# Children with impairments learning numbers

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This research investigates children's understanding of numbers when solving addition and subtraction word problems. A comparative approach – studying children who are blind, children with a hearing impairment and children without these impairments aims at illuminating and describing the differences and similarities between the three different groups. Thereby, the investigation contributes to the understanding of critical aspects characterising development of numerical competence in each of the three groups. The design of the study also makes it possible to describe children's development of number concepts in the context of solving addition and subtraction word problems on a more general level.

Children's early forms of mathematical understanding develop gradually and must eventually be integrated with an understanding of the abstract symbolic language of mathematics if the child is to be able to develop mathematical skills (Resnick, 1994). The idea that mathematical knowledge develops through sequence of stages characterised by changes in representational content – that is, the kinds of conceptual entities that are recognized and reasoned about – is shared by a number of theorist (Fuson, 1988; Steffe, Cobb & von Glasersfeld, 1988; Steffe, von Glasersfeld, Richards & Cobb, 1983). Most of these analyses have focused heavily on the representations underlying counting and the transitions from counting objects to reasoning about cardinalities of sets (Resnick, 1994).

Different views on children's development of number concepts that emphasise various aspects can be identified on a general level. As well as looking upon the act of problem solving itself and the use of language

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and interaction to be an important tool (Ahlberg, 1992, 1998; Cobb, 1995; Ekeblad, 1996; Ginsburg, 1983), a genetic point of view can be taken where the development of number concepts is seen as a biologically anchored (Gelman & Gallistel, 1978, 1983; Gelman & Meck, 1986; Gallistel & Gelman, 1992; Wynn, 1992ab). Abilities such as subitizing and pattern recognition are outlined and considered to play an important role (Bermejo et al., 2004; Brissaud, 1992; Fischer, 1992; Klahr & Wallace, 1973; Neuman, 1987). Hannula (2005) shows that within a child's existing mathematical competence, it is possible to distinguish a separate process, which refers to the child's tendency to spontaneously focus on numerosity. However, the notion that number concepts arise from the activity of counting and reflective abstraction also is carefully described (von Glasersfeld, 1993, 1996; Steffe et al., 1988; Steffe, 1992) and so is the significance of understanding the number sequence (Fuson, 1988, 1992ab; Ginsburg, 1983; Resnick, 1983). Furthermore, the relationship between conceptual and procedural knowledge is outlined (Hiebert, 1986) as well as the importance of an interaction between concepts and procedures (Gray & Tall, 1994).

## Children with impairments and mathematics

In studies on children with impairments the research interest often is directed towards mapping the children's mathematical skills to identify the specific difficulties children have and in figuring out the delay in acquiring knowledge compared to children without impairment. According to Best (1992) children who are blind – as they do not have access to the co-ordinating sense of vision – do not experience phenomena in the world simultaneously. To explore the world they need to examine elements in a sequential manner and use their hands and fingers to touch the objects around them. Another characteristic is that there are many concepts which can only be developed through logical thinking and many researchers emphasise that the most important thing for children who are blind to learn arithmetic is to develop powers of logical thinking and the ability to remember (Deloche & Seron, 1987).

In the last 20-30 years most of the studies concerning mathematical skills in blind children deal with classification and conservation, using tasks adopted from Piaget's work. The main interest of the research has been to answer whether cognitive development of blind children is behind that of sighted ones (Hatwell, 1985; Stephens & Gruve, 1982). Many studies point to a delay of two to four years. However, blind children almost catch up with sighted ones when they are between eight and eleven years old (Warren, 1984, 1994).

Ahlberg and Csocsán (1994) conducted a case study in a phenomenographic research framework on how congenitally blind children aged seven and eight solved mathematical problems. The results show that none of the children who are blind spontaneously used their fingers when solving arithmetic word problems. When trying to grasp a number of elements by touch, the children used three different strategies: (1) counting, (2) counting and grouping, and (3) structuring. The results show that being able to group and structure the numbers in a part-whole relation is strongly linked to the development of number concepts among blind children.

Research on hearing impaired children and mathematics has also pointed to the inferiority of hearing impaired schoolchildren in arithmetic achievement compared to age mates with normal hearing (Allen, 1986; Frostad, 1996; Wood, Wood & Howarth, 1993). However, in contrast to the children who are blind, the difference will not disappear as the hearing impaired children grow older. Some explanations have been suggested that often fall in one or two main categories. Researchers are either focusing on the comparative effect of different teaching methods on learning (Heiling, 1994; Kluwin, 1993; Wood, 1991) or they are focusing on how deaf children's cognitive profiles influence their academic achievement (Emmorey, Kosslyn & Bellugi, 1993; Furth, 1966; Myklebust, 1964). Frostad and Ahlberg (1999) took children's experience as a point of departure in describing how children with a hearing impairment master three different types of elementary arithmetic problems presented in a non reading format. They outlined the effect of task-specific factors on the level of difficulty and the children's understanding of problem structures. The results show that the semantic structure of problems affected the level of difficulty. Foisack (2003) studied how deaf pupil's way of expressing themselves in sign language influenced their concept formation. It was found that the structure of sign language could be of help but it could also be an obstacle in mathematics.

## Numerosity from an experiential perspective

In the current investigation the aim was to study the different ways in which children experience and discern numbers focusing on different aspects and possible qualities, which in this context is called numerosity. The research interest was not to compare the outcome of children's problem solving attempts, instead it was to reveal the variation in the ways in which the children are handling and experiencing number [1]. The investigation was conducted within the phenomenographic research tradition. Phenomenography is the empirical study of the different ways in which people understand and experience various phenomena in the world around them from a non-dualist view of human cognition. This approach depicts experience as an internal human-world relationship where understandings of different phenomenon are constituted in the relationship between the individual and the world (Marton, 1981, 1994; Marton & Booth, 1997). The phenomenographer seeks to interpret and to understand, and primarily to describe people's experiences in categories, so called categories of description, that represent the relationships between the subjects and the phenomena. These categories describe the variation in ways of experiencing and encapsulate the distinctly different ways in which the subjects experience the phenomena in the sense of what they see as its meaning, and how they distinguish it from other experiences. The research interest is directed towards exploring variation applied to a group or population as a whole. Individuals are seen as the bearers of different ways of experiencing a phenomenon and the description reached is a description of variation on the collective level.

According to Bowden and Marton (1998) to experience something implies discerning it from the context of which it is a part relating it to other contexts. Marton and Booth (1997) suggest that the capability for experiencing a phenomenon in a certain way can be understood in terms of a discernment of aspects, the simultaneity of discerned aspects and a potential for variation in discerned aspects of the phenomenon. The aspects of a phenomenon and the relations between them that are discerned and simultaneously present in the individual's focal awareness define the individual's way of experiencing the phenomenon.

Ahlberg (1992) adopted this approach in a study examining nine-year-olds solving verbal problems in a classroom context. The dynamic nature of the problem solving process was analysed in terms of what different aspects of the problem were focal and simultaneously experienced. The children managed to different degrees to discern and keep all relevant aspects of the phenomenon – and of the situation – in focal awareness simultaneously. The children's simultaneous discernment was central in the problem solving process and determined whether the children were able to solve the problems.

When exploring qualitatively different ways of experiencing something you have to deal with differences in meaning and difference in structure – referred to in terms of referential and structural aspects (Marton, Beaty & Dall´Alba, 1993; Bowden & Marton, 1998; Marton & Ramsden, 1988). Describing in which way a phenomenon – such as number – is experienced by children, could imply revealing the various referential and structural aspects of number children are focally aware of in the situation. In this context the structural aspect refers to what the child focuses on when solving problems. The referential aspect refers to the meaning of numbers – structured in a certain way – constituted by the child. The two aspects are dialectically intertwined. The way the phenomenon is structured influences the meaning of the phenomenon, and the other way around.

