Education of lower secondary mathematics teachers in Denmark and France

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This paper[1] presents a comparative study of two surprisingly different systems of preparing teachers for lower secondary level teaching of mathematics, namely those of Denmark and France. We first describe these systems differences succinctly. The main part of the paper reports on a qualitative study of how final year teacher students in the two countries handle two hypothetical situations of mathematics teaching. Then we discuss how the findings could be related to the systems of formation described first.

In this paper, we address the following questions:

Q1 What are the main characteristic differences between Danish and French education of lower secondary teachers of mathematics?

Q2 Are there systematic differences between final year students in the two systems with respect to abilities and strategies for approaching demanding teaching situations?

Q3 Could answers to the two previous questions be meaningfully related to each other?

As regards question Q1, we rely on the fact that each of the two systems are extensively described in the literature (the Danish system e.g. Elle, 1996, 1999; the French system e.g. Comiti & Ball, 1996; Henry & Cornu, 2001; Pimm, 2003); without repeating all this detail, this question requires a choice, definitions, and determination of 'crucial variables' of teacher

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Winsløw, C. & Durand-Guerrier, V. (2007). Education of lower secondary mathematics teachers in Denmark and France. *Nordic Studies in Mathematics Education*, 12 (2), 5–32.

training systems. In particular it involves an explicit epistemology of 'teacher knowledge'.

For Q2, we propose a simple qualitative method for comparative studies of *novice teachers competencies*, namely that of organising and analysing a discussion about *hypothetical teachers* tasks (HTTs) among pairs of test persons. Again, the design and analysis of HTTs are based on explicit epistemological assumptions. We report on results pertaining to two HTTs and five pairs of novice teachers in each country.

Behind Q3 is indeed much of the motivation to study teacher education from a comparative viewpoint, namely that such a study could provide insights into the possible results of systematic differences (or changes) of the teacher training systems – *insights that the study of one system could not give*. In this way, a comparative study could contribute to the knowledge on which one may base reforms.

To sum up, our study is comparative in a double sense: it aims to compare two comparisons, as illustrated in figure 1.

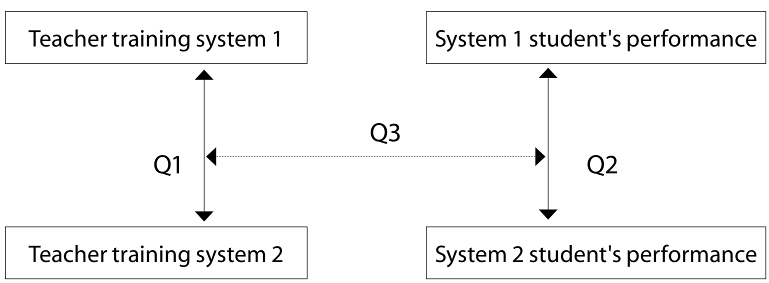

Figure 1. *Double comparison design.*

Besides exhibiting crucial differences between *systems* of teacher education and their *products* in the two countries we studied – and the relation between these two comparisons – we also hope that elements of our theoretical and methodological framework may be useful for similar comparisons among other countries. For instance, the comparison of secondary teaching in Denmark and Japan (Winsløw, 2004, cf. Winsløw & Emori, 2006) should be complemented by a comparison of teacher education. We are currently working with data from Japan to extend the present study to cover this country as well (forthcoming work in collaboration with H. Yoshida).

Theoretical framework

In this section, we briefly introduce the notions we have used for our comparison of how mathematics teachers are educated in the two countries; they concern the *contents* of the education, and its *context*. The contents categories naturally come up also in the analysis of teacher trainees competencies. We have opted to describe the theoretical categories in this separate section because they could be useful for other comparisons.

Three components of teacher knowledge and training

It is common to describe the pertinent knowledge of mathematics teachers using the following chief categories (Bromme, 1994, cf. also Comiti & Ball,1996):

- *Content knowledge* (pertaining to mathematical concepts: use of techniques, theories etc.).
- *Pedagogical knowledge* (concerning education, learning and teaching in general; includes sociological, psychological and ethical aspects of education and its functions, and general principles for managing classroom teaching).
- *Didactical knowledge* (regarding the conditions and mechanisms of *mathematics* teaching and learning, requiring an analysis specific to the target knowledge; an important example for our study is the theory of didactical situations, cf. Brousseau, 1997).

Each component may occur with different emphases on *theory* and *practice*. For content knowledge, this could mean at least two things: more or less emphasis on mathematics which actually appears in lower secondary level teaching (as opposed to other, particularly more advanced, parts of mathematics); or more or less focus on applications of mathematics (as opposed to pure and deductive mathematics).

The precise delimitation of the three components, as well of their mutual relations, is a highly controversial issue. Notice that these components are sometimes integrated in concrete elements of a teacher education programme, such as the American 'methods' courses which could be said to combine all three. Moreover, such integrated knowledge is not a simple juxtaposition (see e.g. Ball & Bass, 2000). In the French and German traditions of didactics of mathematics, there is a tendency to regard the second category as important but not as fundamental as the other two; more precisely, content knowledge is prominent in didactic analysis of these traditions. In other traditions – including mainstream

Anglo-Saxon research which is also influential in Scandinavian countries – the last category seems to be considered as more or less derivative of the two first. Finally, apparently regardless of country, the idea that *only* the first category represents true knowledge, while the rest is a (possibly highly developed) 'craft', is present among teachers and dominant among mathematicians. Indeed to some education researchers, the essentials of a teachers' knowledge lies in a profound and teaching-oriented understanding of the mathematics that the teacher is teaching (e.g. Ma, 1999), and in particular there is a remarkable tradition in East Asian countries to develop the craft of 'good lessons' to perfection (cf. Stiegler & Hiebert, 1999).

This paper takes a *descriptive* stand on these issues, and we use the above categories as broad orientation points to identify how the two systems in question organise the development of the three components, and how they appear in novice teachers' performance on hypothetical teachers' task. On the other hand our position is clearly not *neutral*: we do consider the three elements together as domains of *knowledge* which can be developed through initial teacher training and which are, together, potentially crucial for the *know-how* that goes into delivering actual and efficient mathematics teaching.

