The use of reasoning in the resolution of geometric problems

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The cognitive processes of 15-year-old students, when they solve geometrical problems involving the construction of isosceles triangles, and the different forms of reasoning which they use, are investigated in this paper. First we explore the large variety of reasoning processes which appear, categorize them in three approaches (visual, heuristic and theoretical) and look at the language which is used by each one. Then we focus on the weaknesses of students' reasoning and examine their reasons. The analysis of the data intends to support teachers to recognize and understand the relationship between students' reasoning (nature of justification) and their geometrical thought.

The aim of this study is to produce knowledge that can help mathematics teachers comprehend the cognitive processes of their students and realize the difficulties, which they face, when they solve geometric problems. The more fully a teacher can understand the thought processes of the student, the more s/he is able to support the student's learning. "Geometry, more than other areas in Mathematics, can be used to discover and develop different ways of thinking" (Duval, 1998, p. 51). For Duval, the development of thinking parallels the differentiation and the co-ordination of semiotic systems of representation, which are used. Moreover, the development of thinking is achieved through the independent development of cognitive processes, one of which is reasoning (Duval , 1998, p. 49).

Our goal is to investigate the forms of reasoning, which are used by students in the 10th grade, when they solve geometry problems involving

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isosceles triangles. The components of reasoning are analyzed with the goal of understanding and then categorizing the large variety of processes, which appear.

Reasoning is any process, which enables us to draw new information from given information (Duval, 1998, p. 45) and therefore, it facilitates the use of mental representations during thinking and learning. As there are many ways of looking at a figure, so there are various kinds of reasoning (Duval, 1998, p. 39). Differentiation of the multitude of processes of reasoning is a prerequisite for the coordination of the student's cognitive functioning.

We adopt the cognitive approach and we seek to determine the cognitive functioning underlying the diversity of mathematical reasoning processes (Duval, 2001, p. 1). We examine the hypothesis that the process of reasoning develops, and development is multimodal. As a result of the investigation of reasoning, we expect to distinguish the way in which this cognitive process takes place, the factors, which influence it as well as the parameters which indicate its development.

Reasoning

In order to solve a geometry problem, we activate three cognitive processes which form the basis of geometrical thinking, namely, visualization, construction and reasoning (Duval, 1998, 38). Each one, with its specific characteristics, performs a different purpose.

Visualization provides an overall conception of the situation, beyond what vision can conceive, as an organization of relationships. The image required by the problem is created. Its elements and the relationships between them are recognized (visually and verbally), while the relationships considered important (within the context of the given problem) are selected.

Construction focuses on the elements of geometric figure, constructs and, with successive efforts, actualizes scenarios. The result of this process is the geometric figure, while the geometrical instruments and the instructions for construction constitute the "technical support".

Reasoning is the process, which is used to substantiate consistency between the hypothesis of the problem (verbal and figural), the mental representation of the problem and the geometric figure, which has been constructed. In everyday language it could be said that reasoning is "a logical explanation of what we believe to be true".

In Mathematics the meaning of reasoning encompasses a substantial number of ”justification terms” such as proof, verification, explanation, elucidation, etc. which differ in the degree of rigidity, the language which they use and the way in which they present the elements. Rodd, based on earlier and more recent studies (Bell, 1976, p. 24; De Villiers, 1996, p. 24) refers to different views as to the meaning of ”mathematical proof” such as:

- Justification, verification and discovery (that).
- Illumination, explanation (why).
- Systematization, communication (how) (Rodd, 2000, p. 224).

In every activity, the role of natural, everyday discourse is essential for cognitive control and understanding. In Geometry, it is used mainly, a technical, theoretical discourse which differs from the natural. This theoretical discursive process only uses geometrical propositions, according to their status (i.e. theorems, axioms). Duval elucidates concerning the make up of theoretical discourse:

> In Geometry, to reason in order to prove requires two critical conditions:
> 1. To use propositions, each one having beforehand a specific theoretical status: axiom, definition, theorem, hypothesis, conjecture, etc.
> 2. The exclusive use of theorems, axioms and definitions in the process of reaching the conclusion (Duval, 1998, p. 47).

Besides the different nature of the propositions, which natural and theoretical discourse use, they differ essentially in the way statements are joined for the creation of a proof. In natural discourse, propositions are linked using correlation or contrast, as in everyday talking, while in theoretical discourse propositions are linked according to their status, according to the axiomatic system of Euclidean Geometry and in accordance with the rules of logic. Finally, the difference between natural discursive process and theoretical discursive process is the one between spontaneous everyday thinking and the artificial, strict reasoning, which is known as typical proof.

Theoretical discourse, though necessary for mathematical proofs, is a source of problems for students who require a considerable period of time to conquer it (and many never manage it). As stated by Fischbein, ”The meaning of typical proof is entirely out of the prevailing behavior” (Fischbein, 1982, p. 17).