### *Three studies*

The investigation includes three studies – each focusing on a specific group of children. In the first study 38 Swedish children *without visual or hearing impairments* in their final year at one of three different pre-schools participated. At the time of the interviews the children had not received any formal instruction in mathematics (Ahlberg, 1997).

The second study included 25 Hungarian children who are *congenitally blind*. All of the children were born with visual impairment and did not have any other disability. The children attended the Hungarian school for blind children (Ahlberg & Csocsán, 1997).

In the third study 31 Norwegian children with a *hearing impairment* were interviewed. The participants were selected from all available children attending special educational units in the two largest cities in Norway. Only children with Norwegian Sign Language (NSL) as their first language participated in the study [2] (Frostad & Ahlberg, 1996).

The ages of the children in the three groups were not quite the same. All children without impairment were between 6 and 7 years old, while the age of the children who are blind varied from 5 to 9 years and the hearing impaired from 6 to 10 years, as shown in Table 1.

Table 1. *Number of children in different age ranges*

| | Children without impairments | Children who are blind | | | | | Children with hearing impairments | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| age | 6-7 | 5 | 6 | 7 | 8 | 9 | 6 | 7 | 8 | 9 | 10 |
| *n* | 38 | 1 | 6 | 6 | 10 | 2 | 5 | 6 | 7 | 7 | 6 |

There were two reasons for selecting children of different ages. Firstly, the selection of participants with impairments was limited because these groups of children were rather small. Secondly, the research interest was to describe a variety of ways of experiencing, and as there is a gap in arithmetic skills between children without impairment and the other two groups, it would benefit to the study if older children with impairments were participating.

### *Data collection*

Data was collected through task-based video taped interviews in all three studies. The interviews with the children without impairments were carried out over a two-week period in a separate room at the children's preschools and lasted between 25 to 60 minutes. The interviews with the blind children lasted about 40 minutes and took place in a separate room at the school for the blind. Each child with a hearing impairment was interviewed in two sessions of approximately 30 minutes each, on two different days within the same week. The interviews were held during a period of 2 months. In the study with the hearing impaired children two video recorders were used during the interviews, in order to have an ”en face” recording of both the interviewer and interviewee. The two recordings were later edited, so that the picture of the interviewer was integrated as a small picture in the upper corner of the picture of the interviewee. Based on the edited video recordings the dialogue was translated into Norwegian oral language, and transcribed. Each child was interviewed in two sessions of approximately 30 minutes each, on two different days within the same week.

The problems in all the interviews were introduced by the researchers, who also critically followed up the answers to the questions by a careful questioning and listening approach trying hard to understand and respond to the children's acting and thinking by asking accurate ”follow up” questions. The intention was to get the children to expound upon and elucidate their answers and explain in more detail how they went about solving the problems. When solving the problems the children did not have any manipulative objects to handle.

### *The tasks*

The children in the three groups were given the same problems for exploration, however the mode of the presentation was somewhat adapted to the children's differing sensuous experience. The numerical content in the tasks was imbedded in three different contexts in which the semantic content and structure of the problems varied. The first kind of task involved ”every day problems” well known to the children, such as getting or losing money, fruit, or crayons. They were asked to solve one addition, one subtraction, and one missing addend problem, in the number range 1-5. Thereafter the children were presented with problems of the same structure in successively larger number ranges. This kind of problems was presented to the children from all three groups in the same way.

The second kind of tasks, ”decomposition problems” aimed to show if the children were able to decompose a specific number into parts and wholes. In the study with the *children without impairments* and the *hearing impaired children* the interviewer initially put 9 buttons and 2 boxes – one white and one black – on the table and asked the child: Can you count how many buttons there are altogether? Then she said: Now I want you to help me to figure out what we can do, if we want to put these 9 buttons into these 2 boxes. How many buttons can we put in each box? In the interviews with the *children who are blind* the interviewer used her hands instead of boxes. When asking how many coins in each hand she made the children touch and feel her hands. The children who were not able to give an answer with nine buttons were presented with the same task, but with only five buttons. The *hearing impaired children* who easily solved the task were also asked to divide 13 and 18 buttons.

The third kind of problems, ”contextual problems” was preceded by a talk about the activities of a baker and a thief who stole buns. To make it easier for the children to understand the content of the problems they were initially shown a plate with a number of paper buns and allowed to solve the first problem of this kind by using these manipulative objects. This way of introducing the problems differs from the other tasks and is the reason why it is called contextual. The problem used to introduce the tasks was: There are 8 buns on the plate. When the baker comes back there are only 2 buns. How many buns has the thief taken? It was only when solving this very first problem that the children had access to the buns and these answers were not included in the empirical material. After solving the first problem the interviewer gave the children other problems concerning the story of the baker using increasingly larger number ranges. In the study with the hearing impaired children these tasks were presented through an animated computer game, so that the level of linguistic competence should not affect the children’s understanding of the content of the problems.

### *Analyses and interpretation*

The intention in the interpretation was to reveal the children’s acts and words exposed while solving the problems. These analyses served as a basis for trying to reveal the meaning of numbers constituted by the children and what aspects of numbers children were focally aware of in the problem solving situation. The preliminary analyses showed that there was a wide variety of different ways in which children handle numbers and that a single child might shift between ways of handling. These interpretations were explored in more depth by analysing the different ways in which the children understood the meaning of numbers in an integrated and interactive analysis relating the child's way of handling numbers solving one specific problem to how she/he goes about solving all of the given problems.

### *Reliability and validity*

Phenomenography is one of many different research approaches used when studying children's development of numerical competence. Unlike most earlier research an intention in the current investigation was to make a difference between what children do when solving arithmetical problems and how they experience the meaning of numbers. In order to capture meaning an interpretative approach had to be used, in this case phenomenography seemed to be an appropriate choice. However, when using a theoretical approach grounded in understanding and interpretation you can not expect to find the one and only truth, but to make the findings as credible as possible. The following description of how children are handling and experiencing numbers is one among others dealing with number concepts. Analyses made from different perspectives can however complement each other and contribute to a broader and richer understanding of children learning numbers.

Looking at this investigation in terms of credibility the intention has been to follow the guidelines set by Säljö (1988) in validating the conclusions by discussing the internal logic of the children's different ways of experiencing and to verify the results arrived at by conducting a comparative discussion of the findings and the results of other research in the field. A number of extracts and quotes are presented in order to make it possible to judge the validity of the description and to open the findings to scepticism and alternative interpretations [3].

## Various ways of handling and experiencing numbers

The results from the comparative analyses show that there is a rich variation in children's ways of handling and experiencing numbers between and within the three groups and that the ways in which children experience numbers are related to their ways of handling them. However, this is not in a one to one correspondence. Some ways of handling numbers may be linked to more than one way of experiencing.

The children's experience of number is revealed by exploring what aspects of numbers they are focally aware of in the problem solving situation. These are described in terms of *referential* and *structural aspects*. The structural aspect refers to how the child understands the structure of numbers and the referential aspect refers to what meaning numbers – structured in a certain way – has for the child. The two aspects are dialectically intertwined and the relation between them constitutes the children's experience. Five different ways of experiencing numbers are identified: Numbers are experienced as Number Words/Number Signs – Extents – Positions in Sequence – Grouped Units – Composite Units. Within these categories there is a substantial amount of different ways in which the children are handling numbers, as shown in Table 2.