Describing teacher training systems

We consider, for our comparison of systems, the following 'variables', most of which refer to the three elements considered in the previous section.

1 *General aspects*: What are the institutions? What is the overall structure of the programme? What are the requirements to enter the programme? And to complete it?

2 *Organisation*: how do the three elements appear in the programme? (e.g. separately/integrated, simultaneously/consecutively) How is the programme determined? Who are teaching?

3 *Volume*: how much work (quantity, quality) is required by students on the three elements?

4 *Contents*: What is emphasised? And what is not? Emphases on theory and practice?

5 *Working modes*: how do the students work with the elements? How are they assessed?

6 *History, ideologies and traditions*: what are the main circumstances and ideas behind the present system? How do they affect or explain the previous variables?

While we think these are important variables for comparing any systems of teacher education, they are of course not the only conceivable ones. But for the comparison of Denmark and France – both relatively homogeneous countries, for instance from a language point of view – we have found that the above minimum list is sufficient to highlight essential differences.

Comparison of lower secondary teacher training systems

There are quite eye catching differences between the two systems, the most obvious perhaps being that in Denmark, teachers at this level are trained to teach 4 different subjects, while in France they should teach only mathematics. The programmes we are considering are also different in scope: the Danish programme prepares simultaneously for teaching in primary and lower secondary schools; the French prepares for teaching in both lower and upper secondary school.

The differences in terms of teacher preparation are to some extent related to significant differences between the two countries' current programmes for mathematics at lower secondary level. Roughly speaking, the Danish programme is quite open and emphasises 'real life' mathematics, while the French curriculum requires specific topics and methods to be covered, and focuses more on scientific aspects of mathematics, such as precise definitions, reasoning, etc. And both when it comes to teacher training and lower secondary school, the 'institutional traditions' are quite different, and can be roughly summarised as 'centralised' (France) versus 'local autonomy' (Denmark).

The following analysis is based on discussion between the authors (both have extensive experience with the systems in their own country, as well as some encounters with the other country's systems) and on separate descriptions of the two systems in the literature.

General aspects

The French mathematics teacher takes a 3 years bachelor degree in mathematics at a university, followed by (normally) 2 years of further studies at an IUFM (university institute for teacher education). The first year, which is not compulsory, is spent to prepare for a national competitive examination (*concours*) which is needed to gain access to the second year. The *concours* consists in two written and two oral examinations. One of the oral examinations, prepared at the IUFM, is about choosing and presenting a set of exercises on a given theme. The other three examinations, which concern academic mathematics and are prepared in the university, consist in two long written problems and an oral synthetic presentation of a mathematical theme as it could be done at secondary level. Students who succeed are assigned to a school. During the second year, the students are responsible for one class in a secondary school. They follow courses at the IUFM during this year, and receive formative and assessment inspections from the IUFM. They also practice for about 40 hours in the classroom of an experienced teacher so that they face both lower and upper secondary school. After successful completion of this second year, students are guaranteed to be recruited as teachers somewhere in the country (except for exceptional cases). Notice that the decisive point of selection occurs before the last year of training, at the national *concours*.

To become a lower secondary level mathematics teacher in Denmark, one must take a 4 year study programme at a teacher training centre (CVU) that includes mathematics among the four subjects studied. No academic mathematics courses (of the type normally given in universities) are included in the programme. The CVUs are national institutions that are independent of the universities. Furthermore, the content of the education is decided by each individual CVU, within the frames of rather broad national guidelines. Among the mandatory elements are some weeks of practice each year, during which students teach in a school under supervision from the regular teacher; the longest practice of about seven weeks is placed in one of the two last years. Examinations have to be passed every year, in particular one oral and one written examination in mathematics and its teaching; these exams could be placed in different years of study, depending on the CVU and the students' individual choices. After completion of the programme, the graduates apply for positions at public or private schools; the recruitment procedure is, within general rules regulating the labour market, entirely left to the individual school. There is little unemployment among teacher graduates, and particularly those certified in mathematics or science subjects may easily find a job.

In both countries, completed upper secondary school is the only essential requirement for entering the programmes (at university and CVU, respectively). There is a slight difference, though: in France, only students with a scientific upper secondary school exam may enter the mathematics programme, while a similar requirement exists but is not strictly enforced for the Danish training of mathematics teachers.

Organisation

The two programmes both seek to develop all three elements of teacher knowledge defined above, however within very different organisations. In France, mathematics comes first, in the form of a three year curriculum in pure mathematics, taught at university by mathematicians. At the IUFM, especially during the first year (preparation for the *concours*) one attempts to relate students' knowledge of mathematics, particularly at school level, to the task of preparing mathematics for pupils; and during the second year, elements of pedagogy and didactics are taught and related to the students' experiences during the part time teaching assignment. In Denmark, the training includes general pedagogy disciplines (taught separately) and a unique study unit on school mathematics and didactics of mathematics, which must be spread over two years, along with other courses. Practice is also spread over the entire study period. Thus, while French teacher education has a clearly *consecutive* and *separate* structure for teaching mathematics on the one hand, didactics/pedagogy on the other hand, the Danish programme teach them in a *partially simultaneous and integrated* structure (partial because the general pedagogy courses are taught separately and mainly in the beginning of the programme). However, in the French system, didactics is closely connected to mathematics, and the professional dissertation to be written during the second year contributes to the integration of the three elements: mathematics, didactics and pedagogy (Henry & Cornu, 2001, pp. 491–492). In Denmark, there is a considerable variation in both the volume and contents of the didactics part of the 0,7 study years (cf. table 1 below) and both depends largely on the individual teacher educator.

In both countries, a combination of centrally formulated requirements and local autonomy govern the formulation and implementation of actual programmes.