Reasoning moves between two extremes of behavior: pre-mathematical and mathematical behavior. These differ, apart from reasoning, in basic points, as in the use of geometrical figures and the language, which they utilize. These extremes can be described briefly as follows.

In *pre-mathematical behavior*, reasoning is based on the visual relationships between the elements of the geometric figure and is expressed in terms of everyday discourse without specific mathematical expressions (Duval characterizes this as naïve behavior). Here, "visualization and spontaneous verbalization are very close to each other" (Duval, 1998, p. 47). The role of the geometric figure is decisive, as through it comes the recognition of the elements which are related to the problem (the student compares the figure to his mental representation and responds to the potential incompatibilities between the two).

In *mathematical behavior*, to the contrary, reasoning is based on the whole framework of the axiomatic system of Euclidean Geometry and the rules of logic. Retrieval of information is facilitated by its hierarchical organization (Anderson, 2000, p. 223). The role of the geometric figure is clearly indicative and is used heuristically in the context of operative apprehension, which "does not provide the steps and the organization of deductive reasoning for the proof, but it shows some key points, or an idea, which can allow the student to select the main theorems to be used" (Duval, 1998, p. 48).

Different conditions operate in parallel in the mind of the student, in a slow and strenuous effort to coordinate visual and theoretical reasoning. The successful correlation of the two is a factor, which leads to mathematical understanding. Managing the complex relationship between intuitive and theoretical dimension constitutes one of the main difficulties of geometry teaching/learning (Mariotti, 1997, p. 1).

In geometry, there are some statements, which appear directly acceptable as self evident (axioms) while all the others are accepted on the basis of a logical proof. The intuitiveness of a certain property tends to obscure in the student's mind the mathematical importance of it. For example, in an isosceles triangle ABC, with AB=AC, students do not feel the necessity of proving that $\angle B = \angle C$. Apparently, trivial properties seem to discard the necessity and utility of mentioning them explicitly, of proving them, or defining them (Fischbein, 1999, p. 22).

In the context of school (although not exclusively) reasoning is closely linked with understanding. For teachers, the way a student reasons usually indicates the degree of his understanding. This common perception is not always certain. Often students are seen justifying a proposition by reproducing a proof of it, without really understanding the whole process. They simply repeat mechanically the reasoning of someone else,

Table 1. *Various types of relationships between intuitive and formal knowledge.*

	Intuitively accepted	Not intuitively accepted
Accepted without a formal proof	Axioms (for example, 2 points determine a straight line and only one)	
Accepted with a formal proof	**Coincidence** (between intuition and mathematical proposition): In an isosceles triangle the angles adjacent to the basis are also equal. **Conflict** (between intuition and mathematical proposition): The quotient of the areas of two similar triangles equals the square of the quotient of two respective sides.	In a parallelogram the sum of its angles is equal to 360°

either of the book or of the teacher. On the other hand, the student may know something, without being able to justify it with arguments. Rodd summarized this problem with the following epistemological question,

> what is the relationship between the method a student uses to justify the truth of a mathematical proposition and his/her claim that he knows the proposition's truth? (Rodd, 2000, p. 242)

The answer, which he gave himself, is that

> a student's mathematical knowing requires a mathematical way of thinking ... In particular, deductive reasoning may be a way in which a mathematically-minded student can attain personal conviction ... though it is not the only way ... (ibid., p. 242)

The essence of Rodd's thinking is that "proof may not warrant, and a warrant may be other than a proof" (ibid., p. 242) (for example, it may be visualization).

Trying to comprehend the reasons which cause students' failure in geometry, research brings up the relationships between intuitive and formal knowledge (Fischbein, 1999, p. 18-21) (Table 1), the arranging of students' thoughts, the building of logical arguments, the dealing with deduction, or the formal proves (Shriki, 1997, p. 4-152).

Methodology

We tried to investigate the forms of reasoning used by 15-year-old Greek students, when they solve geometry problems involving isosceles triangles. In fact, we attempted to understand the methods that students use to justify their solutions and the large variety of processes, which appear in the cognitive process of reasoning. We hoped our study would open a window to the growth of student's reasoning and would trace regularities useful to teaching.

We observed a group of ten male and ten female 15-year-old pupils, from four different high schools in Athens, for the duration of one school year. These 10^{th} grade students have to create a more systematic representation and organization of their knowledge, concerning geometrical concepts, and are involved with classic geometrical proofs for the first time. The sample was chosen out of 120 students, on the base of a pretest that investigated their basic knowledge. Different abilities were represented in the sample.

We designed seven construction problems (which were related to isosceles triangles). Each pupil participated individually in three sessions, for a period of one academic year. We gradually gave them the problems to solve, asking them concurrently to describe their thinking and offering them all the time needed to solve each problem (on average 100 min /session). The whole process was audio recorded and transcribed, while the researcher kept notes of the pupil's non-verbal behavior. Then, each student's discourse was analysed and categorized according to:

- the nature of justification used by the student (whether it was based on images, on student's own figures or on theoretical prepositions).