### *The experience of numbers as number words/number signs*

Children experiencing numbers as *number words/signs*[4] give answers without counting or using some kind of procedure for exact quantification, even though they do not know the correct answer. They are *saying numbers* referring to an unprecise numerosity – referential aspect – and focus on individual or successive numbers in the standard number sequence – structural aspect – when solving the given problems. Children are aware of the fact that the answer should be a number and they recognise the numbers in the given problems. Although, they know that numbers refer to a numerosity, they do not have an understanding of either an approximate nor a precise numerosity and they do not understand the cardinal meaning of the particular number words.

Wynn (1992ab) outlines that even by the age of eight months children know that the counting words refer to a numerosity, though they do not yet know to which numerosity each word refers. According to Wynn the two essential components for understanding the cardinal meaning of a particular number words are: (1) the knowledge that a number word refers to numerosity, regardless of which it refers to, and (2) the knowledge of the precise numerosity a word refers to – its cardinal meaning. The experience of numbers as number words is found in connection with all the different types of problems presented. The experience includes three different ways of handling numbers – *saying random numbers – saying equal numbers – saying successive numbers*.

#### **Saying random numbers**

One way of handling numbers when experiencing them as number words is to *say random numbers* either inside or outside the given number range. This is an uncommon way of trying to solve the problems and it is mainly the younger children who give a random number as an answer, except

Table 2. *Children's ways of handling and experiencing numbers*

| |
|---|
| THE EXPERIENCE OF NUMBERS AS NUMBER WORDS/NUMBER SIGNS |
| **Ways of handling numbers – Saying numbers** |
| Saying Random numbers |
| Saying Equal numbers |
| **Referential aspect** – Unprecise numerosity |
| **Structural aspect** – Individual numbers |
| Saying Successive numbers |
| Referential aspect – Unprecise numerosity |
| Structural aspect – Successive numbers |
| THE EXPERIENCE OF NUMBERS AS EXTENTS |
| **Ways of handling numbers – Estimating** |
| **Referential aspect** – Approximate numerosity of parts and whole |
| **Structural aspect** – Size of numbers |
| THE EXPERIENCE OF NUMBERS AS POSITIONS IN SEQUENCE |
| **Ways of handling numbers – Counting** |
| Double counting |
| Counting all |
| **Referential aspect** – Precise numerosity of parts and whole |
| **Structural aspect** – Individual and successive numbers |
| THE EXPERIENCE OF NUMBERS AS GROUPED UNITS |
| **Ways of handling numbers – Counting and grouping** |
| Counting and touching |
| Counting and hearing |
| Counting and seeing |
| **Referential aspect** – Precise numerosity of parts and whole |
| **Structural aspect** – Individual, successive and grouped numbers |
| THE EXPERIENCE OF NUMBERS AS COMPOSITE UNITS |
| **Ways of handling numbers – Structuring** |
| Seeing |
| Using derived facts |
| Using experienced number fact |
| **Referential aspect** – Compositional relationship of numbers |
| **Structural aspect** – Composed and decomposed numbers |

where the number range is unfamiliar where also the older children may use this way of handling numbers.

In focusing on individual numbers, the children do not reflect on the quantitative relations between the parts and the whole and the relations of the numbers in the problem. Sometimes the answers are given very quickly, Anna, a girl without impairments[5] answers rather quickly when she is trying to solve an addition problem in the larger number range.

I: If you have 13 kronor and get 5 more, how many kronor do you have?
A: 10.
I: Did you use your fingers?
A: No.
I: How did you do it?
A: Thought.
I: Did you think in any special way?
A: Just said it.

**Saying equal numbers**

Another way of handling numbers which is related to the experience of numbers as number words is to *Say equal numbers*, such as when the child answers with a number already mentioned in the problem. According to the content and structure of the problems, the *first number* or the *last number* mentioned in the presented problem could be the answer. When solving the ”decomposition problems” the equal numbers turns out to be equal parts.

Occasionally *children without visual or hearing impairment* try to use their fingers, but they do not really know how to go about this and therefore they are not helped by using their fingers to arrive at an answer. Most often they count the first part using their fingers and then give this as an answer. Anna (below) gives a correct answer to some of the addition problems by counting on her fingers. However, she has difficulty performing a subtraction problem – despite the help she got by counting on her fingers – and just focuses on the *first number*.

I: If you have 4 kronor and lose 2, how many do you have left?
A: 4.
I: Can you show me with your fingers?
A: 1, 2, 3, 4. [Counts four fingers on right hand].
I: You had 4 and lose 2?
A: [Still holds up the hand with four fingers].
I: How many do you have?
A: 4.

Other children focus on the *last number* mentioned in a in either addition and subtraction problems. An example is Rachel, a girl who is blind (5.5 years).

I: Imagine the buns. The baker baked 8 buns and finds only 2 buns when he comes back. How many were taken away?
R: 2.

When children solve the decomposition tasks it is not unusual for them to focus on *equal numbers* and divide in ”equal parts”. The size of the parts is most often within the whole set, although their sum could be greater. Janet, a hearing impaired girl (6 years) answers with equal numbers as a possible decomposition of five buttons.

I: If we have these 5 buttons and are going to put them in these two boxes, how should we put them?
J: 3 and 3 and 1 and 1.

In the decomposition tasks some children are not primarily aware of the numbers, when reflecting over their part-whole relations but of a social situation where sharing means dividing into equal parts. The children focus on dividing the coins or buttons equally and in this context it means that there have to be two equal number words.

**Saying successive numbers**

The third way of handling numbers when experiencing them as number words is to *say successive numbers*, where children say a number next to one of those mentioned in the given problem by taking one step upwards or downwards on the number sequence. It is not unusual for children to count the number words of one of the parts of the problem aloud or in their head before giving *a successive number* as an answer. When a child is aware of the words next to each other on the number sequence the sums often end up with a number outside the number range. In his answers, Richard, a boy who is blind (6.5 years) answers with many numbers next to the number mentioned in the problem.

I: We are only playing with 5 coins. I will put some of them in one of my hands and the rest in the other. How many shall I put?
R: 5.
I: How many in the other?
R: 6.
I: If I put 3 in one hand?
R: 4.
I: If I put 4 first, how many are in the other?
R: 5.
I: If I put 3 first?
R: Then 4.
I: Do you know how many there are together?
R: 5.

Children sometimes say *successive numbers*, going one step upwards or downwards on the number sequence, and know that the quantities mentioned in the problems are not the answer. There must be some difference but they do not have access to an accurate procedure with which to perform an operation. However, when children use successive numbers they have a growing insight that one may count on the number sequence in order to perform numerical operations. In this way, saying successive numbers can be the embryo that grows to the later use of the number sequence as a tool for doing operations. Children do not use their fingers to perform numerical operations when saying numbers, although the ones without impairment often move their fingers and touch them when trying to come up with an answer. Some children start to count one of the parts in the problem on their fingers, but do not continue, and simply give the number of fingers they have counted as an answer. This way of trying to use fingers for solving problems when saying number can be the basis for their later use of finger counting.

**Saying numbers related to number words**

When children are *saying numbers* they experience numbers as number words. In the problem solving situation they do not simultaneously discern several aspects of numbers and they do not have access to any problem solving procedure. The meaning of numbers as number words were experienced by some of the children in every group and there were no differences found between the ways in which children from the three groups were handling numbers when experiencing number words. One reason for this is that the handling is a mental act and the children's different sensuous experiences do not greatly influence their thinking.

