Mathematically productive discourses among student teachers

Raymond Bjuland

This article reports research that focuses on the characteristics of mathematically productive discourses (MPD) while student teachers are working collaboratively on a geometry problem in a problem-solving context. Analyses from the discourses of two groups of students are presented in order to illustrate mathematically non-productive and productive discourses respectively. A definition of MPD is presented and used as an analytical tool to identify critical characteristics of sequences of productive discourses. This definition involves the following five criteria: 1) Student utterances, stimulating a monitoring utterance, 2) the monitoring utterance, 3) student responses, stimulating a second monitoring utterance, 4) the second monitoring utterance, 5) further elaborations, advancing the mathematical discussion among the students. The article also discusses the difficulty of concluding when a discourse is productive or not, especially when students are challenged to work on complex problems in which a solution is not usually reached within a school lesson.

As a teacher educator working with student teachers for about ten years, I have been concerned with the teaching and learning of mathematics through problem solving in collaborative small groups. Throughout these years, I have seen the importance of identifying student teachers' heuristic strategies used in group discourses while solving mathematical problems (Bjuland, 2002). This paper aims at identifying more critical characteristics of mathematically productive discourses (MPD). The monitoring questions identified in Bjuland (2007) are one promising attempt at finding indicators of such discourses. However, there is a

Raymond Bjuland
Agder University College

Bjuland, R. (2007). Mathematically productive discourses among student teachers.
Nordic Studies in Mathematics Education, 12 (2), 33–55.

need for a careful discussion of how MPD can be defined and identified in group discussions.

Current research has revealed that students have difficulties in communicating when solving mathematical problems in groups (Kieran, 2001; Ryve, 2004, 2006; Sfard & Kieran, 2001). Research in the didactics of mathematics has therefore recently focused on how to identify mathematically productive discourses among students (Kieran, 2001; Ryve, 2004, 2006). As a continuation of this research, my aim is to extend the field of discourse analysis in mathematics by identifying MPD among student teachers working on problem solving in geometry. Based on findings from students' reflections on their experience as learners of mathematics (Bjuland, 2004), analyses of discourses have revealed that it is useful for future teachers to engage in collaborative reasoning processes to obtain better understanding of the whole process of doing mathematics. Reflections on their own learning processes have also triggered reflections on their preparation for the teaching profession.

This paper addresses the following research question, *what are the characteristics of mathematically productive discourses (MPD) when student teachers are working on a geometry problem in collaborative small groups*? When elaborating on this question, I firstly identify whether the student teachers produce monitoring utterances in their group discussion or not. One sequence of discourse from one of the groups is chosen to indicate a non-productive mathematical discourse. Two sequences of discourses (one episode) are chosen in another group to illustrate MPD.

Theoretical framework

According to Blum and Niss (1991, p. 37), a mathematical problem is "a situation which carries with it certain open questions that challenge somebody intellectually who is not in immediate possession of direct methods/procedures/algorithms etc. sufficient to answer the questions". Schoenfeld (1993) introduces a similar definition. He makes it clear that it depends on the students' prior knowledge whether a task is a real problem, and most exercises in textbooks are not real problems since they can often be solved by means of a well-known method.

Drawing on Polya's (1945/1957) four-stage model of a problem-solving process, Borgersen (1994) has expanded this process into seven main stages that are not linear, but dynamic and cyclic. Geometry problems in my study were designed in accordance with Blum and Niss' definition of a mathematical problem and Borgersen's seven stages. An indication of mathematical progress in a solution process could then be related to this seven-step problem-solving model.

Mathematical- and collaborative monitoring in problem solving

The identification of monitoring utterances is a necessary step in fulfilling my aim to characterise MPD in group discussions. *Monitoring utterances* are defined as questions or statements that involve the two following important functions: *mathematical-* and *collaborative monitoring.*

Mathematical monitoring comprises the students' mathematical focus, and their heuristic strategies produced in the group discussion that stimulate mathematical progress in the solution process. The focus is on a specific mathematical content, involving mathematical concepts, or the attempts at modifying or rejecting a conjecture and so on. The heuristic strategies are defined as strategies produced in the group discussion, for instance the use of if-then structures in order to build up a logical cause-effect argument, questioning, monitoring, going to an extreme location, or introducing counter examples. The monitoring strategies involve the students' attempts at monitoring their solution process, and their reflection on a solution. These strategies are linked to Schoenfeld's (1992) third component of his framework (*monitoring and control*) and Polya's looking-back step (1945/1957).

Collaborative learning is characterised by unstructured processes in which participants negotiate goals, pose problems, develop procedures, and produce socially constructed knowledge in small groups (Springer, Stanne & Donovan, 1999). In the present study, the term *group collaboration* has been used to denote the dynamics of groups of four-persons or five persons, bringing the perspective of every single student into the mathematical discussion. A crucial feature of group collaboration can be defined as mutuality, a reciprocal process of exploring and challenging generated ideas in the students' discourse in order to construct a shared understanding of the problem (Damon & Phelps, 1989; Goos, Galbraith & Renshaw, 2002).