- the kind of language used by the student (natural/imperfect discourse/theoretical discourse).

Thus, we had the opportunity to record the whole procedure and to understand the learner's full rationale, avoiding judgments based solely on results.

In this article we will focus our attention on two of the problems that were used in the study, selected accordingly to their significance in terms of the objectives of this article.

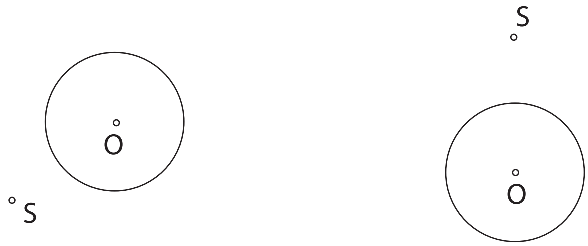
The first problem intends to investigate the recognition and use of isosceles triangle in a context that seems to present few similarities to it, that of a circle. The student has to construct the isosceles triangle out of very limited elements – the circle and a point. It is interesting to detect the role of the prototype of isosceles triangle in this construction.

On the contrary, the second problem intends to investigate the recognition and use of isosceles triangle in a similar context, that of the isosceles trapezium.

The individual elements of a geometric figure have definite and distinct ”roles” determined by the relationships that exist between them. For

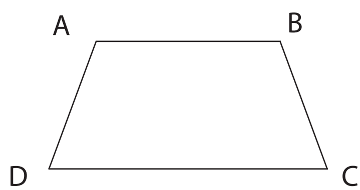
Problem 1

You are given a circle with center O and a point S. In each case, construct an isosceles triangle SAB, in which A and B are points on the circle. Be sure that the triangles you construct are different. Write analytically the method of construction and justify why the triangles are isosceles.



Problem 2

Starting with the isosceles trapezium ABCD construct whatever isosceles triangles you are able to. In each instance, write the construction scenario which you followed and explain why the triangle, which you made, is isosceles. Which of the constructions, which you described in your scenarios, do you think was better and why?



example, in the case of an isosceles triangle the basic roles are: "equal side", "base" and "height" (the height-axis of symmetry is implied). Becoming aware of the roles requires lexical and semantic comprehension of the elements of the geometrical figure, involving a different form of identification that we can call the "role yield"(Kabani, 2003, p. 161).

Isosceles triangle and isosceles trapezium share several roles (base, equal sides /angles, height) and therefore, we would like to see if students can discern them and use them effectively in their reasoning.

There is a common ground under these two problems. They are designed to investigate the establishment of links between discrete items of knowledge/schemata and the factors that influence them. We considered two diametrically opposite cases, with the hypothesis that students would perform better in the case of isosceles trapezium, than in the case of the circle.

Presentation and analysis of findings

We followed and analyzed the nature of the processes, which were used by each student, and specifically, the form of presentation and method of organization of the information (Duval, 1998, p. 45). On this basis, we noticed regularities and we grouped the various processes of reasoning into three different approaches: visual, heuristic and theoretical. In the following, we refer analytically to each approach, describing the processes of reasoning that are categorized to them.

The visual approach

The Visual Approach is adopted by students with pre-mathematical behavior, who do not know and consequently do not utilize theoretical propositions. Their reasoning is based on their mental representation and on the figure they have constructed. Their main concern is the accurate transcription on the paper of their mental representation. This is the answer to the problem, which they are facing. Accuracy of construction is a decisive factor.

In the visual approach, reasoning takes place in a simple way. The students cannot see any reason why they should have to rationalize something differently from how they perceive it, in other words, through the figure (Chazan, 1993, p. 360). Thus their reasoning is structured around the comparison of their mental image and the geometric figure. The language which they use is everyday language, which describes the figure and draws the arguments from it. Here we encounter comments such as "I am making a tic-tac-toe game", referring to two pairs of intersecting parallel lines.

In this approach, the comprehension of the figure has perceptual characteristics. Specifically, the visual approach is implied by what the students see, construct or measure, as we can see in their arguments, which may be:

- "It looks like ... (i.e. perpendicular)" referring to a relationship (perpendicularity, equality, etc.) which is obvious in the figure.

- ”It is isosceles because I constructed it that way”, or ”they are equal because of the way they have been constructed”. In other words, reasoning is a description of the method they used to construct the geometrical figure. This description is given in natural language and can include the steps of construction, so that someone would be able to repeat them.

- ”If we measure, we will see that they are equal ...”. Measurement does not constitute a proof (Chazan, 1993, p. 370), as it never exceeds the stage of ”approximately” which define the conditions of construction. In spite of this, there are students who use measurements as arguments, even when they are inaccurate. Thus we confront the phenomenon of students who claim that two elements are equal, though they are ”approximately equal”.