### *The experience of numbers as extents*

Children who experience numbers as *extents* use *estimating* in order to give an answer. Similar to handling numbers when saying number words the children do not use any procedure to come to an answer. They refer to a number, but in contrast to children, who experience numbers as number words, they give a plausible answer close to a correct one when using addition as well as subtraction. When experiencing numbers as extents children have a diffuse awareness of the numerosity of the numbers in the problem. In the problem solving situation the children make estimations close to the correct answers because they refer to an *approximate numerosity* – referential aspect – and focus on the *size of numbers* involved in the present problem – structural aspect.

In contrast to when experiencing numbers as number words, when experiencing number as extents children have an approximate understanding of numerosity and do not only focus on numbers as individual or successive number words. They realise that there is a relation between the addends and the sum. However, as the children's awareness is directed towards comparing and relating the parts and the whole in the problem without considering their precise numerical relations, it is only by chance that estimating leads to a correct solution.

The experience of numbers as extents does not include an understanding of the cardinality of numbers outlined by Piaget, (1969). However, it is possible that a single child may have this understanding of cardinality. Furthermore it is not unusual that children who estimate have access to procedures, but prefer to estimate so as to solve the problem fast enough. Instead of using a procedure to solve the problem Mikael, a boy without impairments, makes a rather fair estimation when trying to solve an addition problem without counting on his fingers as he often does.

I: If you have 2 crayons and get 7 more, how many do you have?
M: Mm, 2 first and then 7, mm ... that's 11.
I: How were you thinking?
M: I'm fast. I don't think, I just say it.

When the number range is larger, it is not unusual that a child estimates even if she/he uses some procedure for solving the problems in smaller number ranges. Estimating is not very common way of handling numbers, although children use it in solving all kinds of problems. Renata, a girl who is blind (9 years), gives an answer not far from the correct one. She estimates two numbers within the number range of 9.

I: Could we divide the nine forints in another way? How many should I hide in one of my hands?
R: 3.
I: How many will be in the other one?
R: 5.

The children with hearing impairment also make quite plausible estimations experiencing numbers as extents. In the decomposition tasks the parts of the divided buttons are one or two more or less than the whole set. Ken (6-years) has two suggestions.

I: If we have these 9 buttons and are going to put them in these two boxes, how should we put them?
K: 3 and 5 or 2 and 8.

**Estimating related to extents**

When children are *estimating* they experience numbers as extents. They do not use any procedure to come to an answer, even if they do not know the exact answer. There are no differences observed in the ways in which children in the three different groups are handling numbers when experiencing numbers as extents. Similar to when children experience numbers as number words this way of handling is a mental act and sensuous experience does not influence their thinking to any great extent.

### *The experience of numbers as positions in sequence*

When experiencing numbers as *positions in sequence* children give an answer to a problem by counting on the number sequence to make an exact quantification. They refer to a precise numerosity of parts and the whole – referential aspect – and focus on the number sequence constructed by individual and successive numbers – structural aspect. The children have an understanding of the cardinality of numbers in that the end points in procedures are discerned and the positions refer to a precise numerosity and they use *double counting* or *counting all*.

#### Double counting

The most characteristic way of dealing with numbers when experiencing them as positions in sequence is *double counting*. When double counting, children count on two parallel number sequences keeping track of the counted numbers by linking the number words in one sequence to the number words in the other sequence. The children have an understanding of the cardinality of numbers in the sense that numbers are referring to a precise numerosity and they grasp the totality of the numbers in the parts and whole by focusing on the two different number lines. This way of handling numbers could be very demanding for the memory and is sometimes thought to be fairly developed, because it is at a high level of abstraction (Fuson & Hall, 1983; Steffe et al., 1983). Only two of the 38 children without visual or hearing impairments once used double counting and then only when performing addition.

In contrast children who are blind very often use double counting and they are often very skilful. However, if they have to say many pairs of numbers in order to reach an answer, it can prove difficult for them to remember and to keep the order of the counted number words and arrive at a correct answer. Children count up on one number sequence and down on another, or more often count up on both sequences as Rebeka, a girl who is blind (7 year).

I: If we add 5 forints to the 13 forints how many forints do we have?
R: This is more difficult.
I: How could you begin?
R: 14,1; 15,2; 16,3; 17,4. So then there are 18.

It is rather common for children with hearing impairment to use double counting and they use two qualitative different ways. They count orally simultaneously using a sign language keep-track count or they just use sign language. Pia (9 year) solves the problem through counting on two sign language counting sequences.

I: The baker has baked 9 buns and put them on the plate. Then the bun thief comes and takes some so that when the baker comes back there are only 5 left. How many buns did the bun thief take?
P: ... 4.
I: 4? How did you think?
P: I counted.
I: How did you do that?
P: Like this [showing the sign language number symbol for 9 on her left hand] and then I counted like this [counts down in Sign Language from 9 to 5 on her left hand and simultaneously counts in Sign language up from 1 on her right].

When Pia has counted down 5 steps on her left hand (the 5 remaining buns), she could ”read off” the answer 4 on her right hand.

**Counting all**

When children are *counting all* they use their fingers – counting them like objects. This way of handling numbers is only used by the children without impairments and is similar to the ”direct modelling strategy” outlined by Carpenter and Moser (1982). The child starts counting the first set one by one, followed by the second set, and thereafter counts all. Also, when working with subtraction children follow the structure of the problem. They start with the whole and count all numbers on their fingers and then the two parts, as Niclas, a boy without impairments does below.

I: If you had 10 sweets but eat some up so you only have 3 left, how many have you eaten?
N: 10. [Counting quickly his 10 fingers] 1, 2, 3 so I'd eaten 1, 2, 3, 4, 5, 6, 7. [Counts on fingers].

Counting all is rather common among the six-year-olds without impairment, and they tend to be more successful solving addition than subtraction problems.

**Counting related to positions in sequence**

When children are *counting* they experience numbers as positions in sequence and take one step at a time on one or two number sequences. When using double counting their understanding of numbers does not include any sensuous experience and aspects of numbers besides numbers as position in sequence, are not focal in their awareness. However, they simultaneously experience two number lines. Children in all three groups use double counting even though it is very uncommon among the children without impairment. When counting all is used by the children without impairments their understanding of numbers evolves from counting on the number sequence. They use their visual sense at the same time as their *focal awareness* is directed towards numbers as positions in sequence. But they do not simultaneously experience different aspects of number.

### *The experience of numbers as grouped units*

When children experience numbers as *grouped units*, they give an answer to a problem by *counting and grouping.* They refer to a precise numerosity of parts and the whole – referential aspect – and focus on individual, successive and grouped numbers – structural aspect. The children's awareness is simultaneously directed towards numbers in sequence and towards different parts or/and the whole. Thus when experiencing numbers as grouped units children also experience them as positions in sequence. The analyses show that there are several ways of handling numbers that are related to this simultaneous experience of numbers as positions in sequence and grouped units. Three different main categories are identified – *counting and touching – counting and hearing – counting and seeing.*

#### Counting and touching

When children use *counting and touching* they touch something in the surroundings trying to perceive the counted numbers to know when to stop counting. They can touch anything, a table, a chair, their knee, whatever, or they may press their fingers together. The children do not look at their fingers and often only make a very small movement. Children's awareness is directed towards both the numbers as single units – when counting on the number line – and towards the totality of numbers in the part and whole – when trying to grasp the numerosity of the counted numbers. Counting and touching is a rare way to handle numbers in all groups of children, however, Eva, a girl with a hearing impairment (8 years) is counting and touching her fingers when counting in Norwegian Sign Language (NSL).