From a methodological perspective, I have adopted a framework, particularly used for mathematical discourses (Goos et al., 2002) in order to capture the interactive nature of the student teachers' collaborative monitoring. These authors have produced an operational definition of collaboration. *Collaborative monitoring* involves the categories *self-disclosure*, *other-monitoring* and *feedback requests*. This is exemplified by considering a pair of students working on a problem. The utterances from one of the students may become the subject of discussion in four ways (Goos et al., op. cit., p. 220):

1 Spontaneously, and initiated by the student (*self-disclosure*: here is my idea).

2 Spontaneously, and initiated by the partner (*other-monitoring*: here is what I think of your idea).

3 Through an invitation issued by the student (*feedback request*: what do you think of my idea?).

4 Through a partner's challenge (*other-monitoring*: what do you mean?).

In the analysis of the student discourses, I have distinguished between points 2 and 4 in the following way:

2 Other-monitoring, following up.

4 Other-monitoring, requesting clarification.

Mathematically productive discourses

Before focusing on the term MPD, there is a need to define some key concepts. The term *communication* is defined as "the use and production of means intended to make an interlocutor act or feel in a certain way" (Sfard & Kieran, 2001, p. 47). *Discourse* is defined as "a stretch of concrete, situated and connected verbal, esp. spoken actions" (Linell, 1998, p. 6). A *mathematical discussion* can be conceived of "as a polyphony of articulated voices on a mathematical object that is one of the motives for the teaching-learning activity" (Bartolini Bussi, 1998, p. 68). Bussi uses the term *voice* in the sense of Bakhtin, to mean "a form of speaking and thinking that represents the perspective of an individual (e.g., his or her conceptual horizon, his or her intention, and his or her view of the world) as a member of a particular social category" (p. 68). This definition is in accordance with the present study, indicating that the discussion is focused on a mathematical object in which there are real student contributions and interaction.

A prerequisite for a mathematical discourse to be productive is the effectiveness of the communication among partners (Sfard & Kieran, 2001). These authors regard communication to be effective if "the different utterances of the interlocutors evoke responses that are in tune with the speakers' meta-discursive expectations" (p. 49). This means that all the participants are aware of what they are talking about and they "refer to the same things when using the same word" (Sfard, 2001, p. 34). There must be a correlation between the students' intended foci. Following Ryve (2004), I present the following definition of productivity:

> The term productivity, in turn, refers to discourse which can be proved to have had some concrete lasting effect: the discourse has led to the solution of a problem, it influenced participants' thinking and ways of communication, it changed their mutual positioning, it became richer in rules and concepts. In the case of mathematics discourse, an interaction will be regarded as educationally productive if it is likely to have durable and desirable impact on students' future participation in this kind of discourse.
>
> (Sfard & Kieran, 2001, p. 50)

From this definition, it is not easy to examine whether mathematical discourses among the student teachers are productive or not. The students are working on two geometry problems during four group meetings, indicating that the problem-solving process is rather complex and long lasting. It is therefore difficult to measure productivity in the students' discourses based on some chosen sequences of discourses from two groups of students respectively. With these limitations, the analysis will therefore focus on some local mathematical discourses, revealing indicators of mathematical productivity. Based on the definition introduced above (Sfard & Kieran, 2001), MPD is defined as a discourse that advances the mathematical discussion among the students. More specifically and derived from the data, productivity in a *sequence of discourse* is defined in the following schematic way:

1 Student utterances, stimulating a monitoring utterance.

2 The monitoring utterance, bringing the *collaborative- and mathematical monitoring* into the discourse.

3 Student responses, elaborating on the monitoring utterance, also stimulating a second one.

4 The second monitoring utterance, indicating a more focused direction.

5 Student utterances, elaborating on the second monitoring utterance, advancing the mathematical discussion.

The unit of analysis is the exchange of utterances between the double monitoring utterances (points 2–4). However, point 1 is important for identifying student utterances that stimulate the first monitoring utterance. This utterance could be stimulated by some prior questions or statement in the discourse, but the monitoring utterance could also be spontaneously initiated by one of the students. Point 5 makes it clear that students have to elaborate on the second monitoring utterance if discourses could be characterised as mathematically productive.

Method

In this case study a detailed analysis of the students' mathematical discussion has been conducted in an authentic classroom setting.

Data collection

The empirical material was collected at a teacher-training college in Norway. 105 students attended the first year of the four-year teacher education programme in order to become teachers in primary (elementary) school or in lower secondary school. My research project was carried out on a problem-solving course in geometry in the students' first semester at the college. This course consisted of two parts: the first part, teaching over a month in September, including group work assisted by the teacher, and the second part, collaborative small-group work (21 groups) without teacher intervention over three weeks in October. The data collection was documented over four meetings throughout this period in October. The corpus of data consists of fieldnotes from the observation of three small groups of student teachers (randomly chosen) and utterances registered on an audiotape (8 lessons in each group) when they worked on two problems of classical geometry. The three groups for observation were called A, B and C respectively. The empirical material also consists of the students' group reports from this collaborative small-group work.

I had two different roles during the research project: being a teacher in the first part of the problem-solving course and a researcher in the second one. In fact there were two of us (a colleague and myself) who carried out the teaching programme in the first part. After having finished the collaborative small-group work of the second part, the geometry problems were discussed and elaborated in some plenary lessons.

Procedure

The first part in September consisted of three didactical activities: lecturing to the whole class (14 lectures, 105 students), class teaching (3 lessons in each class, 35 students), and small-group collaboration (11 lessons). The aim was to focus on basic classical geometry, prepare students for working on problems in small groups by focusing on monitoring training and collaborative learning, and stimulate students to experience mathematics as a process, as described by Borgersen (1994).

In the lectures to the whole class (105 students) during the first part, the students were introduced to how collaborative learning can be used in mathematics in order to stimulate social scaffolding (Johnson & Johnson, 1990). The collaborative and monitoring activity was linked to the students' writing process: they were supposed to write about how the problem-solving process developed, how obstacles and openings emerged in the discussion, and how every idea and suggestion was introduced or presented. After each small-group session in the first part, we used the students' writings and their reflections on the group work as a starting point for a discussion in the consecutive lectures. All these monitoring activities were designed in order to develop monitoring training among the students.

The small-group work of the second part in October finished with a group report. This report was to consist mainly of three different elements: the solution to the problems, the process writing in which the students were to write down their ideas and strategies throughout the problem-solving process, and a reflection part in which the students were to reflect on their group work and their own learning processes.