In France, the main responsibles for teaching in universities are researchers with a ph.d. of mathematics; in the IUFM the responsibility of teaching is shared between researchers with a ph.d of mathematics or didactics and secondary school teachers, often with a with a part time assignment to teach at this level. Some of these assistants will have a ph.d. or another post-graduate degree, but it is not a requirement. Some of them are engaged in the IREM (Institute for Research in Mathematics Education).

In Denmark, the CVU teachers are not researchers, although there is a tiny minority of teachers holding a ph.d. Most (about 85 %) of them have a masters degree from a university, either in pedagogy or mathematics, while a minority have just a teachers diploma of the type delivered at CVUs. Recently, CVU teachers are encouraged to take a ph.d. in didactics or educational studies, and there are various incentives to strengthen relations between university researchers and CVU-teachers.

Volume

Another main difference is the volume of study time assigned to each of the three elements in the programmes, particularly 'mathematics'. A rough overview is provided by table 1, where the numbers indicate stipulated work load expressed in full time study years:

Table 1. *Study time (yrs) devoted to different elements.*

<table>
<tr><th></th><th>Mathematics</th><th>Didactics of mathematics</th><th>General ed. (pedagogy)</th><th>Teaching practice</th><th>Other</th></tr>
<tr><td>DENMARK</td><td colspan="2">0,7</td><td>0,7</td><td>0,6</td><td>2,0</td></tr>
<tr><td>FRANCE</td><td>3,7</td><td>0,4</td><td>0,3</td><td>0,6</td><td>–</td></tr>
</table>

Here, 'other' includes, for the Danish teachers, three other disciplines and their didactics. Notice that the total length of the French programme is 5 years with the last year being a kind of semi-employment in a school, while the Danish programme lasts 4 years and has the practice periods spread over each of these years, as well as over the four school subjects studied.

Although it can be quite different among individual students, we think that the actual workload corresponding to a study year is similar for the two programmes. In both countries, many students have part time jobs along with their studies, and the most intense work occurs prior to main examinations.

Contents

The differences in contents are less easy to describe shortly and exactly, in part because the contents taught vary also within each country and even institution (according to teachers' choices). However, some clear differences remain for each of the three elements previously defined.

As for the *mathematics* part, the Danish programme is essentially restricted to strengthening students' competence related to the themes that are taught in primary and lower secondary school, which include: arithmetic, simple equations and algebraic symbolism, elements of plane and spatial geometry, graphs and functions, empirical probability and statistics. Considerable emphasis is put on simple application of mathematics and mathematical models in everyday life and in other disciplines. The training contains no post-secondary material, for instance, no calculus and no linear algebra. Thus, the teaching of mathematics is entirely motivated by, and structured according to, the needs for school teaching. By contrast, the French students spend three years as mathematics students in a university environment still influenced by Bourbaki. During the first two years, they study the standard diet of linear and non-linear algebra, real and complex analysis etc., and in the third year, more advanced topics such as general topology, functional analysis and projective geometry. Probability and statistics is not compulsory and is, like plane geometry and other topics taught in school, not part of the university programme. However, at the IUFM, school mathematics is explicitly attended to, particularly in preparing the *concours*. It is often a challenge for educators at the IUFM that students seem to acquire, during the three years at university, a rather academic view of mathematics as a set of monuments to be visited and explained.

The didactics part of the Danish training is, according to the organisation, closely linked to the study of school mathematics, but is also frequently referring to elements of general pedagogy and cognitive theory, such as principles of Piagetian constructivism. Didactics is thus, mostly, understood as principles and ideas relating to the aims and practice of teaching mathematics on the one hand, and to general pedagogy and epistemology on the other. In France, didactics of mathematics implies, usually, elements of: epistemological analysis of concepts (Artigue, 1990), the theory of situations (cf. Brousseau, 1997), and the anthropological theory of didactics (Chevallard, 1999), as well as methods for constructing and analysing lessons and exercises, partly in preparation for the *concours*. In both countries, a final dissertation is written based on both theoretical parts of the training and the experience gained through teaching practice.

Finally, the contents of pedagogy courses in the two countries vary considerably among institutions but include elements of educational psychology and principles of the schools' functions in society. The texts read tend to be written in the national language (Danish or French) and relate to national conditions of the school.

As a kind of general tendency, the Danish programme is explicitly focused on *practice* both in terms of approach to mathematics and its teaching, while the French programme – except for the final year – is much more academic in nature, particularly when it comes to mathematics, but also to some extent in its use of didactics as a scientific discipline. This difference is in part related to the fact that the French students are trained to teach only mathematics, with no post-secondary studies in other school subjects, while the Danish students are prepared to teach four disciplines.

Working modes

Most of the training in France is done in a traditional university setting, with lectures, exercise sessions and written examinations. It is only at the *concours* and in the last year of training that the students are required to work autonomously with questions related directly to the teaching profession. Moreover, the students' work at university is almost exclusively done individually. Due to the professionalisation, during the second year at IUFM, the collaborative work is encouraged. The Danish programme involves a more varied range of working modes, with a high level of interaction among students and CVU teachers. Group work and projects are common formats. In both countries, study units are assessed using written and oral exams, but the Danish programme does not include a high-stakes exam like the French *concours*. Parts of the oral exams in Denmark are based on project work done by a group (although recently this has been restricted by new laws); in France, all exams are individual, except the professional dissertation that can be elaborated and written in pair.

History, ideology and traditions

In both countries, lower secondary school is (as primary school) part of the free, public education offered to all citizens. However, these have developed on very different historic and ideological backgrounds. It is not possible within the frames of this article to give a reasonable (not to speak of comprehensive) account of these differences, so we just indicate two general tendencies which we believe to be of particular importance:

– In France, the public school has been an important element in changing political agendas, from the revolution 1789–1793 to the separation of Church and state in 1905. In Denmark, reforms have been more consensual, and they tend to be slower and less abrupt.

– Mathematics and science are traditionally important elements in the 'enlightenment' ideal of a well-formed citizen in France. In Denmark, one finds a more 'humanistic' view of *Bildung* like in other Anglo-Saxon countries, and a more utilitarian view of mathematics and science.