I: Suppose that you should divide these 13 buttons into these two boxes, how many can you put in each?

E: 9 in that one [pointing] and ... then I must count. ... 10, 11, 12, 13 ... that is 4.
I: 9 and 4, very good! What did you do with your fingers?
E: I just counted like this [starts with 9 in NSL on her right hand and counts on until she reaches 13. She touches the fingers on her left hand for each count step].

The production system in sign language made this one-to-one correspondence between fingers in the symbol and the fingers used as object easy to execute. Thus counting and touching is a rather easy way of handling numbers for hearing impaired children – in contrast to children without hearing impairment who easily loose track of the counted numbers if they do not have a specific pattern for their touching which can be recognised and repeated.

**Counting and hearing**

When children handle numbers by *counting and hearing*, they count upwards or downwards, listening to how many numbers they say without any procedure to keep track of the numbers. They perceive the numerosity through hearing by counting out loud or listening to their inner speech without any keeping track procedure. Sometimes they make a body movement, such as nodding their head, to follow the counting. By getting a sensuous experience of how many numbers they say, or hear in their head, they grasp the totality of these numbers using subitizing. When solving problems by counting and hearing, the child's awareness is directed both towards the numbers on the number sequence – in that they count one by one – and the totality of numbers in the sets they are saying – simultaneously experiencing numbers as positions in sequence and grouped units.

Children with a hearing impairment of course do not use counting and hearing, however, it is very common among children without impairment when the number range in the problems is small. Daniel, a boy without impairment is counting and hearing from the largest number. He starts at 7 and then continues counting.

I: If you have 2 crayons and then get 7 more, how many do you have then?
D: 9.
I: How did you work it out?
D: I turned it round so that I had 7 from the start and then counted 2. 8, 9.

Counting and hearing is very common among children who are blind and some of the children are very clever and can perceive a great amount of numbers without keeping track of the them. Valery, a girl who is blind (7.5 years) mumbles when she is counting on, and perceives the seven number words she says.

I: You have 12 forints and you would like to buy a chocolate bar for 19 forints. How many forints do you need?
V: ...
I: Out loud please!
V: May I count to myself?
I: It would be better out loud!
V: 13, 14, 15, 16, 17, 18, 19.
I: How many are there altogether?
V: 7.

Children who are blind often discern the number sequence and successively include or exclude the numbers they are going to add or subtract by counting out loud, adding or subtracting one by one. The children's awareness is directed towards numbers as positions in sequence. However, they grasp the numerosity of the counted numbers, knowing how many numbers they have added or taken away without counting that part and consequently have an simultaneous experience of numbers as grouped units. The difference between adding and subtracting and double counting is that – without any keeping track procedure – the children perceive the manyness of the set they add or subtract. Only the children who are blind use adding or subtracting. The children without impairments have not yet started learning formal mathematics in school and this could be one reason for not using adding and subtracting.

Rebeka (7 years) adds one by one when she is solving a non canonical problem.

I: You have 3 forints and you would like to have 7. How many forints do you need?
R: I need 4.
I: Why?
R: I add one, now there are 4. I add one there are 5. I add one now there are 6. And then I add one, now there are 7.

Rebeka counts four steps one after another on the number sequence knowing that she must stop counting when she says 7. She knows through hearing that she has added four, even though she does not count this part on a number sequence.

**Counting and seeing**

When solving problems by *counting and seeing* the child's awareness is also directed both towards the numbers on the number sequence – in that they count one by one – and the totality of numbers in the sets they are saying – simultaneously experiencing numbers as positions in sequence and grouped units. Children group their fingers and look at them to perceive the counted numbers.

The variation in using fingers for counting is immense among the children without impairments and a single child can go about in various ways when solving different problems. Children's hands can be turned up or down and they may count on the left, the right, or both hands, starting from the left solving one problem and from the right solving another. The same child may begin to count on the little finger, the thumb, or some other finger when solving different problems. In this investigation children's sensuous experience of numbers is emphasised, therefore the ways of handling numbers described do not focus on "counting on strategies" – "counting on from first" and "counting on from largest" as identified by Carpenter and Moser (1984) – even though the participating children use these strategies. When counting and seeing the children count some numbers one at a time and group some others, seeing the numerosity on this part. They have a sensuous experience of the numbers by seeing the numerosity of the grouped numbers and therefore know when to stop counting. Through subitizing they *immediately grasp the numerosity* grouped on their fingers. However, they do not grasp the complete part-whole relation of all the quantities in the problem by seeing their fingers.

The *children without impairments* use their fingers in two different ways. The child counts some subset or the whole on their fingers and groups the other set and/or the whole without counting these fingers one by one. When children group some fingers they do not have to use "counting all" but just look at the grouped fingers to know how many they are. The grouped numbers are seen as a unit referring to a specific number. There are several ways in which children are "using fingers seeing". They can see "one part", "two parts", "the whole", or "one part and the whole". However, they do not see the complete part-whole relation of all the quantities in the problem.

Sofia counts the largest number first when solving an addition problem seeing one part. She continues counting from the largest number and by "seeing" the manyness of the small part she knows when she has been adding this without counting it one by one.

I: If you had 2 crayons but got 7 more how many do you have?
S: 9.

I: How did you think it out?
S: I counted to 7 first. 1, 2, 3, 4, 5, 6, 7. [She counts 7 fingers and then continues] 8, 9 [and sees that she has added 2 again].

Some children count the two parts – one number at a time – and thereafter *see the whole* without counting again. This is not a very common way of handling numbers among the children interviewed, but Ninni uses it.

I: If you had 2 crayons but got 7 more how many would you have?
N: 9. [Counts the fingers one at a time. First 1, 2 followed by 1, 2, 3, 4, 5, 6, 7 and she sees that it is 9].

When the children are counting their fingers one by one they use their fingers as references without distinguishing between fingers and numbers – creating a unit "a oneness" between the number word and the fingers. Similar to this they experience the grouped fingers as a unit corresponding to a number. The findings show that children do not give their fingers specific "names", but create an experienced unity between the number word and one finger or a specific group of fingers. Obviously, it is a complicated thing for some children to learn to count on their fingers. They have to learn by practice and become familiar in the use of their fingers in arriving at a correct answer to the given problems.

The *hearing impaired children* also see different combinations of the parts or the whole when they are counting for exact quantification. Sometimes they have a procedure for keeping track and like the children without hearing impairment they are counting and grouping the fingers. However, they count in two different ways: (1) *oral counting* – counting the fingers as objects – creating "a oneness" between the number word and the fingers, and (2) using *sign language*.

Sometimes they use a combination of oral counting and sign language (NSL). This way to deal with numbers however, turns into double counting as they use one of the modalities for keeping track of the numbers.

Paula (8 year) groups her fingers – when using oral counting.

I: If you have 15 apples and give 7 away. How many do you have left?
P: I don't know. I'll have to count.
I: That's all right, you can count if you like.
P: 15, 14, 13, 12, 11, 10, 9, 8. ... It is 8. [Counts orally].

Paula counts orally back from 15 to 8, simultaneously raising one finger at a time from 14. When 7 fingers are raised, she stops counting, sees that she has raised 7 fingers and states 8 as her answer.