Subjects

Here I focus on two groups. Group A comprises one male student, Roy, and four female students Unn, Mia, Gry and Liv (ages: 24, 20, 21, 21, and 22 respectively). The names are pseudonyms. The four female students have only attended a compulsory course in mathematics in the first year in upper secondary school. The male student has attended one voluntary course in the second and third year in preparation for further studies in natural sciences.

Group C comprises one male student, Tor, and three female students Aud, Lea and Pia (ages: 23, 19, 20, and 21 respectively). The four students have all attended the compulsory course in mathematics in the first year in upper secondary school. The three female students have also attended a voluntary course in both the second and third year in preparation for further studies in social sciences or economics. The male student has attended a voluntary course in the second year in preparation for further studies in natural sciences (see Bjuland, 2002 for more details).

Problem selection

During the four group meetings in the second part, the students had to work on two problems (see Bjuland, 2004). One of the problems was compulsory while the other one was to be chosen from two other problems. Here I focus on part B of the compulsory problem:

> Problem 1B
> Choose an arbitrary equilateral triangle ΔABC. Let P be an interior point. Let d_a, d_b, d_c be the distances from P to the sides of the triangle (d_a is the distance from P to the side opposite of A, etc.)
>
> a) Choose different positions for P and measure d_a, d_b, d_c each time. Make a table and look for patterns. Try to formulate a conjecture.
> b) Try to prove the conjecture in a).
> c) Try to generalise the problem above.

Problem 1B is an open form of Viviani's theorem, which states that in an equilateral triangle, the sum of the distances from an interior point to the sides of the triangle, equals the altitude. By giving it an open form (Borgersen, 2004), my aim is that the student teachers find the distance sum based on drawings, measurements, constructions of conjectures, and finally give a proof for it. The problem stimulates the students to go through the seven stages of the problem-solving process presented by Borgersen (1994).

The dialogical approach and unit of analysis

The dialogical approach (Cestari, 1997; Linell, 1998, 2007; Marková & Foppa, 1990) is chosen as an analytical tool in order to divide the students' discourses into elementary building-blocks. This approach in the analysis of a particular sequence of discourse, permits the identification of interactional processes expressed in each utterance by the participants.

The term *dialogue* is characterised by an "interaction, in temporal and spatial immediacy, between two or more participants who face each other and who are intentionally conscious of and orientated towards each other in an act of communication" (Schutz, in Marková & Foppa, 1990, p. 6). In the present study, the main concern is to focus on a particular dialogue, *the mathematical discourse*, in which the students' communicative process is a focused mathematical discussion that is produced by the students based on their collaborative efforts to solve the mathematical problem.

Inspired by Wells (1999), episodes are analysed at three levels. At the first level, the episode has been divided into thematic sequences. At the second level, each sequence has been divided into exchanges. At the third level, each exchange has been divided into utterances. The exchange level comprises an initiating monitoring utterance, one or several response utterances, and a follow-up monitoring utterance. This corresponds to points 2, 3 and 4 respectively in my definition of MPD and is chosen as the unit of analysis.

Below, Pia shows the student who initiates the monitoring utterance (616). The two functions, *collaborative-* and *mathematical monitoring*, are illustrated in the right column.

616	Pia:	But it also has something to do with ... then I have found out that in an isosceles ... then it's maybe correct too ... (6 sec.) ...	Self-disclosure [collaborative monitoring]; Recalling some earlier work on generalisation (mathematical focus), Looking back on past experience (strategy), [mathematical monitoring].

Making sense of a conjecture emerged in the solution process

Empirical material from the group discussion of two groups of students will be presented to illustrate how the students approach and make sense of a mathematical conjecture just produced in the solution process, and their attempts at proving it. One sequence of discourse is chosen from group A in order to identify an example of a non-productive mathematical discourse. In group C an episode, consisting of two sequences of discourses, is presented with the aim of identifying MPD. Even though mathematical productivity should be revealed in one sequence of discourse, it is important to analyse a second sequence, related to the first one. It is then easier to claim that the mathematical discourse stimulates the students to make progress in the solution process.

Group A: a sequence of a non-productive mathematical discourse

The students in group A started to work on problem 1Ba in the beginning of the second meeting. There was some discussion in the group about how to measure the line segments from P to their intersections with the sides of the triangle. Two of the students, Roy and Liv, suggested that they should measure the lengths of the line segments from P along the perpendiculars to their intersections with the sides of the triangle. Unn had an alternative way of doing it. She wanted to measure the distance along the line through P parallel to the base of the triangle. However, the students gradually come round to a single way of interpreting the concept of distance from a point to a line (see Bjuland, 2007). After having spent about 32 minutes on problem 1Ba, the following conjecture emerged in the discussion: $d_a + d_b + d_c = constant$.

786	Mia:	Isn't that a proof when we all have quite a few locations of P? ...	Other monitoring, requesting clarification.
787	Roy:	Well ... the formulation we came up with ... is dependent on ...	Response, focusing on the formulation of the conjecture.
788	Mia:	No ...	Negation, does not provoke any discussion about this topic.
789	Roy:	Is it ... it's not necessary to write ... well it depends on the fact that ... that the equilateral triangle is constant ... but this is the case when we work on such a (triangle) ...	Following up (787).
790	Gry:	Yes ...	Confirmation.
791	Unn:	But how should we prove this? ... do we have to start with *P* then? ...	Feedback request; Moving from a conjecture to a proof, 1. Monitoring – how question, 2. Yes/No question, suggesting a direction.
792	Roy:	Yes wait ... let's see ... hmm ... it must have something to do with the fact that it's an equilateral triangle where all the sides are of equal length ...	Response with elaboration, recapitulates the characteristics of an equilateral triangle.
793	Liv:	I think that it has something to do with those quadrilaterals ... I can't give ... give that up ...	Following up, looking back on the solution process, the three cyclic quadrilaterals (see figure 1 below).
794	Roy:	Yes ... maybe that's the case ... (7 sec.) ...	Response, silence.
795	Gry:	Unn ... can we say that you made a sort of a table? ...	Monitoring question, recapitulation of the solution process.
796	Roy:	Yes ... last time when we ...	Brief response, continuation in (798).
797	Unn:	No ... we all did that ... (laugh) ...	Response to (795).
798	Roy:	Last time when we measured on the blackboard ... then we found out that ... this angle is 120 (degrees) ... that is 120 ... and that is 120 ...	Looking back on their past experience, recapitulating the three angles of 120 degrees with *P* as a common point (see figure 1).

