematics in the development of technology, as Smestad mentions. Technology today is very much part of students' every day life, take for instance mobile phones, MP3 players, and laptops – all the mathematics used in these technological devices have a history.

Regarding the two purposes of involving history (tool and goal) Horng seems to see the purpose of 'history as a tool' as the main one (confer the quote in the end of the previous section). And in the story of Yu 'history as a goal' merely seems to serve as a road towards 'the true commitment of HPM', i.e. to use history to teach mathematics more efficiently. So if Smestad could be said to play down the role of 'history as a goal' in the sense of general education, Horng can be said to do so tenfold since he plays down the entire purpose of 'history as a goal'.

## The case of Danish upper secondary school

In contrast to this strong focus on 'history as a tool' in the papers let me present a Nordic case, and another one than Smestad's Norwegian, namely the Danish case of upper secondary school (the one I am most familiar with). In the Danish upper secondary mathematics programme the students now are to "demonstrate knowledge about the evolution of mathematics and its interaction with the historical, the scientific and the cultural evolution" (Undervisningsministeriet, 2006, p. 34, my translation from Danish). The official regulations for the Danish upper secondary mathematics programme of 2006 [13] are to some extent based on the Danish report on *Competencies and learning of mathematics* (title translated from Danish) where it says:

> In the teaching of mathematics at the upper secondary level the students must acquire knowledge about the historical evolution within selected areas of the mathematics which is part of the level in question. The central forces in the historical evolution must be discussed including the influence from different areas of application. Through this the students must develop a knowledge and an understanding of mathematics as being created by human beings and, in fact, having undergone an historical evolution – and not just being something which has always been or suddenly arisen out of thin air. [translation from Danish by the author]
>
> (Niss & Højgaard Jensen, 2002, p. 268)

As a way of manifesting these aspects of (meta-)mathematics Niss and Højgaard Jensen (2002, p. 268) mention, amongst others, prime numbers and the related number theory as a way to illuminate "how pure mathematics all of a sudden becomes applied mathematics and why it is sensible to invest in basic research".

Now, this clearly concerns 'history as a goal' rather than 'history as a tool'. And further more it concerns 'history as a goal' in terms of general education. As a matter of fact such an approach to the involvement of history of mathematics has been part of the Danish upper secondary mathematics programme since 1987. However, it still does not seem to be quite clear how to fulfil the goals presented by Niss and Højgaard Jensen (2002). How would one go about testing whether or not the students actually acquire this kind of knowledge and understanding? And in what manner they are able to acquire it at all. For instance, in what way are their knowledge and understanding of the meta-issues of mathematics anchored in the related in-issues of mathematics? Will they, from a mathematics point of view, be able to truly understand how and why prime numbers found their way into applications in cryptology and why this application didn't come about until the 1970s when the study of prime numbers goes back all the way to Euclid? Or will they just be able to repeat cliché-like statements about the use of primes and the necessity for basic research? Questions like these, it seems to me, are still open regarding the case of 'history as a goal' in the Danish upper secondary mathematics programme. And they are questions which, if they are to be answered, call for empirical studies.

## Conclusive remarks and further perspectives

In conclusion: The proceedings HPM2004 & ESU4 are rich in ideas and thoughts on why and how to integrate the history of mathematics into the teaching and learning of mathematics, what pieces of the history of mathematics to integrate and when to do this. The proceedings also contain a few empirical studies on the use of history in mathematics education, some of them on evaluation of the effectiveness and some not. In the papers discussed above we have heard about empirical data supporting the use of 'history as a tool', but not as much about the students' profit of the 'history as a goal' approach. This seems to be somewhat typical of the empirical studies within the field. In the light of the results of Smestad (2007) it is, I think, quite paradoxical that the majority of the actual involvement of history in classrooms concerns 'history as a goal', but yet the few empirical studies which exist within the field of using history in mathematics education almost all deal with 'history as a tool'. Conferring the case of the Danish upper secondary mathematics programme empirical studies on the use of 'history as a goal' are highly relevant as well.

There are no doubt that empirical research studies on the use of history in mathematics education are important, whether they concern history as a goal or as a tool, since they may tell us how positive experiences can be transferred from one teacher, class, or school to another. The studies of Tzanakis and Kourkoulos, Su, and Horng actually provide some insight into this problem area. Unfortunately, such studies are rare in the proceedings HPM2004 & ESU4 and to a large extent the critique by Gulikers and Blom (1991), as quoted earlier, still apply to the literature in the field, that is if the proceedings can be taken as a representative for this literature. So regarding a real movement in the community towards more empirical studies I'm sad to say that the proceedings do not seem to reveal such one being on the way. In an interview with one of the board members of HPM, Abraham Arcavi, which I conducted at the NoGSME [14] summer school in Iceland in June 2007, I asked about this matter as well as what to expect in the future from HPM (and ESU). Arcavi revealed:[15]

> The community of HPM has been successful in at least two fronts: it called the attention to the potential of history of mathematics in mathematics education and it also provided a lively 'home' to learn from each other for all the professions (teachers, mathematics educators, mathematicians and historians) who work with history.