Comparison of outcomes of teacher training

The ultimate goal of a teacher training program is, of course, to contribute to the quality of teaching in schools. However, many other factors are obviously in play there, both of an individual and institutional nature (cf. Skott, 2000 for an extensive case study). Hence if one wants to study the results of initial training, looking at novice teachers' teaching may not suffice. This is why we have chosen to consider students who have just finished their studies but have not yet taken up a regular teaching position and make a sample – however modest – of their competencies. We now explain how we did that.

Methodology

The hypothetical teachers' tasks (HTT) are constructed so as to introduce, in a concise yet recognisable way, a teaching situation which could reasonably arise in both countries, and where the teacher would have to mobilise considerable aspects of the knowledge components to act appropriately. The mathematical contents of the situations are both standard and elementary, and would in principle be addressed in some way within both programs. The two HTT can be found in appendix 1. One is about a geometry problem, which can be interpreted as an example of similar triangles, the other concerns the reason why $(-2) \cdot (-3) = 6$, a classical problem of elementary teaching discussed e.g. in Glaeser (1981).

In each country, 10 teacher students where asked to participate. They did the tasks in pairs, with some time for individual work (see appendix 1), and a researcher observed and audio taped their performance (besides introducing the tasks according to precise instructions). At the end, the students were asked a question about *where* they had acquired the knowledge which they had drawn upon in solving the tasks (suggesting lower or secondary school, teacher training, elsewhere), and in some cases a shorter conversation on points related to this or to the tasks was recorded as supplementary evidence. But we consider mainly, as our data properly speaking, the first 'planned' part, where the researcher did not intervene.

We do not claim that the five pairs are representative of their country or institution in some formal sense, yet we don't think they are 'special' as there was no difficulty to find volunteers. However, the Danish pairs came from a rather popular teacher training college (i.e. one which tends to get students with better grades from secondary school) while the French pairs came from an IUFM which has no such particularities.

The data obtained consist of students' notes, audiotapes of discussion, and researchers' accompanying notes. In order to analyse these data we developed a scheme of possible approaches for each HTT (described in the next section) in order to gain a rough overview of the performance of each pair. We also analysed the transcribed discussions (and supplementary notes) in order to find characteristic (or outstanding) ways of using and combining different elements of knowledge, some of which are selected for discussion in this paper.

A priori analysis of the tasks

The HTT's require the teacher students to analyse the task from a mathematical point of view, and to reflect on how to proceed in a hypothetical class situation. The latter will of course depend on the informants' analysis of the mathematical situation, as a part of a didactical analysis and also in order to motivate the more pedagogical aspects of the response (e.g. how to manage the classroom).

These two parts of the task are *explicit* in HTT1 (which has two separate parts) while it is implicit in HTT2. Nevertheless both can be seen clearly in the student responses. The a priori analysis suggested in the subsections below therefore has two parts:

- A specific enumeration of possible approaches to the mathematical problem, which of course may be both the students' understanding 'as teachers' and their ideas for explanations to be given to, or found by, the hypothetical pupils in class.
- A general typology of approaches to the classroom.

When we look at these two parts of the answers separately, we consider mainly the students' mathematical and pedagogical knowledge and competence in their pure form. But in school reality – and as a consequence of the design of HTT's – they do not *appear* separately and the true interest is in the interaction and relation between the two, which can be said to be where the students demonstrate *didactical* knowledge and competency.

Specific analysis of HTT1

In HTT1, the mathematical question concealed in the pupils' responses is *according to what principle(s) do the sized of a figure (a triangle) change when it is enlarged in such a way that the 'form' (the angles) is preserved?* As the task is concrete, with two different responses given, the most basic mathematical analysis is to recognise the right answer among the two; and one might then, in addition, be able to justify the choice by some principle, and, as further steps, to justify the principle by reference to institutionalised mathematical knowledge (e.g. a theorem), and to give elements of proof in such a setting. Finally, students may use analogies (rather than principles that could directly justify the answer) either using aspects of the concrete task or similar examples, or from other parts of mathematics or its application in a broad sense; we include three categories of this type that we found likely to occur, based on experience and knowledge of the curricula and practices of school and teacher education in the two countries. More precisely, the student answers (as done separately and jointly) could involve:

(1) Explicitly recognize the right answer (4,5cm and 4,5cm).

(2) Recognize and state a correct principle behind the right answer (multiplication, congruence, magnification), but with no justification.

(3) Justify the right principle by reference to a mathematical result (Thales' theorem or similar knowledge), without providing an explicit proof of this result.

(4) Justify the right principle by proving it explicitly (something like proof of Thales' theorem).

(5) Recognize and state the wrong answer (5cm and 5cm).

(6) Explicitly recognize a principle behind the wrong answer, with no counterexample.

(7) Give counterexample to additive principle (besides recognising it).

(8) Discuss the concrete aspects of the photograph and the map to be constructed from it.

(9) Refer to other applications of magnification.

(10) Proceed to talk about other aspects of triangles and plane geometry.

Some of these are logically dependent, e.g (2) contains (1) in the sense that one would not give a principle justifying an answer without actually providing the answer. In fact, we have

$$(1) \subseteq (2) \subseteq (3) \subseteq (4) \text{ and } (5) \subseteq (6) \subseteq (7)$$

and so, for instance, stating that a student pair obtained (3), implies automatically that they got (1) and (2), which is then omitted in the rough analysis of their answers. The difference between (2) and (3) is mainly that while a 'principle' or technique might be taught as a natural or empirical fact, without any mathematical explanation, we mean by (3) that reference is made to a named, explicit result that could be proved elsewhere in the curriculum. To make a direct proof (4), the mathematical results referred to (3) would most likely include more elementary ones, such as the area formula for a triangle in the Euclidean proof of Thales' theorem. Notice that for assigning the code (7), we require mainly that an example is given where adding 'obviously' does not preserve form; typically this would imply exhibiting a triangle which is quite far from being isosceles. We expected that some of the French teacher students would refer to the 'puzzle situation' (Brousseau, 1997, pp. 177–179) which is a classical didactic situation allowing to detect and reject the additive principle.