799 Liv:	That will be the case for the other points as well …	Elaboration on (798).The same 120-degree angles will emerge from all the three points P_1, P_2 *and* P_3 (see fig. 1).
800 Roy:	Yes	Agreement.
801 Liv:	120 (degrees) added by 60 (degrees) …	Following up and elaborates on (798), (799). Focus on the sum of opposite angles in the quadrilaterals (120° + 60°).
802 Roy:	That's 90 … and that's 90 … and that's 90 …	Focus on the sum of opposite angles in the quadrilaterals (90° + 90°).
803 Liv:	120 added by 60 … those are opposite … it's … it's a cyclic quadrilateral …	Elaboration on (798) – (802, identifying the characteristics of a cyclic quadrilateral.
804 Roy:	Yes but how will you use this as a proof … to prove the conjecture? …	Other monitoring, following up; Moving from an idea of cyclic quadrilaterals to a proof, Monitoring how question.
805 Unn:	It's a kite …	Not attuned (804). Introduces a new idea of considering the quadrilateral as a kite.
806 Liv:	No that's not necessarily the case …	Disagreement, dependent on where the points P are located in the figure.

This sequence of discourse reveals that the students are communicating with another, and they are engaged in a mathematical discussion. The question is then, how is it possible to examine the quality of the mathematical content of the discussion? It is possible to argue that the first utterances (786)–(790) stimulate the monitoring utterance (791). However, it is also maybe the case that Unn has an inner discourse with herself, bringing her own thoughts into the discussion. Both monitoring utterances could be categorised as intrinsic properties (IP) discourses (Ryve, 2006) since there seems to be a clear initiative between both Unn (791) and Roy (804) to dig into the problem, to move from having found a conjecture to focusing on reasons why the conjecture is correct.

The students are really concerned with mathematical activity when they elaborate on the first monitoring utterance, bringing the concepts of equilateral triangles (792) and cyclic quadrilaterals (793) into the discourse (see figure 1, below). Gry's monitoring question (795) triggers a recapitulation of the solution process in which Roy focuses on the 120-degree angles identified at the previous meeting (796), (798). The students follow up and focus on the sum of opposite angles in the quadrilaterals, observing the characteristics of a cyclic quadrilateral, that the sum of opposite angles is 180 degrees (799)–(803). In one respect, it is possible to argue that this exchange of discourse (792)–(803) comprises many IP discourses since the discourse is based on intrinsic mathematical properties of mathematical concepts. On the other hand, the students are only recapitulating established experience, indicating EE discourses (Ryve, 2006).

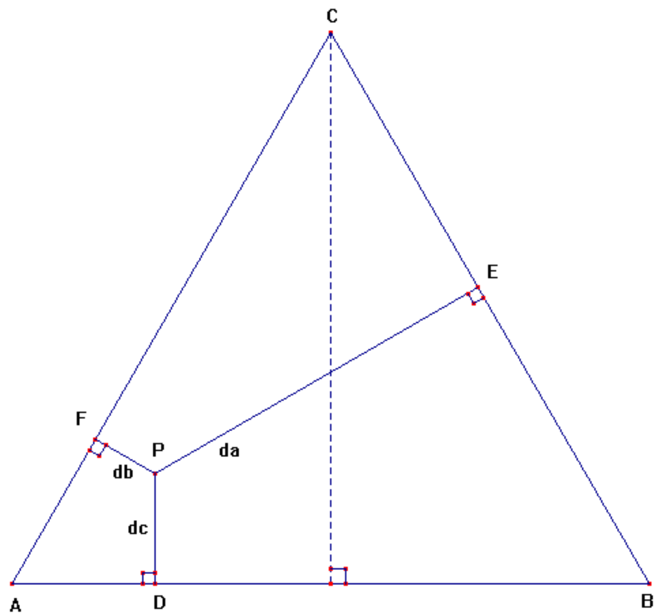

Figure 1.

In figure 1 the cyclic quadrilaterals are called *ADPF*, *DBEP* and *FPEC* respectively. These notations are not made by the students.

So why is this sequence of discourse not productive even though the students are focusing on characteristics of mathematical concepts? The discussion has so far shown that the students have come up with some ideas in their process of making sense of the conjecture. The second monitoring utterance invites the students to consider how they can link the idea of the cyclic quadrilaterals to the proof (804). This initiative challenges the students to take a more advanced step in their solution process. However, the lack of response to this particular question indicates that it is difficult for the students to elaborate on this initiative. Instead, another suggestion is brought into the discussion, focusing on the idea of a kite (805). The students are concerned with the idea of the kite (805), (806), bringing the characteristics of those quadrilaterals into the discussion (see Bjuland, 2002). The students' elaborations on the second monitoring utterance do not seem to stimulate mathematical progress in the solution process.

As a continuation of the mathematical discussion, the students reject the idea that the cyclic quadrilaterals look like the special quadrilateral kite. They go on conjecturing about the cyclic quadrilateral being similar (Bjuland, 2002). The students spend about 25 minutes working on problem 1Bb, when they decide to start working on the second geometry problem. The students have successfully gone through the first four stages of Borgersen's problem-solving model (1994) while they solved problem 1Ba. However, they do not manage the fifth stage, to develop a proof for the conjecture $d_a + d_b + d_c = constant$. The analysis of this particular choice of discourse sequence leads to the conclusion that the discourse is mathematically non-productive.