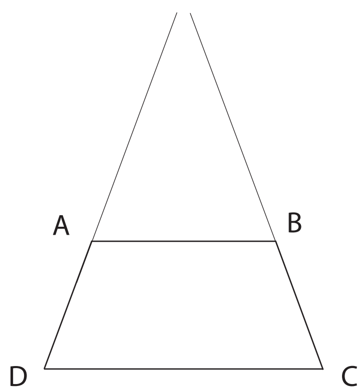


Figure 1.

An example of argument depending on the construction of the figure is the following. In the second problem, a significant number of students attempt to create an isosceles triangle, extending the equal sides of the isosceles trapezium two times (Figure 1). In the case that these two extended segments do not meet, that is to say, when a triangle is not formulated, as they had expected, they make minor corrections so that the extensions of DA and CB finally meet. The students give excuses for their corrections such as ”it didn’t turn out very well before ... the ruler should have been a little lower”.

The findings of this study highlight two disadvantages of the visual approach, which affect the ability of the students to solve problems:

- In a geometrical figure, it is not obvious what is the given and what is the required of the problem. In other words, all of the information appear to have the same importance. This is a serious disadvantage in the progress toward theoretical discourse, where the known and the required information must be clearly distinct. The only method, which can be used to define the given elements on the figure, is the use of symbolism (for equal elements, perpendicular lines, etc).

- Relationships, which have general application cannot be discerned in the figure. In other words, for each relationship, it cannot be seen if it applies generally, to the whole range of situations of which the geometrical figure is one representative, or specifically in the particular instance. In order to make this distinction, one must have the ability to transform the figure dynamically in his mind, maintaining the elements given by the problem constant, so as to determine the invariables. However, this ability, as shown in the study, is not always developed in 15-year-old children.

In the following example (problem 1, Figure 2) the student's arguments are based on the figure. He does not recognize that the proposition "a line drawn from the center of a circle perpendicular to a chord of this circle, bisects it" holds in every case. Furthermore, he tries to justify by measuring the figure, since he doesn't know that every chord has this property.

S: I'm going to construct a chord (of the circle), in such a way that ... SO will be perpendicular on its midpoint.
I: How are you going to do that?
S: ... I really don't know. Afterwards, I'll draw the segments SA and SB and ... there is the isosceles triangle SAB!
I: can you give me a reason why this SAB is isosceles?
S: if I measure ... SA = SB.

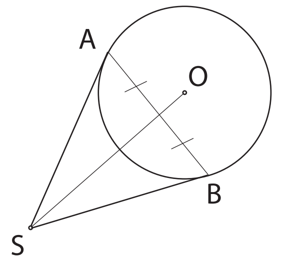


Figure 2.

However, the next student, attempting to solve the same problem (Figure 3), has the ability to dynamically transform the figure and to understand her error (of course, she does not adopt a visual approach).

S: I am going to construct a triangle SAB such that, A,B will be on a diameter.
I: You are drawing a diameter AB ... and now what will you do?
S: What will I do now ... (she connects S to A, B).
I: Why do you believe that this triangle is isosceles?
S: It is isosceles.
I: Are you sure?
S: ... I'm not sure, forget it (she erases her figure).
I: Why?
S: If I had drawn the diameter differently, then the triangle would have turned out differently and it would not have been isosceles!

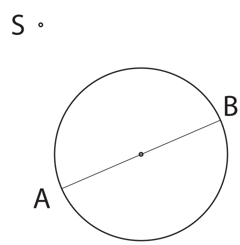


Figure 3.

The Heuristic Approach

Justification in visual approach relies on visualisation while reasoning in theoretical approach relies on the discursive apprehension of the figure and is independent from visualisation. There is a gap between these two approaches. In order to step over this gap, the cognitive processes of the students must develop and become more complex. Students' approach, in this transitional period, is characterised as "heuristic" and can be described and defined, to a large degree, through the comparison to visual and theoretical approach. The mapping out of the development of the child's functioning, which appears naturally in a transitional situation such as the heuristic approach, is of special interest.

Reasoning, in this approach, is implied by major improvements (compared to the visual approach) which are, the student's ability to re-organize the figure, the use of geometrical tools and instructions for the construction of the figure, and finally, the gradual development of a theoretical discourse. In particular:

- Students adopting heuristic approach demonstrate the ability to visually re-organize the figure in order to see other forms, which cannot be seen at first glance. We refer to the ability that Duval calls ”figural change” or ”operational apprehension” (Duval, 1998, p. 44) and has great importance in the searching for a solution (Mesquita, 1998, p. 190). It involves complex processes, which often take place unconsciously, such as the ability to think of drawing some units more on a given figure. In other words, the ability of the student to draw a new (helping) line on the figure indicates operational apprehension of it.