Specific analysis of HTT2

Again, this HTT is about a classical, central and difficult problem located somehow between arithmetic and algebra. It is sometimes called the *mysterious law* (e.g. Allenby, 1983, p. 12) because it is hard to justify at the level where it's normally introduced (between 5 th and 7 th grade in Denmark and France) and quite impossible to relate meaningfully to real life use of multiplication and negative numbers. As a result, it's often taught is a convention, a rule one has to accept. We believe a teacher in lower secondary school should know more, exactly because we know that the situation in HTT2 can arise. In fact, it is often taken up in Danish teacher education, it might be in the French programme (although perhaps just in a second year algebra course at a level of abstraction which the students don't easily connect with school arithmetic). And certainly the explanations that the student teachers would think of are likely to be related to their knowledge of school arithmetic and algebra (the latter being taught to some extent in both countries).

Based again on our knowledge of programmes and experience with teacher students, we arrived at the following inventory of explanations that the students might suggest:

(1) Justification by 'number patterns', e.g. look at $n \cdot (-3)$ for $n = 2,1,0$.

(2) Justification by drawing lines, such as $y = -2x$ or $y = (-2)(x - 3)$.

(3) Justification by considering the equation $(-2) \cdot (x - 3) = 0$, which is equivalent to $x = 3$ and to $(-2) \cdot (-3) = 2x$, so that $(-2) \cdot (-3) = 2 \cdot 3 = 6$. This is close to (5).

(4) Justification by 'parenthesis magic': $(-2) \cdot (-3) = -(2 \cdot (-3)) = -(-6) = 6$. A variant involves a reduction to

multiplication by -1:
$(-2)\cdot(-3) = (-1)\cdot(2\,(-3)) = (-1)\cdot(-6)$, and this is 6 'because (-1) changes the sign'.

(5) Justification by 'distribution' [in fact, this is a 'proof' if accepting $2\cdot(-3) = -6$], such as:
$0 = (2-2)\cdot(-3) = 2\cdot(-3) + (-2)\cdot(-3) = -6 + (-2)\cdot(-3)$, so
$6 = (-2)\cdot(-3)$.

(6) Justification by 'technology': try it out on your calculator. It's surely cleverer than you.

(7) Justification by 'tradition' or 'authority': This is a well-established convention, trust me.

(8) Justification by (false) real life example, such as: "Twice, you remove a *deficit* of three euro from my bank account. What is my net result?"

(9) Other false explanations, including those working for addition but not multiplication in general, such as visualisations using a number line.

(10) Like (8) or (9), but realising explicitly that such explanations don't work.

Clearly, more than one explanation might come up in the discussion.

General approaches and interaction with mathematical analysis
The teacher students' discussion on *approaches* for how to proceed in class may include, besides providing directly one of the explanations given in the specific analysis, one or more considerations involving the pupils. Among which we particularly noted the following types that could indeed be relevant for both HTT's and many situations where a problematic question is to be tackled :

(a) provide several explanations to pupils

(b) organise a class discussion

(c) organise activities involving technology (calculator, software...)

(d) make pupils work on more examples

(e) build on previous knowledge of the pupils

(f) use different representations (e.g. algebraic and geometrical) for the same objects or relations

Again, none of these possibilities are mutually exclusive, and could occur more or less explicitly during the students' discussion.

Results and analysis

As already mentioned, we gave the HTTs to 5 pairs of teachers students in each country. Table 2 gives a rough summary of the results, according to the coding scheme previosly discussed; numbers in parenthesis indicates that the corresponding strategy or argument was merely mentioned or just vaguely suggested, without any detail that could be used to identify its exact nature. For each pair, we also indicate their answer to the final question about the principal sources of knowledge they had drawn upon.

Table 2. *Overview of outcome of HTTs.*

Respondents country/initials	HTT1, math. task analysis [1]	HTT1, class approach [1]	HTT2, math. task analysis [1]	HTT2, class approach [1]	Sources of knowledge [2]
DK / C&J	2, 8, 10	c, d	(9), 10, 7, (5)	(e), (b)	TT, OE, (HS)
DK / P&R	1, 7	d	8, 10, 7	(f), a	TT, TE
DK / L&I	10, 9, (2)	b, e	9, 7	(f)	TT
DK / L&J	2, 7	d, c	10, 7	(a)	TT
DK / P&A	2, 6	d, (b)	9, (5?), 7	a, e, f	TT, TE
F /T&B	2, 7, 9	a, b, c, d	1, 4, 5, 3	a, e	TT, TE, LS
F /W&P	(1), 3, 7, 8	a, b, c, e	1, 3, 4, 10	a, e	TT, TE
F /O&G	2, 7	a, d, (e)	1, 7, (8), 10	(e)	TT, TE, LS
F /K&H	2, 7	a, b, e	4, 10	(e)	TE
F /C&D	2, 7, 8, 9	a, c, d, (f?)	1, 4, 5, 7, 10	a, (e)	TT, TE, LS, (OE)

1 See previous sections for explanations of the coding
2 LS = lower secondary school, HS = Upper secondary school, TT = teacher training, OE = Other tertiary education, TE = teaching experience [eg. in practice periods within TT]

Main differences between the two countries

There are some striking differences between pairs from the two countries which are suggested already by looking at table 2 and further confirmed when looking more closely at the protocols:

1 As for the mathematical analysis of HTT1, the Danish pairs refer to the technique of multiplication as a natural fact, with no attempts of justification or link to a mathematical result. The French students want to present a generalised principle using tables and symbolic representations. One French pairs refer to the Theorem of Thales, but they do not consider to provide a justification of that theorem (probably they consider this is elsewhere in the curriculum). However, all French pairs provide explicit counterexamples to the additive principle for magnification, which appears only for two of the Danish pairs. In short, while all pairs – except one of the Danish pairs – recognise the correct result for the *pupils'* task, there is a considerable difference in the quality and detail of the mathematical analysis. This is also reflected in the notes prepared by the students individually during the first part of HTT1.