Group C: the first sequence of a productive mathematical discourse

The first sequence of discourse is selected about 60 minutes into the first meeting of group C. The students have analysed and defined the problem 1Ba and constructed triangles with a compass and straight edge. They have chosen different points for *P* and measured d_a, d_b, d_c each time and compared their results and looked for a pattern. The following conjecture has just emerged in the discourse: $d_a + d_b + d_c = constant$. In this group there was no discussion about how to measure the line segments from *P* to their intersections with the sides of the triangle.

1130 Aud:	I also got ... 8.6 ... 8.6 and 8.5 ...	Measurement of $d_a + d_b + d_c$.
1131 Lea:	Yes but then we have to find out why this is true ...	Self-disclosure; Moving from a conjecture to a proof, Focusing on the problem.
1132 Pia:	Mmm ...	Agreement.
1133 Tor:	Do we? ...	Other-monitoring, requesting clarification.
1134 Lea:	Yes, you can't just write something without proving it ...	Clarify problem 1Bb: The conjecture needs to be proved.
1135 Aud:	Yes but the next thing is that we should prove... or that we should formulate a conjecture... then... yes it has to be... conjecture... that's just	Following up (1131), continuation (1137)

1136 Lea:	I wrote ... why? ...	The process writer's action of writing the why-question in the log.
1137 Aud:	No matter which point we choose inside here ... then it will be	Recapitulating the formulating of the conjecture.
1138 Tor:	But the question is ... is this true just for a right-angled triangle? ...	Feedback request; Extending their conjecture from equilateral triangles to a larger class of triangles, Posing an open generalising question.
1139 Lea:	No it's an equilateral ...	Correcting Tor's mistake.
1140 Tor:	Equilateral ...	Linked to 1138, 1139.
1141 Lea:	Yes ... no that's ... I don't know ...	Agreement, disagreement and uncertainty.
1142 Pia:	Should we go on trying on all the triangles (laugh) ... that we think of just now? ...	Linked to 1138. Generalising question: extending the conjecture to all triangles.
1143 Tor:	Now ... I am going home (laugh) ... that was a burning question ...	Challenge – avoid answering the question (1142).
1144 Lea:	We have just an equilateral ... but we can try ...	Invitation to try out the conjecture for a larger class of triangles.
1155 Aud:	Yes is it maybe the case that we can try out different triangles and ... or different? ...	Linked to 1138, 1142. Generalising question: extending the conjecture to other figures.
1156 Pia:	Maybe circle and quadrilateral? ... and ... (laugh) ... yes you never know ...	Linked to 1155. Generalising question: extending the conjecture to circle or quadrilateral.

The discourse shows the crucial mathematical move in the students' solution process from the activity of comparing their measurements in order to find a conjecture (1130) to the *monitoring utterance* (1131) that provokes the students to focus on the next step in the solution process. The monitoring idea seems to be spontaneously initiated by Lea (self-disclosure), the process writer, but the use of the personal pronoun we, suggests a shared initiative to elaborate on the mathematics. The monitoring utterance provokes the short follow up question (1133) that could be rephrased: 'is that necessary?', requesting a clarification (other-monitoring). All the students (1131)–(1137) participate in the mathematical discussion with mutually coordinated initiatives and responses.

The first *monitoring utterance* (1131) brings the new idea of proving the conjecture into the discourse, while the second *monitoring utterance*

(1138) triggers a more focused direction for the solution process. This double monitoring and the elaborations on these monitoring utterances are promising indicators for identifying MPD. The students are invited by Tor's monitoring utterance to elaborate on an extension of the conjecture (feedback request). This is further stimulated when the 'right-angled triangle' is introduced into the question by mistake (1139), (1140). The mathematical discussion is focused around, and generated by, the monitoring question (1138), stimulating the genesis of a local generalisation. The generalising questions (1142), (1155), (1156) are clearly linked to Tor's initiative of extending the conjecture to a larger class of triangles. There is a gradual development in extension of the conjecture from right-angled triangles via all triangles and other figures to circle or quadrilateral. This sequence of discourse illustrates MPD since there are signs of mathematical progress, from a conjecture to an attempt at modifying the conjecture to a larger class of triangles.

At the end of the first meeting the students formulated two conjectures for equilateral triangles: (1) $d_a + d_b + d_c = constant$, (2) *The perimeter of* ΔABC / ($d_a + d_b + d_c$) = 3.5. At the beginning of the second meeting, they made some attempts at finding out if there is a connection between conjecture (1) and (2). This discussion led to the modified conjecture for equilateral triangles $d_a + d_b + d_c = h$ (the altitude of the triangle, see Bjuland, 2002 for more details).

Group C: a second sequence of a productive mathematical discourse

Between the first and second meeting, Pia tried out the conjecture $d_a + d_b + d_c = constant$ for isosceles and right-angled triangles. The conjecture was not correct for right-angled triangles. However, her measurements of the distances d_a, d_b and d_c respectively for different locations for P in an isosceles triangle showed that the sums of corresponding distances were almost a constant number. The discussion towards the end of the first meeting seemed to have inspired Pia to continue the work of generalising the conjecture. She brought this information to the other students about 35 minutes into the second meeting, illustrated by the following sequence of discourse.

612	Tor:	But ... we have to find out why those ... the sum of those ... is the same no matter where we place the points in the triangle ... isn't it? ...	Monitoring question. Looks back on the problem. Recapitulation of monitoring utterance in previous sequence.