- In this approach, the use of geometrical tools and instructions for the construction of a geometrical figure guarantee to the student the correctness of his solutions. Here there has already been a serious developmental step, compared to the visual approach, in reasoning. Elaborations are not based on what the student himself can see or construct, but on what is included integrally in a given procedure of geometric construction (i.e. the construction of the bisector of an angle, using compass and ruler). The subjective opinion, as to the validity of construction (in the visual approach), is replaced by the application of geometrical tools and instructions, which, according to their capabilities or limitations, often affect the process and results of construction. In this way the constructions are validated and consequently the solutions of the student are validated.

- Another characteristic of heuristic approach is the development of a theoretical discourse, not perfect yet. For a correct organization of information, it is necessary for the student to start his reasoning from the hypothesis of the problem. In this approach, it is often noted that students do not use the hypothesis, which is given in the problem, but rather start their reasoning from relationships, which ”appear in the figure”. Ambiguities result as well from erroneous transformations between systems of representation or the dissimilarity of different representations of the same entity.

Students distort for various reasons (incorrect hypothesis, ambiguous terms, etc), theoretical propositions. Thus, instead of using *proposition p*, they quote and use the *proposition pseudo-p*. For example, the pseudo-p could be a generalization of p:

p: The median AD of the isosceles triangle ABC (AB=AC) is also its height and bisector.

pseudo-p: Every median of an isosceles triangle ABC is also its height and bisector.

This kind of generalization appears in the following dialogue. The student has drawn the three medians (BZ, CE, AD) of the isosceles triangle ABC (AB=AC) (see Figure 4).

S: Since BZ, CE are medians, they are also the bisectors of the angles.
I. What makes you believe this?
S: In an isosceles triangle, ... every median is also height and bisector (pseudo-p).
I: What's your problem now?
S: No ... it can't be a bisector. Only this (the median AD) that I can draw from the apex A is.

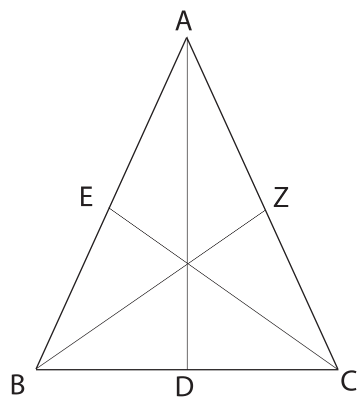


Figure 4.

A proposition pseudo-p can also be the result of an erroneous transformation between two systems of representation. For example, some students refer to the property of the points of the perpendicular bisector; however, in their minds the perpendicular bisector is identified with the median, a fact that is revealed only when they construct it. These students fail to transform the verbal representation to the proper figure. Thus in reality they use, instead of proposition p, a pseudo-proposition.

p: Every point of the perpendicular bisector of a segment AB is equidistant from A and B (becomes unconsciously).

pseudo-p: Every point of the median of AB is equidistant from A and B

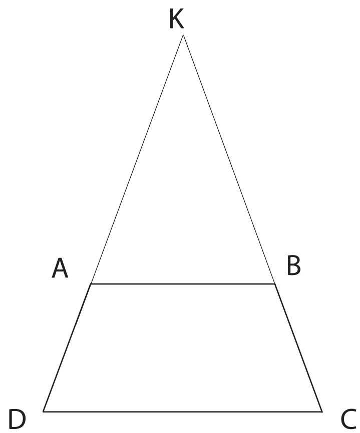


Figure 5.

The following student refers to a pseudo-proposition in order to justify that the triangle, which she created, is isosceles (problem 2). The student has extended the equal sides AD, BC of the isosceles trapezium doubling their lengths and, with some distortion; he has drawn the triangle KCD (Figure 5).

S: BC is 1/3 of KC. If we make use of the theorem which states that "if we join the mid points of two sides of a triangle, this segment will be parallel to the third side" (theorem) ... Therefore, "if we take this point higher or lower on one side and take a corresponding point on the other side, the same conclusion will be valid" (pseudo-theorem of her own).

Malfunctions in theoretical reasoning, according to the data from our study, may be owed to deficient organization of information. The student wants to use logical forms, which he has been taught or has seen in his book, but he does not yet have the ability to establish any logical relation between the propositions, which he uses. Consequently, his discourse "looks theoretical", however is often erroneous. From the protocols of the students the three following causes, which create problems of organization, are highlighted.

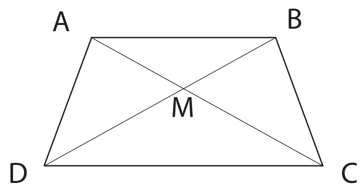


Figure 6.

The linking of information

The most common logical structure which is encountered in the heuristic approach to reasoning is the structure ”if ... then” without the two propositions always having a logical connection. The structure is used, even in the case of two unrelated statements. The first statement usually states something which is true (from the hypothesis or it is obvious in the figure), so as to give a flavor of truth, while the second is the required conclusion.