2 When it comes to the ideas for how to proceed in the class in the situation of HTT1, the main difference is that the French pairs discuss only the task at hand, and how to make the pupils understand it better; they all want to provide more than one explanation, in most cases as outcome of class discussions or group work with the two answers and more examples of the same principles. Three of the pairs explicitly refer to Brousseau's puzzle situation and other concepts of didactics. The ideas of the Danish pairs go in more diverse directions; besides letting the pupils discuss this and more examples, some would go on to other applications of maps and magnification or to other topics of geometry; none of them consider to provide more than one explanation of the principle.

3 As for the teacher students' own explanation of the mysterious rule (HTT2), the difference is quite striking. None of the Danish students' arrive at some form of a mathematical explanation (the codes 1–5), except that two of the couples talk loosely about 'the algebraic explanation' without giving any details (code (5) for C&J, P&A). All of the French pairs arrive explicitly at such an explanation, although some of these are informal, e.g. using number patterns (code 1); one of the pairs gets only to a reduction of the problem to multiplication by -1 (code 4). Three of the French pairs produced a genuine proof (codes 3, 5). All of the Danish pairs attempted, as did two of the French pairs, to produce an explanation using the number line (and corresponding visualisations valid for addition). When it comes to the conclusive step bearing on what to say to the pupil in HTT2, another striking difference is that *all* of the Danish pairs proposed to say that this is a convention one must accept

(code 7), while only two of the French pairs suggested that. It must be noted here that the all of the Danish pairs had studied this particular problem (how to explain product of negative numbers) during teacher training, while this does not seem to be the case for the French students; those who indicate precise sources of knowledge point at textbooks for lower secondary school with which they have been working in the last year of teaching practice.

4 In both countries, most of the discussion of HTT2 is spent on the mathematical problem of explaining the product, which – for all teacher students in both countries – turns out to be quite challenging. Thus, in both countries the inventory of approaches to class intervention is somewhat sparse. In Denmark, most of the pairs will only discuss the problem with the inquiring pupil; as mentioned, they will tell him that this rule is a convention one must accept. The French pairs consider explanations to give to the whole class (as part of the *cours*), to the extent they are explicit about an approach. All of the French pairs refer, at least implicitly, to what the pupils could be expected to know in 9^{th} grade, while this is only the case for one of the Danish pairs.

5 When it comes to class approaches, the French students are generally more likely to emphasise the need of providing several explanations to the pupils (this is said explicitly in all cases except for two pairs on HTT2 who do not find more than one explanation for this task). In HTT1, the Danish students are more in favour of enlarging the perspective and add more examples for the pupils to work on, while in HTT2 they emphasise the use of both visual and symbolic representations.

6 A more general difference, partially reflected by the previous points, can be found in the teacher students' main focus in the discussions. The French students mainly focus on the mathematical problem and, especially for HTT1, on how to organise a presentation of solutions; the Danish students use considerably more time to discuss how to engage and activate the pupils, particularly in a real life context, and often with rather vague relation, if any, to the hypothetical situation. To illustrate this, we provide two characteristic passages from two pairs' discussion of HTT1. First an excerpt from the discussion of a Danish pair:

D1: ... one may work more on maps, can you, can you develop that in general, for instance to ratio of measures, to, for instance, if we look at different maps, right ...

D2: Yes.
D1: The distances on maps, not only, yes, that is, one could of course begin with working with lengths on the map we have here ...
D2: Yes ...
D1: And say, can we find the length of a road, or something, right? And then simply, work on into, that is, if we must find the distance between Copenhagen and Paris... on the map of Europe, one could work with ... year, and, as I said, lastly, drive them over into geometry in the end.
D2: Yes.
D1: And try to talk of, yes, what we could call triangles ...
D2: Pythagoras, perhaps ...
D1: Yes, good old Pyth, or something.

And, a French pair:

F1: If it's 8 cm on the photo, how much on the map with +2cm, 10 cm. 4 cm gives 6, 8 gives 10. The length is twice more. See that one cannot add 2 because one will not respect proportions.
F2: You take 4
F1: If you have 8 on the photo, 10 on the map; will you preserve the same proportions?
F2: It's logical. Change the didactic variables
F1: It's interesting to see they mistook, that they made this error.
F2: Give them that in a moment of activity
F1: You give them that, you make them search; you take one of each type to the blackboard. One recalls, to remember, one sums up: one establishes that it's about proportionality. Maybe someone hasn't noticed that at all.
F2: I don't know if they have seen proportionality.
F1: They have seen that in 5ème [the French label for *7th grade*]. But the notion of scale, I don't know. Understand that a scale, it's related to proportionality ...

It should be noted that the French curriculum defines more strictly what is taught at each grade, and indeed several of the French pairs refer to what the pupils will surely know at the given grade for each task. The Danish students don't refer (explicitly) to the official curriculum.

Similarities and subtle differences

For both tasks, it's remarkable that all of the 10 pairs have to work considerably on the elementary mathematical problems involved. In fact, they are just a few months from assuming regular teaching duties where such mathematical and didactical challenges can be assumed to arise.

It seems that the geometry problem is easier for all of them, even if there is, in both countries, one or two of the teacher students who fail to identify the right answer (two Danish students in the same pair, and one student in France who is corrected by the other in the pair). But none of the 20 students has a 'ready made' strategy for this situation. And none of the pairs produce (perhaps, know?) a complete argument for the relation between multiplication and proportion, neither for themselves nor as something that could be explained to, or developed with, the pupils. In fact, only one of them (a French student) expresses the need for a direct, positive argument in favour of the 'multiplication principle'. The main explanation that is proposed to decide between the two pupil suggestions is a *counterexample* to the 'addition principle' where the distortion of geometric form is to be 'seen' directly. As already mentioned, some of the French students refer to a mathematical result (theorem of Thales) that might be proved elsewhere; for those one may recognise indirectly an acknowledgement that there is and should be a positive argument.