613	Aud:	Yeah ...	Confirmation.
614	Pia:	It might have something to do with the fact that it's an equilateral ...	Related to (612), stimulates the monitoring utterance (616).
615	Aud:	Mmm ...	Confirmation.
616	Pia:	But it also has something to do with ... then I have found out that in an isosceles ... then it's maybe correct too ... (6 sec.) ...	Self-disclosure; Recalling some earlier work on generalisation. Looking back on past experience.
617	Lea:	And was it correct too? ...	Linked to 616. Other monitoring, requesting clarification.
618	Pia:	Mmm ... I have tried three points ... they weren't quite ...	Response.
619	Lea:	What does that triangle look like? ... was it very obtuse? ... no but then we'll try a very obtuse ...	Other-monitoring, following up; Elaborating on a possible generalisation for isosceles triangles. Trying out a worst case counter-example.
620	Pia:	I got 95 ... no 9.5 ... 9.2 and 9.3 ...	Measurement of $d_a + d_b + d_c$ for isosceles triangle.
621	Aud:	Mmm ... (16 sec.) ...	Agreement. Silence, indicating the beginning of measuring activity.

It is possible to argue that Tor's monitoring question (612) could be categorised as a monitoring utterance. However, he only repeats the monitoring utterance in the sequence of discourse, analysed above, and recapitulates the contents of the conjecture that stimulates the students to establish common ground for the group discussion. The utterances (612)–(615) are therefore triggers for the monitoring utterance (616) in which the students are challenged to focus on the conjecture for a larger class of triangles.

The two monitoring utterances (616), (619) have different functions. Pia's initiative (616) brings the idea of generalisation into the discussion by looking back on her previously acquired measurements for isosceles triangles. The students' sense-making on a possible extension of their conjecture (first sequence analysed) together with Pia's work between the two group meetings seem to have been a starting point for a more focused discussion on this topic. Lea's following up question (617) requests clarification. The second monitoring utterance (619) is also provoked by Pia's initiative. By applying the strategy of trying out a worst-case counter example, the students are stimulated to elaborate on this possible extension of their conjecture. This is exemplified by choosing an isosceles triangle in which the obtuse angle C is close to 180 degrees (see figure 2). The attempt at modifying the conjecture is tried out for an extreme case in order to be considered, modified or perhaps rejected.

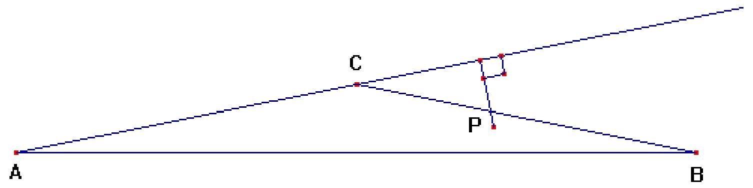

Figure 2.

Aud informs the other group members about her measurements of d_a, d_b and d_c for different locations of *P*, indicating that the sums of corresponding distances are almost a constant number (620). The students follow up Lea's monitoring utterance, and they are concerned with measuring activity.

This sequence of discourse comprises the five different elements in order to be characterised as MPD. In addition, both the monitoring question (612) and the two monitoring utterances (616), (619) seem to represent IP discourses (Ryve, 2006) or object-level utterances (Kieran, 2001) performed by different students. The *mathematical*- and the *collaborative monitoring* indicate the interactive and mutual willingness among the students to participate in the focused discussion, to generalise the conjecture from equilateral triangles to isosceles triangles.

It is possible to argue that this discourse is not productive based on the definition by Sfard and Kieran (2001) since the students do not come up with a proper solution. However, the two sequences of discourses both illustrate that the mathematical discussion stimulates the students to make progress in the solution process. These sequences are related to the same mathematical topic, the process of generalising the conjecture to a larger class of triangles. The monitoring utterances play a crucial role as a prerequisite for defining MPD. However, it is the dynamic sequentiality of the five criteria that advances the discourse among the students.

Discussion

The aim of the present study has been to identify critical characteristics of mathematically productive discourses (MPD) through detailed analysis of the mathematical discussion from two groups of student teachers while working collaboratively on a geometry problem. In Bjuland (2007), one promising attempt at illustrating indicators of MPD has been based on the identification of monitoring questions that have been crucial for the students' solution process. Research has suggested that discourses are mathematically productive if they comprise a high rate of object-level utterances (Kieran, 2001), or if there is a high frequency of sections, involving intrinsic properties discourses (Ryve, 2006).

The analyses of sequences of discourses are identified to be mathematically non-productive and productive respectively based on my five criteria for MPD. It is important to emphasise that the student teachers are working on geometry problems (for a long period of time) which really challenge them to go through a complex problem-solving process (Borgersen, 1994). This differs significantly from the pair of 13-year-old students working on smaller problems given in activity worksheets (Sfard & Kieran, 2001). It also differs from the four collaborative groups of engineering students constructing concepts maps in linear algebra (Ryve, 2004, 2006).

Even though the sequence of discourse of group A is identified to be non productive based on my definition of MPD, it is possible to argue that the sequence could be useful as background for developing their knowledge base (Schoenfeld, 1992). The students' use of the heuristic strategy of looking back on past experience and their recapitulations of the characteristics of mathematical concepts like equilateral triangle, cyclic quadrilateral, kite, and similar quadrilaterals (Bjuland, 2002), could be an important starting point for the attempts at solving the second geometry problem.

The students' (group A) attempt at solving the second geometry problem is illustrated in Bjuland (2007). Here the students have analysed the problem and made an auxiliary figure (see figure 3).

The students are asked to prove that $\angle BQC = \angle BPC$. The students succeed in solving this problem by identifying the subconfiguration $QBCP$ in figure 3. They argue that $QBCP$ is a cyclic quadrilateral since the sum of opposite angles is 180 degrees, which means that it is possible to construct a circle that circumscribes the quadrilateral. Then they conclude that the two angles $\angle BQC$ and $\angle BPC$ are equal since they are both angles at the circumference, subtending the same arc BC.