> However, HPM still needs much more empirical research on teaching and learning related to history than it is the case now, and there is no lack of research questions to pursue. This avenue is important in order to strengthen HPM both internally and externally. Internally, research, as I envision it, would provide insights which confirm, extend or challenge some of our assumptions and proposals, it may reveal directions not yet pursued and it would certainly sharpen our own views and future plans. Externally, research can be a way to reach out and communicate with other communities within mathematics education like PME, CERME, and others and would open opportunities for its themes to appear more in journals like ESM, JRME, JMB and many others. Pursuing such opening of the current 'borders' will give history a wider stage and will be instrumental in attracting more people. Probably, HPM should aim at working in a similar way than other 'thematic' communities already do (such as technology in mathematics education, modelling, and the like) – they nurture inner meetings and discussions, but at the same time they pursue a strong presence in general conferences (plenaries, working sessions, discussion groups) and publish in general journals. In my opinion, research is the main way to pursue a wider and visible presence which would make HPM stronger and ever growing.
>
> (Arcavi, 2007)

An emphasis on empirical studies would of course not mean that research based on other research paradigms, for instance logical argumentation, would not become unimportant. Having people come up with ideas and thinking about the use of history in mathematics education is still highly relevant. However, taking into consideration the vast amount of papers dealing with the non- or pre-empirical aspects of using history I think, in accordance with Arcavi, that a shift in emphasis would be in order.

This summer I attended the fifth ESU in Prague from July 19$^{th}$ to 24$^{th}$. A quick look in the programme and abstracts from this summer university does not indicate a true movement towards an increase in empirical studies either. In July 2008 HPM will be held, as usually as a satellite meeting to ICME, in Mexico City. Now, what may we expect from this conference regarding further empirical studies? I don't know. Hopefully there will be an increase. But the way things look now the field of using history in mathematics education has a long journey ahead of it before it will become a research area in a similar sense as PME, the sister permanent study group of HPM, is today. And one might fear that if the community is not willing to go on this journey its contributions to the teaching and learning of mathematics will not themselves go into history.

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## *Notes*

1 Notice, that the recent issue of ESM (Volume 66, Number 2, October, 2007) also deals with the use of history in mathematics education.

2 The five articles which Siu mentions are: (Fraser & Koop, 1978), (Gulikers & Blom, 1991), (Lit, Siu & Wong, 2001), (McBride & Rollins, 1977) and (Philippou & Christou, 1998).

3 In the proceedings the history of HPM is accounted for in an introductory paper by F. Fasanelli and late J. G. Fauvel (Fasanelli & Fauvel, 2007). The history of ESU is presented by E. Barbin, N. Stehlikova and C. Tzanakis (2007).

4 HPM Newsletter 65, July 2007, pages 15–17. http://www.clab.edc.uoc.gr/HPM/ NewsLetters.htm (located on 12th of June 2007).

5 Australia, Czech Republic, Hong Kong SAR, Japan, The Netherlands, Switzerland, and United States.

6 For some reason Horng's correct, i.e. revised, paper does not appear in the proceedings, the version there is an older one. The review here, however, is based on the correct and new version. This version may be obtained from the editors.

7 Both Horng and Su use their empirical studies to produce visual models for teachers' professional HPM development. These studies will, however, not be discussed in this paper, only the results that led to them.

8 Other papers discussing teachers' possible benefits from knowing and learning about the history of mathematics are, for instance, (Freudenthal, 1981), (Burn, 1998) and (Heiede, 1996).

9 Another example of this categorization applied as an analyzing tool, in this case to upper secondary textbooks, may be found in the soon to be published (Jankvist, in press).

10 The title of the chapter is *Contrasts and oblique connections between historical conceptual developments and classroom learning in mathematics*.

11 In the proceedings HPM2004 & ESU4 Schubring has a paper discussing the issue of ontogenesis and phylogenesis as well, a paper in which he also relates the discussion to Piaget's use of psychogenesis (Schubring, 2007).

12 In the proceedings the categories of Tzanakis and Arcavi are developed further for the purpose of curriculum study by Tang (2007).

13 First draft of the reform was published in 2004.

14 NoGSME stands for Nordic Graduate School of Mathematics Education.

15 The quote here is used and printed with the approval of Abraham Arcavi.

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