**Counting and grouping related to grouped units**

When children are *counting and grouping* their awareness is simultaneously directed towards individual, successive and grouped numbers and towards a precise numerosity of parts and whole. Children simultaneously experience numbers as individual numbers with a fixed position in the number sequence – when counting on the number line – and as grouped numbers – when they through various sensuous experiences grasp the precise numerosity of an experienced set using subitizing. In this way they simultaneously experience numbers as positions in sequence and grouped units. Numbers have a cardinal meaning in the sense that they have a precise numerosity and that the last number word used, can be interpreted as an indicator of the counted collection's numerosity.

Experiencing numbers as grouped units is common among all three groups of children when dealing with numbers by counting and grouping. Children are touching, seeing and hearing when solving the problems allowing them to grasp numerosity of the parts and whole. When using counting and grouping children know how many numbers they have added or taken away and they realise when to ”stop counting” without counting once again.

### *The experience of numbers as composite units*

The experience of numbers as *composite units* includes both discerning single units and the unification of elements. When children are focally aware of numbers as composite units they do not count on the number sequence – but structure the numbers in various ways. Numbers have a cardinal meaning, They are related to one another, and can be composed and decomposed. Children's experience of numbers evolves from the relation between the referential aspect – the compositional relationship of numbers – and the structural aspect – individual, successive, grouped and composed and decomposed numbers. There are three qualitatively different ways in which the children go about experiencing numbers as composite units: *seeing numbers – using derived facts – using experienced number facts.*

**Seeing numbers**

When children are seeing numbers they do not count in any way, but rather grasp the manyness of numbers entirely through looking at their fingers. In this way they gain a sensuous experience and they simultaneously see the parts and the whole-ness of the numbers in the problem. Among the hearing impaired and the children without impairments this is a very uncommon way to deal with numbers, and of course children who are blind do not use it. Moa, a girl without impairments (6 years) starts with the whole and just looks at her fingers in order to arrive at an answer when performing subtraction.

I: Moa, if you have 4 kronor but lose 2 how many do you have left?
M: 2. [Shows 4 fingers and bends down 2 of them].

**Using derived facts**

Another way to handle numbers experiencing them as composite units is to use derived facts. Using derived facts does not include any counting on the number sequence. Children (1) deduce an answer setting out from a known number fact and use this as a starting point for performing numerical operations, or (2) they are grouping the numbers in parts and use these as a basis for further operations. Daniel, a boy without impairment knows a ”double” which helps him to solve an addition problem.

I: If you have 2 apples and get 3 more, how many do you have then?
D: 5.
I: Can you remember how you thought about it?
D: Yes.
I: How?
D: I know that 2 + 2 are 4 and that then 2 + 3 are 5.

Children who are blind frequently use derived facts and known facts. Sometimes they group the numbers and add or subtract two, three or even four numbers in different combinations at a time. They do not say the number words one by one nor do they grasp the wholeness immediately. The children with hearing impairment also often use known facts and the relationships between facts to deduce solutions that were not immediately known. Toril (9 years) knows that 5 and 2 equals 7.

I: You have 25 kroner and you buy something for 7 kroner. How many kroner are left?
T: 18.
I: How do you know that?
T: Because 25 minus 5 are 20, 20 minus 2 are 18.

**Using experienced number facts**

A third way to experience the composite nature of numbers when solving problems is using experienced number facts. Children do not count or perform any operation in order to arrive at an answer, but simply give an answer directly. There seem to be two different ways of using ”number facts". Firstly, numbers are isolated, rote learned number facts – number words – understood as an unprecise numerosity. For example, when a very small child knows a "double" by heart, such as two and two equals four. If a child's knowledge does not include an awareness of the part-whole relation of the numbers in the problem, then they experience numbers as number words. Secondly, there are "experienced number facts" – in the sense that children know how to compose and decompose the known understanding of their part-whole relation. These experienced number facts are learned from handling numbers in different ways in school and every day life.

Children without impairments mostly use "number facts" when performing addition in small number ranges. Primarily they know the answers to the problems in the number range up to five. It is rather common that they know that two and two equal four and also that two and three equal five. Using number facts is the most common way of handling numbers by the children who are blind and among the older children with hearing impairment. When children use known facts they do not use any observable procedure or calculation and the children often explain their thinking by saying that they just knew or remember, as Mary (8 years).

I: How can you divide 9 buttons in these two boxes?
M: 5 and 4, 6 and 3, 8 and 1, 7 and 2.
I: How did you do that?
M: I just know.

**Structuring related to composite units**

When children are structuring they are focally aware of numbers as composite units and compose and decompose numbers into unit items. According to Fuson (1992), a genuine understanding of cardinality means that children are able to decompose the cardinal number into ideal unit items, can compare and relate cardinal numbers, and have cardinal numbers embedded within the number word sequence, so that they are seriated and included within each other. When children experience numbers as composite units they have this deep understanding of cardinality. The numbers have a cardinal meaning, they are related to one another, and the numbers are in themselves conceptual units that can be decomposed into compositions of smaller cardinal numbers.

The only way of handling numbers related to a sensuous experience of numbers when experiencing them as composite units is by seeing numbers. When children deal with numbers in this way they get the opportunity to subitize and simultaneously grasp the parts and whole of the numbers. However, this way to deal with numbers is very seldom used.

## *The nature of the variation*

The variation in the ways in which the children without impairment, children who are blind and children with hearing impairment handle numbers is revealed on both an individual and a group level. The number of problems given to the children within the different groups differed as well as there ages so the following description of the variation will be outlined in terms of four levels: (1) not used, (2) unusual, (3) usual, and (4) very usual, as shown in Table 3.

Table 3 shows that children without impairments on a group level use all of the 12 described ways of handling numbers. Children who are blind use 9 and children with hearing impairments use 10. There is a huge variation in how often children use the different ways of handling on a group as well as on an individual level. Some ways of dealing with numbers are not used, while others could be used only a couple of times or very often. The most obvious reason for the difference in use of handling is the children's various sensuous experiences. Two groups of the children lack one sense and the obvious reason for not using some ways is this lack of sensuous experiences. Also, the age of the children is of importance when scrutinising the differences. All of the children without impairment were between 6 and 7 years old, while the age of the children who are blind varied from 5 to 8 years and the hearing impaired form 6 to 10 years (see Table 1).

The findings show that it is rather common for children in all groups to deal with numbers by saying numbers and estimating. Among the children who are blind and the visual impaired it is mostly the younger ones who use these ways of handling the numbers. However, when the number range is big it is not unusual that the older ones do so to. The lack of differences among the three groups may be explained by the fact that children do not have any sensuous experience of numerosity when handling numbers by saying numbers and estimating.