In the following example (problem 2, Figure 6), a student uses an ”illogical” postulate.

S: This is the (required) isosceles MDC.
I: Why do you think it is isosceles?
S: Since the trapezium is isosceles, then its diagonals are equal and bisect each other.
I: ...
S: Isn’t that correct?
I: What does bisect mean?
S: (he reconsiders) The diagonals are equal. If AC = BD then MC = MD as well.

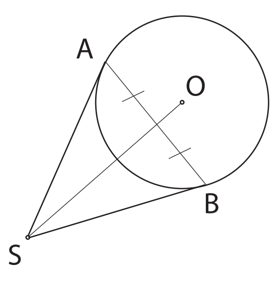


Figure 7.

Another student realizes the difficulties he is facing in the organization of the information of the first problem (Figure 7) and says to himself.

S: I'm trying now to associate too many elements, that SO is perpendicular to AB, that AO = OB, that OA is the radius of the circle. I've got to figure out how to associate them ... (so as to show) that S is equidistant from A and B.

The organization of information

The organization of information may also fail in the instance where the student wants to use some (theoretical) proposition, but is not capable of doing so. Specifically, it is possible for the student to correlate the context of the problem to a theorem but, at the same time, not being able to activate a link between the hypothesis and the conclusion. We call this "inactive propositions".

This particular weakness may give a false impression to the teacher, concerning the knowledge of his student, for example, when the teacher hears him refer to some theorem. A simple reference to a proposition does not automatically imply the ability of the student to use it. Understanding is confirmed only by the use of propositions.

An example of "inactive proposition" is the basic property of the isosceles triangle. Many students, adopting the heuristic approach, report that "one height of the isosceles triangle is its median and bisector too", and yet need to compare triangles in order to show that if such a property is true, it is an isosceles triangle. In other words, they do not activate the intrinsic connection of the basic property, in spite of the fact that they see that the hypothesis applies.

In the following example, the student reverses the direction of the connection between hypothesis and conclusion. Though she wants to prove that the triangle SAB is isosceles, she unconsciously assumes that SAB is isosceles and therefore her reasoning is erroneous (see the section *Comparison of the three approaches*, heuristic approach in problem 1).

The Theoretical Approach

The Theoretical Approach is adopted by students with a mathematical behavior. For these students, the transition from subjective to objective has been accomplished to a large degree.

In this approach, geometrical facts are characterized and distinguished by the way in which they are made acceptable. In particular, a geometrical fact, a theorem, is acceptable only because it is systematized within a theory, with a complete autonomy from any verification or argumentation at an empirical level (Mariotti et al., 1997). Reasoning is not based on whatever the students see or construct, but on an objective reality external from them, specifically on the body of propositions, which constitute the geometrical theory.

Visualization and construction have a secondary, helping, role in reasoning. This becomes evident from the fact that, the students construct their mental images on paper, and even though their figure may be imperfect, they consider that the various properties apply, since they are supported by a generally accepted theory. In other words, a process of idealization of the figure automatically operates. The students' efforts shift to the locating of the appropriate theoretical argument, which will justify the truth.

Characteristic of mathematical behavior is that, the individual, in order to comprehend a mathematical proposition, feels the psychological need to be convinced of its truth with some mathematical reasoning. Later, he uses the same mathematical reasoning to convince others. This *need for proof*, according to Rodd (Rodd, 2000, p. 234), is related to the increasing demand for strictness, as the students progress.

Pre-requisites for the theoretical approach are:

- the knowledge of the terminology (of the required theoretical propositions) and

- the ability to engage in productive, logical reasoning.

In the following example (Figure 8), a student who adopts the theoretical approach thinks aloud. He refers to the appropriate property of the trapezium and correlates it with the demands of the problem.

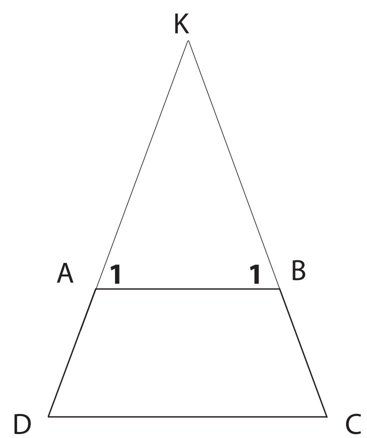


Figure 8.

S: it is an isosceles trapezium, so the angles adjusted to the base are equal, $\angle D = \angle C$. If I extend DA and CB I'll construct a couple of isosceles triangles (he makes the construction). Here, KAB and KDC are isosceles triangles, because the angles at their base are equal. Obviously it is $\angle D = \angle C$, and $\angle A_1 = \angle B_1$ since $\angle A = \angle B$ in the trapezium.