This indicates that a non-productive sequence of discourse within one problem could develop the students' knowledge base which may become important for solving later problems.

The monitoring utterances are critical indicators for identifying MPD since these utterances involve both collaborative- and mathematical monitoring. The use of the personal pronoun we has also been prominent in some of the monitoring utterances, suggesting a shared initiative to

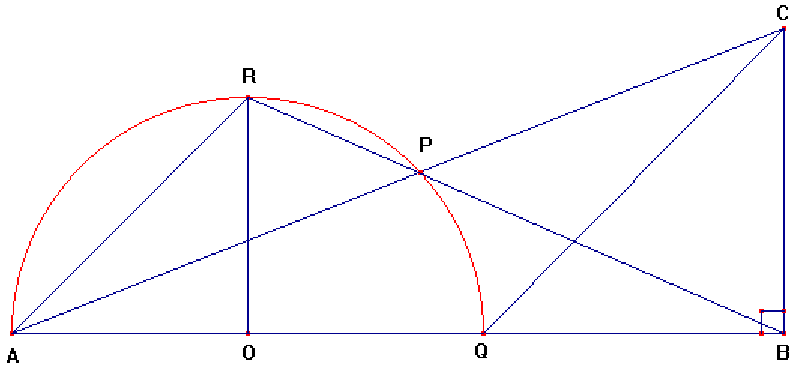

Figure 3.

elaborate on the mathematics. These aspects are all indicators of mutuality, the hallmark of peer collaboration (Damon & Phelps, 1989; Goos et al., 2002). I have also suggested in the analyses of the student discourses that the monitoring utterances are quite similar to the object-level utterances (Kieran, 2001) and the IP discourses (Ryve, 2006).

It is not easy to conclude when a discourse is mathematically productive or not. This is particularly difficult when students are challenged to work on problems that are designed so that they can experience the entire problem-solving process, from doing experiments and making conjectures via attempts at proving the conjectures to generalisations and formulation of new problems (Borgersen, 1994). Both groups of students have found a proper conjecture for equilateral triangles $d_a + d_b + d_c = constant$, which has also been modified by group C for equilateral triangles $d_a + d_b + d_c = h$. By employing the definition of productivity defined by Sfard and Kieran (2001), there are indicators of productive discourses in both groups of students since they find a solution to one part of the problem (1Ba). The students in both groups do not find a solution to problem 1Bb, to give a proof for the conjecture $d_a + d_b + d_c = constant$. In this respect the analysed discourses, presented here, should therefore, by Sfard and Kieran's definition (2001), not be characterised to be mathematically productive. However, when students are working on complex problems in which they do not manage to solve the whole problem, it is important to emphasise that discourses, in my definition, could still be mathematically productive.

The analyses of two sequences of discourses show how the students in group C are concerned with generalising their conjecture for equilateral triangles to a larger class of triangles, showing how the communication is brought mathematically forward between the first and second group meeting. The first sequence of discourse has been identified to be productive based on my five criteria for MPD, since there is a clear elaboration on what it means to generalise the conjecture, suggesting different directions for possible generalisations. The second sequence of discourse is also mathematically productive, illustrating how the students are looking back on past experience and bringing the topic of generalisation into the discussion. They test out their conjecture for isosceles triangles by trying out a worst-case counter example. A stronger claim of mathematically productive discourses in the problem-solving process is made by presenting two sequences of discourses that are separately productive.

Conclusion

Five criteria have been introduced as crucial aspects for characterising MPD. 1) The initiation of a monitoring utterance, 2) the monitoring utterance, 3) student responses, stimulating a second monitoring utterance, 4) the second monitoring utterance, giving the solution process a focused direction, 5) further elaborations on the second monitoring utterance. Each of those criteria may indicate characteristics of mathematical productivity. However, MPD is only found in sequences of discourses if all these criteria are identified. When students are involved in complex problem solving, it is not easy to conclude about mathematically productive discourses. However, the five criteria for MPD, could be used as a methodological framework for analysing sequences of discourses in order to identify productive or non-productive discourses.

The students' monitoring questions seem to be important triggers for the monitoring utterances which could lead to MPD. As a pedagogical implication, it is therefore important that students in teacher education are introduced to *monitoring training* in combination with *collaborative learning* while working on geometry problems in small groups. These two components were critical in my design of the teaching programme over a month described in the method section. By presenting sequences of discourses from students working on geometry problems in collaborative working groups in courses at teacher-training colleges or in in-service training of teachers, this can provide opportunities for students or teachers to observe how MPD can develop based on the five criteria presented in this article.

One possible direction for future research would be to use the five criteria for MPD in this study as a methodological framework for analysing sequences of discourses among pupils at different grade levels working on mathematical problems in small groups. Even though this study has identified MPD and recent studies (Sfard & Kieran, 2001; Ryve, 2006) have identified non-productive discourses, there is still a need for a critical discussion about the term mathematical productivity in student discourses.

Acknowledgements

Special thanks are due to the management at NLA College of Teacher Education that provided the grant for doing this research. I would in particular like to thank my colleagues, Maria Luiza Cestari and Hans Erik Borgersen at Agder University College, and the anonymous reviewers for the constructive help they provided for improving this article.

References

Bartolini Bussi, M. G. (1998). Verbal interaction in the mathematics classroom: a Vygotskian analysis. In M. B. Bussi, A. Sierpinska & H. Steinbring (Eds.), *Language and communication in the mathematics classroom* (pp. 65–84). Reston, VA: NCTM.

Bjuland, R. (2002). *Problem solving in geometry. Reasoning processes of student teachers working in small groups: a dialogical approach* (Ph.D. Thesis). University of Bergen, Department of Applied Education.