Double counting is very common amongst children who are blind and also rather common among the hearing impaired children. However, it is a very rare way to deal with numbers among children without impairments. They do not have the same need to keep track of the numbers by counting on two number sequences as they have access to both their visual and hearing senses. Most of the variation is to be found in *counting and grouping* where the differences in the children's sensuous experiences are very evident. *Counting and touching* is only used a couple of times by the children with or without hearing impairment and it is not very common. Counting and hearing is for obvious reasons not used by the hearing impaired children, but it is a very common way to deal with numbers among children who are blind. Also the children without

Table 3. *Different ways of handling numbers used by the children*

| WAYS OF HANDLING NUMBERS | Children without impairments | | | | Children who are blind | | | | Children with hearing impairments | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 1 | 2 | 3 | 4 | 1 | 2 | 3 | 4 |
| SAYING NUMBERS | | | | | | | | | | | | |
| Random Number words | | X | | | | X | | | | X | | |
| Equal Numbers | | | X | | | | X | | | | X | |
| Successive Numbers | | | X | | | | X | | | | X | |
| ESTIMATING | | X | | | | X | | | | X | | |
| COUNTING | | | | | | | | | | | | |
| Double Counting | | X | | | | | | X | | | X | |
| Counting all | | | X | | X | | | | X | | | |
| COUNTING AND GROUPING | | | | | | | | | | | | |
| Counting and Touching | | X | | | | X | | | | X | | |
| Counting and Hearing | | | | X | | | | X | X | | | |
| Counting and Seeing | | | | X | X | | | | | | X | |
| STRUCTURING | | | | | | | | | | | | |
| Seeing | | X | | | X | | | | | X | | |
| Derived facts | | X | | | | | | X | | | | X |
| Experienced number facts | | X | | | | | | X | | | | X |

impairment frequently use counting and hearing, most often when solving problems within a smaller number range. Counting and seeing is not a way to handle numbers, which is used by the blind children. However, for the children without impairment it is the most common way.

Structuring numbers using *seeing* is very seldom used among the children without impairment. Also among the hearing impaired it is an uncommon way to deal with numbers and of course the children who are blind do not use it. *Derived facts* is often used by the older children with hearing impairment and children who are blind, but also some of the six year old children without impairment use it. Also using *experienced number facts* is a very common way to deal with numbers among the older children who are blind and the hearing impaired children. It is not very common among children without impairments however.

## The dynamics of experiencing number

The comparative analyses shows that there is an extensive variation in children's experience of numbers. It is shown that on the way to understanding numbers sensuous experience is of great importance and that the children frequently use whatever is available of touching, seeing and hearing when trying to solve the problems. The sensuous experience to some extent limits or expands the possible variation in ways to handle numbers. Children without impairment have more opportunities to discern different aspects of numbers by getting a sensuous and simultaneous experience of numbers and in this group the richest variation in ways of handling number is to be found.

### *The simultaneous experience of number*

The analysis of children's ways of handling and experiencing numbers shows that when solving problems various aspects of numbers are focally presented in children's awareness. The different ways in which children handle numbers is intertwined with their interpretation of the meaning of number. Some ways of handling numbers are related to more than one meaning and consequently children may experience several aspects of numbers simultaneously when solving a single problem. When experiencing numbers as grouped units children's awareness is directed towards parts and whole as well as towards numbers in sequence and they simultaneously experience numbers as grouped units and as positions in sequence. The relationship between the ways of handling and the experience of numbers is shown in Table 4.

Some aspects are *focal* in children's awareness (light grey) others are experienced but not focal (dark grey) and others are not in children's awareness at all (black)

When *experiencing numbers as number words* children are simply *saying numbers* as an answer to a given problem. They refer to an unprecise numerosity and focus on numbers as individual or successive number words. When trying to solve the problem in this way they do not simultaneously experience any various aspects of numbers.

When *experiencing numbers as extents* children use *estimating* in order to give an answer. Even though they do not ”know” the exact answer they do not count or use some kind of procedure to solve the problems. They refer to an approximate numerosity, focus on the sizes of the numbers involved in the problem and do not simultaneously experience different aspects of numbers. However, the act of focusing on the sizes of the numbers reveals that they have an understanding of number words as individual or successive number.

Table 4. *The relationship between handling and experiencing numbers*

| WAYS OF HANDLING NUMBERS | WAYS OF EXPERIENCING NUMBERS | | | | |
|---|---|---|---|---|---|
| | Number Words | Extents | Positions in Sequence | Grouped Units | Composite Units |
| SAYING NUMBERS | | | | | |
| Random Number words | | | | | |
| Equal Numbers | | | | | |
| Successive Numbers | | | | | |
| ESTIMATING | | | | | |
| COUNTING | | | | | |
| Double Counting | | | | | |
| Counting all | | | | | |
| COUNTING AND GROUPING | | | | | |
| Counting and Touching | | | | | |
| Counting and Hearing | | | | | |
| Counting and Seeing | | | | | |
| STRUCTURING | | | | | |
| Seeing | | | | | |
| Derived facts | | | | | |
| Experienced number facts | | | | | |

When *experiencing numbers as positions in sequence* children give an answer to a problem by *counting* on the number sequence to make an exact quantification. The child's awareness is directed towards numbers as individual and successive units in sequences referring to a precise numerosity of the whole. The most characteristic way of counting is *double counting* but children use also *counting all*. They are focusing on the end points in the procedure and not on the composition of cardinal sets and the whole. When they use counting they do not have any simultaneous experience of numbers beyond an understanding of numbers as successive number words on number sequences.

When *experiencing numbers as grouped units* children are *counting and grouping.* Their awareness is directed towards both the numbers as individual and successive numbers – when counting on the number line – and towards the totality of numbers in both parts and whole when grouping the numbers by touching, hearing, or seeing. When dealing with numbers in these ways children simultaneously experience numbers as positions in sequence and grouped units.

When *experiencing numbers as composite units* children focus on cardinal numbers which can be decomposed in both single units and in smaller composite units. Here, contrary to the focal experience of numbers as positions in sequence and grouped units, experience of numbers as composite units includes an awareness of the compositional nature of numbers. When children experience numbers as composite units they are also aware of numbers as number words, extents, and as positions in sequence, although in the problem solving situation these aspects of numbers are not focal in awareness. The primary focus is upon numbers as composite units. On their way to grasping numerosity children handle and experience numbers in various ways. When estimating, counting and grouping children successively learn to grasp patterns and gestalts, and develop their understanding of numbers. However, it is evident that children do not pass through different understandings in a graduated, ordered and requisitioned manner. The pathway from experiencing numbers as number words to understanding them as composite units is not straight forward. On the contrary, in various problem solving situations a single child may shift between various ways due to the number range, the structure of the problems or the problem solving situation. Also, when solving a specific problem a child may shift from one way of experiencing to another – for example a child could start with saying numbers and thereafter go on making an exact quantification using double counting.

The embryos of later understanding of the meaning of numbers may be found in the earlier phases of the development. The seed for the children's experience of numbers as positions in sequence may be found in their awareness of numbers as number words and extents. When children are saying successive numbers they have a growing insight that one may count on the number sequence in order to perform numerical operations. Saying successive numbers can therefore be the embryo that grows to the later use of the number sequence as a tool for doing operations and an understanding of numbers as positions in sequence may shift into an experience of numbers as grouped units. In the simultaneous experience of numbers as positions in sequence and grouped numbers children may gradually learn to understand the part-whole relation of numbers and can then experience them as composite units.

### *The sensuous experience of number*

The analysis of children's ways of handling numbers shows that using the senses is of great importance for all children. However, in the problem solving situations there are some ways of handling numbers that do not include any sensuous experience. When dealing with numbers by saying numbers and estimating, children do not perceive the numbers through a sensuous experience. Children who are structuring have a sensuous experience of the compositional nature of numbers only when composing and decomposing numbers using their fingers perceiving the numerosity by seeing. Experiencing numbers as composite units is most often is purely mental, as when using derived facts or experiencing number facts.

On an overall level children who are counting handle the numbers in two qualitatively different ways. One way is to use double counting another is to count on the number sequence, grouping numbers and getting a sensuous experience of numbers. When counting and grouping, children grasp the numerosity of the grouped units by touching, hearing and seeing.

Mostly children's sensuous experiences of numbers are intertwined, although one sense is focal, such as the visual sense when counting out loud using counting and seeing or when counting and touching. When touching some object in the surroundings children may simultaneously listen to the number words they say and obtain a tactile sense or even a visual experience of the numbers.