By contrast, in the context of HTT2, all 20 students somehow manifest a need for an explanation of the 'principle' (*in casu*, for multiplying negative numbers), and they all work hard in order to find one. As already mentioned, some pairs don't succeed to find a reasonable and explicit explanation (all of the Danish pairs and one of the French). But among those who fail and realise this themselves, there is an interesting difference. The French pair is deeply dissatisfied:

S1: ... one realises that one has a gap. A gifted pupil asks me that, I am unable to respond. In my memory, I don't recall how this was explained to me.

S2: But ... we who did advanced studies!

The two students are perplex that they are unable to explain why $(-2)\cdot(-3) = 6$, after several years of advanced studies in mathematics. In fact, on top of the 3-year *licence* (B.Sc.) degree, S2 also did a masters degree in pure mathematics and maybe will proceed to get a Ph.d. They do not doubt that an explanation could and should be given, and they also realise that their studies have not enabled them to find one. It's interesting that they primarily refer to their studies of pure mathematics, while it would be more plausible for such topics to be treated in the IUFM part of their formation.

It's quite surprising that all five French pairs insist on the importance of finding a 'real-life' situation to explain the sign rule; given their mathematical formation, one could expect them to be searching more for a mathematical explanation of the non-arbitrary character of this convention. A possible explanation is that, contrary to the principle in HTT1, they do not find any 'evident' geometric illustration of the rule, and so an explanation cannot rely on a coordination of geometrical and numerical representations (cf. Duval, 2000).

The Danish pairs have encountered this problem in their own formation, and the two pairs who explicitly realise that their 'explanations' are false or just mnemonic rules, seem to conclude that the problem is somehow too difficult for them and hence for their pupils. As one of these four students say:

> ... I haven't yet read or heard any [explanations why minus times minus is plus] /.../ not any I could totally accept, there was always some trick "then we do like that" ... to be totally realistic, I would get around it ... I would simply say: that you must swallow.

The students in this pair agree that there are many things in mathematics which one must just accept, and their best bet for proceeding is to provide the pupil with some historical reasons for the rule to come about (although they don't mention any explicitly). A few minutes later when they are asked about their studies and background, the student quoted above expresses very critical feelings about her upper secondary school experience with mathematics, because it was "rule-governed" and "about remembering a lot of things".

Overall conclusions of product study

The French students provide a more precise and mathematically rich discussion of the two basic problems proposed through the tasks, and they consider providing the pupils with a variety of explanations. They are clearly superior to the Danish students when it comes to identifying and analysing the mathematical problems faced by the 'imaginary pupils' in the two situations. The Danish students focus more on the classroom situations as such, and they are more imaginative when it comes to developing them in terms of pupil activities, adjacent topics one might treat, etc. They are more likely to present the two mathematical principles as 'rules' or facts, and then use examples (if possible from real life) to help pupils understand what these rules mean in a context that seems relevant and interesting to them.

The two HTTs can be said to have each three levels, corresponding roughly to the three elements of teacher knowledge introduced.

(1) The mathematical problem involved.

(2) The didactical task, to support the pupils' learning in the concrete teaching situation.

(3) The pedagogical task, to manage the classroom.

Clearly (2) depends on (1) and, for the realisation, on (3). The French students do not seem to think much about the motivation of pupils or other aspects related to (3), and so their inventory of classroom organisations is limited: they propose mainly direct explanations to provide, although for HTT1 some propose to organise discussions among pupils of the alternative answers. The Danish pairs consider all three points in every task, but the weaknesses of their mathematical analysis typically prevent them from addressing the didactical problem in a precise way and consequently to propose a didactic intervention with precise aims.

Discussion of possible relations between the two studies

Let us begin by emphasising that the survey conducted does not, by itself, allow for definite conclusions about the teacher training systems in Denmark and France, especially because of the small scale (10 students from one institution in each country). The main point of this paper is methodological: to show how HTT based studies may help to enlighten the relation between mathematics teacher education systems and the knowledge students achieve.

Both HTT1 and HTT2 present teacher students with a situation where their task is to consider how they, as teachers, could proceed to help the (imaginary) pupils learn. The descriptions of the situations include, besides the grade of the pupils, essentially just the local mathematical context the pupils work in, and a problem they encountered. What could be expected from the teacher students is then (1) a reflection on the nature and solutions of this problem, leading to (2) reflections on explanations, questions etc. to propose in order to support pupils learning, and (3) reflections on the modalities of classroom organisation that would be realistic and useful to implement the ideas for learning support. In HTT2, there is no indication of 'subtasks'. In HTT1 we explicitly ask students to analyse the pupils' answers (which mainly points to (1)). None of the tasks presuppose a strict order in attending to those three elements.

It seems clear that the overall priorities of the two teacher training programmes can be recognised in the students' approaches to (1). The French students consistently approach (1) with the aim of finding one or more *reasoned explanations* of the mathematical problems involved. The Danish students primarily search for applied or otherwise concrete examples and illustrations.

The picture is not so clear when it comes to (2). All pairs in both countries have relatively more to say about this point for HTT1. For the French students this can be related to the treatment of similar triangles in the last two years of formation and in the French curriculum for lower secondary school, while didactical aspects of the sign rule are not normally treated in teacher training. The Danish students all say that they have worked explicitly on the topic of HTT2 in their formation, but none of them propose a reasonable explanation or activity to help the 'gifted' pupil who asks why $(-2)(-3) = 6$. Some say explicitly they did not find any of the arguments they saw 'convincing', although they do talk about an 'algebraic explanation'; one may speculate that they simply don't have much experience with situations where a rule may be reasoned by deduction from a more 'elementary' or 'basic' convention. Indeed, proof plays a minor – if any – role in their formation. Clearly, a deduction based on the distributive law may not be the right way to convince a 9th grade pupil, but knowing of such an argument might at least convince themselves that the rule is not an arbitrary convention one must just accept – but this is what they will tell the pupil. It should be noted that also for some of the French participants, the sheer difficulty of (1) is an obstacle to discuss the didactic intervention in the situation.