Comparison of the three approaches.

In order to get a better insight of the reasoning processes and their differences, we quote different solutions of the two problems. In each problem, we follow the reasoning processes of three students, each one adopting one of the three approaches. In our sample, there was not a case of a student who adopted the visual approach to one problem and the theoretical approach to some other.

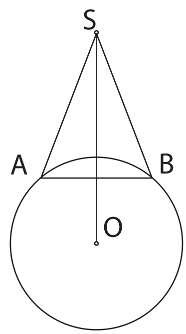


Figure 9.

Problem 1. *Visual approach*

S: I will draw two segments ... two equal segments SA = SB, A and B on the circle, ... this is the triangle SAB (Figure 9).
I: Is this triangle isosceles? Why?
S: As you can see in the figure ...
I: Can you justify it?
S: Yes, it is isosceles because I drew it to be, I drew SA = SB!

Comments: the student's reasoning is based on what he can see in the figure and what he has constructed.

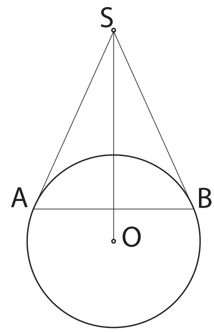


Figure 10.

Problem 1. *Heuristic approach*
The student draws the tangents SA, SB to the circle (Figure 10).

S: This SAB, ... is the requested triangle, because SA = SB.
I: What makes you believe it is?
S: It is ... I don't exactly know why ... if I draw a perpendicular line from S to O ...
I: Yes, you mean you will join S to O?
S: I want to draw a perpendicular line from S to AB, this line will pass through the midpoint of AB. Then I will compare the two triangles. They have two equal sides therefore their third side will be equal as well, so SAB is isosceles!

Comments: 1) Although she draws the two tangents to the circle, she is not capable of using their property. 2) She reverses the direction of reasoning, she assumes that SAB is an isosceles triangle (conclusion) and draws the conclusion that SO is perpendicular bisector (false organization of information).

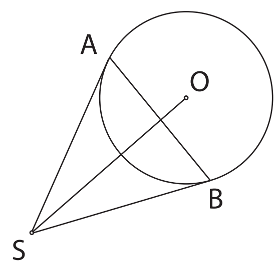


Figure 11.

Problem 1. *Theoretical approach*

S: I draw SO, this will be perpendicular bisector to every chord AB of the circle which is perpendicular to it (Figure 11). SAB is an isosceles triangle because S is equidistant from A and B.

Comments: The student refers to a theorem, relevant to the problem. He knows the property of perpendicular bisector, which he applies in reasoning.

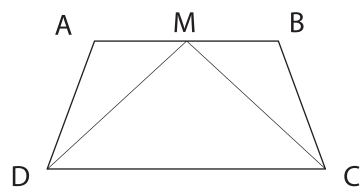


Figure 12.

Problem 2. *Visual approach*

S: I'll take this point M, the midpoint of the base AB, ... if I draw the segments MD and MC then I have an isosceles triangle (Figure 12)
I: You mean the triangle MDC?
S: Yes.
I: What makes you believe that it is isosceles?
S: If I'll measure them, ... I mean measure its sides MD and MC ... I'll see that they are equal.

Comments: The student's solution is based on a perceptive apprehension of the figure and its symmetry. That is to say, she constructs an isosceles triangle having the same axis of symmetry as the isosceles trapezium. She refers or uses none of the properties of isosceles trapezium. Measurement constitutes a proof for her.

Problem 2. *Heuristic approach*

S: I'll draw a line, passing through the midpoint M of AB, perpendicular to CD. Then this triangle MDC is isosceles (Figure 13).
I: What makes you believe that?
S: This line is ... perpendicular bisector of DC.
I: Can you give a good reason for that?

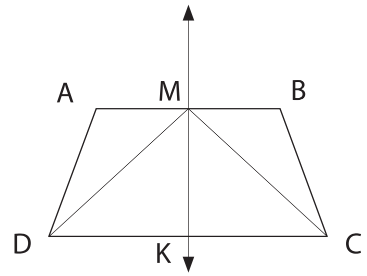


Figure 13.

S: ... there is something, I don't remember ... These two triangles are equal, MKD = MKC because (he compares) K = 90°, MK is a common side of the two triangles and KD = KC. As a result, MD = MC, the triangle MDC is isosceles.

I: I don't really understand, why is KD = KC?

S: This line KM ... is perpendicular bisector of DC, therefore K is the midpoint.

Comments: The student tries to use a theoretical discourse, which presents deficiencies in the completeness of the information. On the other hand, he draws the perpendicular bisector but he is not capable of using it for reasoning. So he compares the two triangles.