Bjuland, R. (2004). Student teachers' reflections on their learning process through collaborative problem solving in geometry. *Educational Studies in Mathematics*, 55 (1-3), 199–225.

Bjuland, R. (2007). Adult students' reasoning in geometry: teaching mathematics through collaborative problem solving in teacher education. *The Montana Mathematics Enthusiast*, 4 (1), 1–30.

Blum, W. & Niss, M. (1991). Applied mathematical problem solving, modelling, applications, and links to other subjects – state, trends and issues in mathematics instruction. *Educational Studies in Mathematics*, 22 (1), 37–68.

Borgersen, H. E. (1994). Open ended problem solving in geometry. *Nordic Studies in Mathematics Education*, 2 (2), 6–35.

Borgersen, H. E. (2004). Open ended problem solving in geometry revisited. *Nordic Studies in Mathematics Education*, 9 (3), 35–65.

Cestari, M. L. (1997). *Communication in mathematics classrooms. A dialogical approach* (Ph. D. Thesis). University of Oslo, Faculty of Education.

Damon, W. & Phelps, E. (1989). Critical distinctions among three approaches to peer education. *International Journal of Educational Research*, 13 (1), 9–19.

Goos, M., Galbraith, P. & Renshaw, P. (2002). Socially mediated cognition: creating collaborative zones of proximal development in small group problem solving. *Educational Studies in Mathematics*, 49 (2), 193–223.
Johnson, D. W. & Johnson, R. T. (1990). Using cooperative learning in mathematics. In N. Davidson (Ed.), *Cooperative learning in math. A handbook for teachers* (pp. 103–125). Menlo Park, CA: Addison Wesley.
Kieran, C. (2001). The mathematical discourse of 13-year-old partnered problem solving and its relation to the mathematics that emerges. *Educational Studies in Mathematics*, 46 (1-3), 187–228.
Linell, P. (1998). *Approaching dialogue. Talk, interaction and contexts in dialogical perspectives*. Amsterdam: John Benjamins Publishing Company.
Linell, P. (2007). *Essentials of dialogism. Aspects and elements of a dialogical approach to language, communication and cognition* (Paper in preparation, version 050625). Retrieved May 16, 2007 from http://www.liu.se/isk/research/Medarbetare/perli.html
Marková, I. & Foppa, K. (1990). *The dynamics of dialogue*. New York: Springer.
Polya, G. (1957). *How to solve it*. Princeton, NJ: Princeton University Press. (Original work published 1945)
Ryve, A. (2004). Can collaborative concept mapping create mathematically productive discourses? *Educational Studies in Mathematics*, 56 (2-3), 157–177.
Ryve, A. (2006). Making explicit the analysis of students' mathematical discourses – revisiting a newly developed methodological framework. *Educational Studies in Mathematics*, 62 (2), 191–209.
Schoenfeld, A. H. (1992). Learning to think mathematically: problem solving, metacognition and sense making in mathematics. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334–370). New York: Macmillan.
Schoenfeld, A. H. (1993). Teaching mathematical thinking and problem solving. In, *Sånn, Ja! Rapport fra en konferanse om matematikkdidaktikk og kvinner i matematiske fag* (Arbeidsnotat 2/93, pp. 67–89). Oslo: Norges almenvitenskapelige forskningsråd, NAVFs Sekretariat for kvinneforskning.
Sfard, A. (2001). There is more to discourse than meets the ears: looking at thinking as communication to learn more about mathematical learning. *Educational Studies in Mathematics*, 46 (1-3), 13–57.
Sfard, A. & Kieran, C. (2001). Cognition as communication: rethinking learning-by-talking through multi-faceted analysis of students' mathematical interactions. *Mind, Culture, and Activity*, 8 (1), 42–76.
Springer, L., Stanne, M. E. & Donovan, S. S. (1999). Effects of small-group learning on undergraduate in science, mathematics, engineering, and technology: a meta-analysis. *Review of Educational Research*, 69 (1), 21–51.
Wells, G. (1999). *Dialogic inquiry: towards a sociocultural practice and theory of education*. Cambridge: Cambridge University Press.

Raymond Bjuland

Raymond Bjuland is Associate Professor in mathematics education at Agder University College. At the moment he has a post. doc. position in the LCM project, Learning Communities in Mathematics at AUC. This project is based on the idea of creating "communities of inquiry" between teachers and didacticians to question existing practices and design new approaches to teaching and learning mathematics. He is interested in classroom research and in problem solving with a focus on students working in collaborative small groups.

Raymond Bjuland,
Agder University College,
Faculty of Mathematics and Sciences,
Department of Mathematics,
Servicebox 422,
4602 Kristiansand, Norway
raymond.bjuland@hia.no

Sammendrag

Målet med artikkelen er å identifisere viktige kjennetegn på en produktiv, matematisk diskurs når lærerstudenter samarbeider i smågrupper om å løse en geometrioppgave i en problemløsningskontekst. Analyser av diskurssekvenser fra den matematiske diskusjonen i to studentgrupper blir presentert for å identifisere både ikke-produktive og produktive diskurssekvenser. En definisjon på en matematisk produktiv diskurs (MPD) blir presentert og brukt som et analytisk redskap i analysen. Definisjonen omfatter følgende fem kriterier: 1) Ytringer som stimulerer en monitorerende ytring, 2) den monitorerende ytringen, 3) responsytringer som stimulerer en ny monitorerende ytring, 4) den nye monitorerende ytringen, 5) responsytringer som fører kommunikasjonen matematisk videre mellom studentene. Artikkelen diskuterer også hvor vanskelig det kan være å konkludere når en matematisk diskurs er produktiv eller ikke, særlig når studenter blir utfordret til å arbeide med kompliserte problemer der en løsning vanligvis ikke blir funnet i løpet av en skoletime.