The Danish students pay considerably more attention in their discussions to the pedagogical aspects of the situation. Even for HTT2, when they decide to say it's a convention to accept – they discuss whether to include the whole class or just deal with the inquiring pupil individually (which most prefer, "in order not to confuse the others" as one student says). Their ideas for HTT1 involve rather precise ideas about the class management, such as organising group work on more examples, including measurements 'in reality' to be used in drawing maps. However, these ideas do not involve very precise examples or strategies related to the principle of magnification, except for the two pairs who exhibit examples that would visualise the defect of using addition to magnify triangles. They are generally well articulated on the general pedagogical principles that are an important part of their formation and outlook on the profession of teaching, but these remain quite loosely linked to the didactical analysis of the two tasks.

The French students are not nearly as focused on the implementation issue. When they talk about how to proceed with pupils, they consider mainly what these 'should know' according to the curriculum; and so the relation to the classroom is also thought in terms of structures of knowledge. Their ideas for classroom activities – to the extent they do get to talk about that – tend to be somewhat timid and vaguely related to their reflections about the mathematical knowledge, even if the latter are quite rich.

To sum up, students in both countries have difficulties to propose precise and argued interventions in the classroom, for different reasons that seem to be clearly related to the differences of their formation. The advanced mathematical formation of the French students is a potential which depends, to be unfolded, on specific didactic and pedagogic experience, as the difference in performance between HTT1 and HTT2 illustrates. The Danish students are more articulate and aware of the general dynamics of the classroom, including general problems and strategies of managing it, but their difficulties with the concrete mathematical challenges involved – particularly in HTT2 – is an obstacle for them to formulate precise strategies for intervention in concrete situations. In both countries, the students have relatively limited 'hands-on' experience with teaching, but even so one could speculate that their potential for becoming efficient teachers would be enhanced by

- an earlier and more comprehensive work with didactics and pedagogy in France, paying attention to the potential of advanced mathematical knowledge, and
- a considerably broader basis in both mathematics and its didactics in Denmark, while maintaining the attention to students' awareness and knowledge about pupil perspectives.

Of course, such modifications would be constrained by institutional conditions, as well as broader conceptions of aims and meaning of the school as earlier described. Educational institutions and norms are still quite different in the two countries, even if the European integration process may in the long run lead to a convergence enforced 'from above'.

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Notes

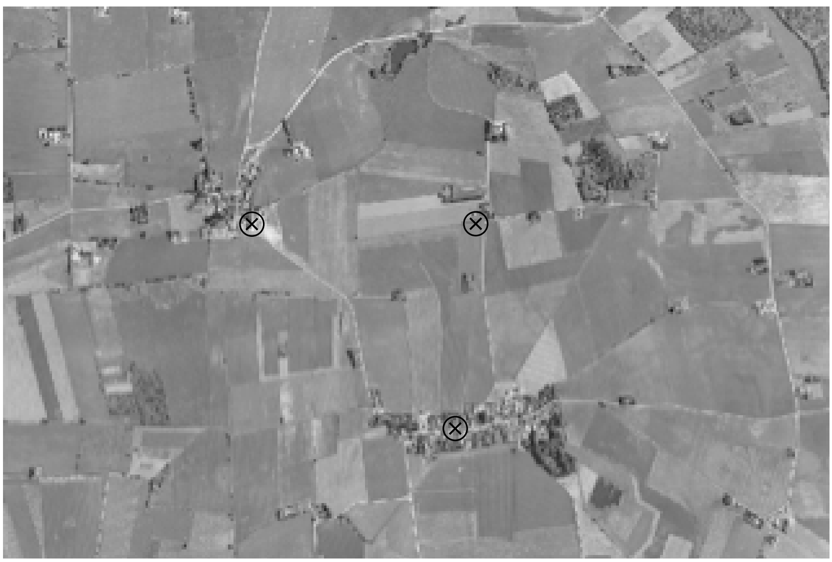
1 A preliminary version of this work was presented at the 15th ICMI study conference in Águas de Lindóia, Brazil, May 2005.

Appendix 1. Hypothetical teacher tasks

HTT 1 (translated from Danish/French)

You assign the following task to your 8th grade pupils:

> An aerial photo is used to draw a map. To begin with, three points are marked on the photo; the distances between these points are 4 cm, 3 cm, and 3 cm. The map must be slightly larger than the photo: the longest distance between the three points should be 6 cm on the map. What should the other two distances be on the map?



Some pupils answer: "5 cm and 5 cm"; others say, "4.5 cm and 4.5 cm".
First task for the teacher (*to be solved individually within 10 minutes*)
Analyse the solutions. What would you do as teacher in this situation? Please take notes.
Second task for the teacher (*to be solved in conversation with the other teacher student, 20 minutes*)
Please, discuss your ideas with respect to using this situation to further the pupils' learning.

HTT 2 (translated from Danish/French)

Your pupils are working in class (grade 9). They encounter at some point the need to calculate the expression $(-2)(x - 3)$ to produce the expression $-2x + 6$. A pupil, whom you know to be rather gifted, calls on you and says:

> "look, I have arrived at this"
>
> [he points the expression $-2x + (-2) \cdot (-3)$ in his notes]
>
> "I know the last term should be 6. But then I began to doubt – why is it so?"

The lesson is about to end. You decide to postpone the question until the lesson tomorrow, and say:

> "Yes, that's a good question. Let us come back to that tomorrow".

Teacher task (*to be discussed in pairs of teacher students, in 20 minutes*). Imagine you are in the teachers' lounge, and discuss the problem with your colleague: how could you make this question an opportunity to learn?

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Sammendrag

Denne artikel præsenterer et komparativt studium af to overraskende forskellige matematiklæreruddannelser til det indledende sekundære niveau (12–15 årige), nemlig den danske og det franske. Vi giver først en kort beskrivelse af de to uddannelser. Hovedparten af artiklen præsenterer et kvalitativt studium af hvordan lærerstuderende i det afsluttende studieår håndterer to hypotetiske situationer i matematikundervisning. Dernæst diskuterer vi hvordan resultaterne af dette studium kan relateres til de to uddannelsers indretning, som først beskrevet.