The most frequent way to deal with numbers among the children are various types of counting – a way to deal with numbers that may generate sequential thinking rather than forming an understanding of numbers as composite units. However, when counting and grouping numbers the children's sensuous experience enables them to grasp the numbers as grouped units in a way which can later on develop into a an understanding of numbers as composite units. Therefore, on their way to grasp numerosity and develop their understanding of numbers, it is of great importance that children are getting a sensuous experience such as touching, seeing and hearing the numbers.

### *Discernment, simultaneity and variation*

Prior experiences and understanding are very important features in the development of numerosity and so are the activities and settings in which the practice of arithmetic is experienced. Baroody (1987) points out that instruction, which is not geared to children, can distort their views of mathematics, mathematical learning and self, and several researchers emphasise that a very early introduction of symbols and formal procedures does not aid the development of arithmetic skills (Ahlberg, 1995; Neuman, 1987). According to Ahlberg (1992, 1998) the main way to develop all children's flexibility of mind and bring about changes in their experience of numbers is to bring their experienced world into the teaching process and *introduce a variation* in the modes of learning. This variation can be along three different dimensions: the content of the problems, the means of expression and in exposing the variation of the children's ideas and thoughts. When bringing children's experienced world into the teaching process, children get the opportunity to use different senses and different modes of expression – learning to simultaneously discern different aspects of numbers.

**Children who are blind learning numbers**

Ahlberg and Csocsán (1994) revealed that children who are blind do not spontaneously use their fingers either to count on the number sequence or to model the numbers as sighted often do. This earlier result is validated in this investigation. None of the children who are blind use their fingers when solving the presented problems and this reveals that no correlation exists between using fingers – in the ways sighted children do – and the development of arithmetic skills among children who are blind. However, Ahlberg and Csocsán (1994) have shown that touching and grouping is of great importance for the growth of blind children's understanding of numbers. In this investigation none of the children use touching and grouping, as they did not have any manipulative objects to handle. Instead they made use of their hearing sense for grouping and perceiving numbers. Most often they are very skilful when counting and hearing and it is not unusual for them to hear five or six units – and some children perceive a lot more.

According to Best (1992), children who are blind need to use their hands and fingers to touch even quite small objects in a sequential manner to get information about the objects around them. However, the findings in this study show that handling numbers counting and grouping contributes to the children's understanding of numbers. Therefore it is important that they get opportunities to group elements in order to perceive their numerosity. The blind children should not only be taught to count on the number sequence using double counting but should also be taught to group numbers in different ways. When touching and grouping elements with both hands or hearing and grouping numbers, children are given the opportunity to simultaneously experience the numbers in a whole-ness consisting of smaller parts. These experiences can contribute to developing their understanding of numbers as composite units. The results of the investigation indicate that one possible way to enhance the blind children's developmental delay compared to the sighted is to give them opportunities to group numbers through hearing and touching.

**Children with hearing impairment learning numbers**

Children with a hearing impairment have inferior achievement in elementary arithmetic compared to others, and in contrast to the blind children, it will not disappear as they grow older. The present study verifies earlier results (Frostad & Ahlberg, 1996; Frostad, 1999) that one reason for this could be sought in the hearing impaired children's ways of handling numbers, which involves both oral counting and sign language counting. The structural aspects of sign language counting may influence the hearing impaired children's thinking and lead them into the experience of numbers as positions in sequence, as the counted number of items can easily be "read off" the fingers. Even though this approach enables these children to solve the tasks, it will not promote insight in the compositional nature of numbers. Indeed, some ways of handling the numbers may guide children towards a position in sequences experience of numbers, which prevents a development towards experiencing numbers as composite units. According to this the implications for the teaching process are that the children should be given the opportunity to work with a variety of tasks that will challenge their experience of numbers as positions in sequences.

**Pathways to numbers**

One main finding in the investigation is that children use various ways of handling numbers even when they experience the meaning of numbers in the same way. Consequently children can develop the same understanding by using different ways of handling numbers. In spite of lacking one sense, on a group level the children with impairments come to understand numbers in the same way as their hearing and seeing mates. The variation in ways of handling numbers among the three groups of children is most obvious when they use counting and grouping. Learning to group numbers in various ways is critical for the children on their way to grasp numerosity. The results seem to show that in order to make all children aware of different aspects of numbers, they should be given opportunities to handle numbers in different ways and especially to group and compose and decompose numbers in various ways.

The pathway for developing children's understanding of numbers is to bring about changes in their ways of experiencing numbers. The focus in the teaching process should be on how to provide opportunities for children to experience new aspects of numbers, by bringing their experiences into the situation and offering possibilities for children to make use of them. Children should be given opportunities to bring available resources to new circumstances when encountering a variation of ways of handling numbers. In order to be aware of the different aspects of numbers when solving problems children should use their sensuous experience of perceiving numbers in a whole-ness consisting of smaller parts.

According to the present study the implications for the teaching process are that in order to develop children's mathematical thinking, they should be encouraged to talk and reflect over various mathematical experiences and they should be given opportunities to bring about changes in their experience of numbers. All children should be given the opportunity to group numbers and compose and decompose them in various ways – ways in which one aspect of the phenomena is kept constant and other aspects are varied. Taking the 'number five' as an example – the whole remains constant but the 'situations of five' – the parts – such as 2 and 3 or 4 and 1 can be varied (Ahlberg, 1997).

The present investigation has illustrated some ways in which children experience numbers and the relation between them. The results show that when trying to grasp numerosity children handle numbers in an array of ways and thereby experience different aspects of numbers. All children bring available resources into the problem solving situation making use of them in different ways. When children with visual or hearing impairments try to solve problems they do not 'compensate' for lacking one sense, but make use of available senses and earlier experiences. Thus numbers may appear in the same way to all the three groups of children and they may experience the same meaning of numbers – in spite of differing sensuous experience.

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## Notes

1 Some parts of this article are earlier documented in Swedish. The investigation is reported in Ahlberg (2000).

2 Two of the children had a less than severe loss (<71 dB better ear average loss), two had a severe loss (71-90 dB better ear average loss), and the rest (27 children) had a profound loss(>90 dB better ear average loss). Five children had deaf parents.

3 An extensive amount of quotes to validate the interpretations are presented in Ahlberg (2000).

4 The two terms ”number words” and ”number signs” refer to the same meaning of numbers. Number words are used to label the experience of children without impairments and children who are blind. Number signs label the experience of children with hearing impairment. However in the following only the label Number words will be used.

5 All children without impairments are between 6 and 7 years old.


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Ann Ahlberg is professor in special education at Göteborg University. Her research interest is directed towards inclusion and exclusion processes in school focusing on participation, communication and learning. A special interest is directed towards mathematics for younger children. In this research she has studied children's different ways of experiencing and learning mathematics as well as the organisation and content of teaching.

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# Sammanfattning

Den redovisade forskningen handlar om hur barn förstår tal när de löser enkla additions- och subtraktionsproblem. I studien deltar barn som är blinda, barn med hörselnedsättning och barn utan dessa funktionshinder. Studiens syfte är att beskriva skillnader och likheter mellan hur de tre barngrupperna hanterar och förstår tal. Den komperativa ansatsen bidrar till förståelsen av kritiska aspekter som karakteriserar utvecklingen av den aritmetiska förmågan för varje barngrupp. Studiens design möjliggör också en beskrivning av barnens förståelse av tal, när de löser aritmetiska textproblem, på en mer generell nivå.