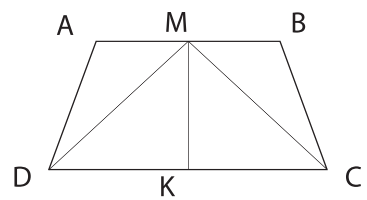


Figure 14.

Problem 2. *Theoretical approach*

S: Trapezium ABCD is isosceles. We know that the line connecting the midpoints of the bases (of an isosceles trapezium) is their perpendicular bisector. Therefore, I'll draw this line MK. Here is the isosceles triangle MDC (Figure 14).

I: Is it really isosceles?

S: Yes, every point on this line MK is equidistant from the points D and C, so MD = MC.

Comments: The student knows and can use productively the properties of the isosceles trapezium. His reasoning is based on the property of the perpendicular bisector.

Conclusions

Our study allowed us to become aware of a number of unconscious reasoning processes, which are used by students, when solving geometrical construction problems. These processes have been categorized into three approaches, namely: visual, heuristic and theoretical. A brief comparison of them is given in the following Table 2. Differentiation is according to the basis of the students' reasoning and according to the kind of language, which they use in their arguments.

Our data support that in the visual approach, reasoning mainly rely upon the geometrical figure and its construction. Measurement, as a method of reasoning, is a particularly important tool for the students, as indicated by their actions:

- Students frequently feel a need to use measurement for their own confirmation, before attempting to express any reasoning.
- Students may persist in the results of measurements, even in the light of conflicting elements.
- Students continue to use measurements, even when they have advanced to more theoretical reasoning. It is characteristic that they revert to measurement whenever they encounter difficulties with other methods.

In the heuristic approach, as our data support, a bridge is created between specific perceptual and abstract thought. Reasoning continues to be based on visualization and on construction. It gets, however, a more complex character. At the same time, reasoning gradually acquires a theoretical character and, as it still contains incomplete and erroneous elements, is characterized as "imperfect theoretical discourse".

This imperfect theoretical discourse of the students indicates two kinds of weaknesses (a) regarding its completeness and (b) regarding the organization of information. The acquisition and organization of information can develop in parallel in the human mind, but the organization takes longer. Specifically in Geometry, difficulties appear in the creation

Table 2. *The three approaches to reasoning*

Approach	Visual	Heuristic	Theoretical
Reasoning is based on	* mental representation * construction	* mental representation (complex) * construction (complex) * theoretical propositions (often incorrect)	* theoretical propositions
The processes of reasoning is	**natural discursive process** which is based on * the figure * measurements * a description of the steps of construction	**natural or imperfect theoretical discourse** which is based on * visual reorganization of the figure (1D or 2D) * instructions for the construction * theoretical propositions (true/inactive/pseudo-propositions)	**theoretical discursive process** which is based on * productive logical thought processes

of links between propositions, the reason being that links must have a direction from the given to the required.

In theoretical approach, students know how to function within the axiomatic system. Moreover, they have developed other, composite abilities, which facilitate the solution of problems, such as pattern imagery or the ability to recall relevant problems, which they have solved in the past, and which facilitate their dealing with the given situation.

According to our data, students' reasoning follow regularities that characterize their approach (visual, heuristic, theoretical), no matter, what the problem they face is. For example, in both the above problems, justifications of all students that adopt the visual approach rely on the construction of the figure or on its measurements. The fact that the isosceles triangle has a lot in common with the isosceles trapezium has a minor impact in their geometrical function.

Weaknesses of reasoning may be due to the organization and to the completeness of the information. The ability to use the theoretical propositions, the way of linking them or the presence of pseudo- propositions, is a gauge, which reveals to the teacher the quality of organization of the information in the mind of the student. On the other hand, a major problem of communication between the teacher and the student is brought about by vagueness in the use of terms. The student may be able to use verbally the correct terminology (an obvious fact) without linking it with the correspondingly correct image (latent fact, which takes place in his mind) or conversely. The only way for the teacher to understand the inconsistency in the mind of the student is to ask him to construct the figure.

The enhanced comprehension of the weaknesses which students confront in reasoning will lead teachers to more efficient ways of teaching Geometry, while much research on the subject of reasoning in Geometry is still needed.

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Sammenfatning

Artiklen undersøger kognitive processer hos 15-årige elever under deres arbejde med geometriske problemer i forbindelse med konstruktion af ligebenede trekanter. Tre forskellige ræsonnementsformer er udviklet til kategorisering af elevernes problemløsning; nemlig visuel, heuristisk og teoretisk ræsonneren. Inden for hver ræsonnementsform analyseres elevernes sprogbrug, grundlaget for og svaghederne ved deres ræsonnementer. Analyserne har som intention at støtte lærere i at genkende og forstå relationen mellem elevernes ræesonneren og deres geometriske tænkning og forståelse. En sådan viden kan anvendes som grundlag for at støtte og udfordre elevernes problemløsning inden for geometri på en relevant måde.