A less radical constructivism

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This paper poses two problems for von Glasersfeld's Radical Constructivism. The first problem concerns the rejection of the idea that it is possible to share meanings. The second problem is that Radical Constructivism rejects the notion of an objectively existing reality of which we can have objective knowledge. Yet with respect to mathematics, von Glasersfeld seems to claim that it is possible to obtain objective knowledge. We propose an alternative position – Constructive Realism – that gives a description of what mathematical objects are and gives an account of why knowledge in mathematics is objective. Furthermore, we argue that some of the assumptions, used in von Glasersfeld's description of how numbers are formed, support the claim that some meanings are objective and that communication is possible. Finally, we consider some of the implications this position has for mathematics education.

Radical Constructivism is a highly regarded theory of knowledge that underlies many approaches to the teaching of mathematics. At a conference celebrating the ideas of von Glasersfeld, H. Foerster recently characterised Radical Constructivism in the following way:

> In fact, I feel that this 'Ernst von Glasersfeld Celebration' was the proper anacrusis, the proper auftact, the proper prelude for a style of thinking that will initiate, and then dominate, the third millennium. Isn't that an overstatement?
> I say: 'No!' (von Foerster, 2000, p. xi)

This paper argues that, although Radical Constructivism may be regarded as a successful foundation for some approaches to mathematics education it remains problematic both as a foundation for a theory of learning

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and as a philosophical theory. Firstly Radical Constructivism rejects the objectivity of knowledge and the idea that knowledge can be shared. At first sight this seems to be problematic for a theory of learning. Secondly, Radical Constructivism generally rejects the view that we can have objective knowledge and the naïve views of ontology, knowledge and truth are given up. Yet in mathematics all these views seem to be reintroduced. In mathematics, knowledge is certain and objective. In some places von Glasersfeld talks about 'abstract' objects, which indicates that there is an ontology of mathematical objects. However, von Glasersfeld writes nothing to indicate of what this realm of abstract objects consists, and there are no general arguments as to why knowledge of this realm is objective.

This paper proposes an alternative position on the nature of mathematical objects – Constructive Realism – where mathematical knowledge is claimed to be objective. This is mainly a position describing the ontology of mathematical objects. It can therefore be regarded as a supplement to von Glasersfeld's Radical Constructivism in the sense that Constructive Realism explicitly addresses the question of what mathematical objects are. We shall show that some of the assumptions, underlying von Glasersfeld's description of how numbers are formed, can be used more generally to argue that some meanings can be shared and that communication is possible.

Radical Constructivism

Radical Constructivism is foremost a theory of knowledge or, as stressed by von Glasersfeld, a theory of *knowing*[1] (von Glasersfeld, 1995, p. 1). Its main thesis is that knowledge is constructed by the individual and that knowledge consists of the individual's re-presentations of the world. This has certain consequences. Firstly, it entails that the individual can not know about the external world independently of the re-presentations that he or she makes of it. This does not mean that von Glasersfeld rejects the existence of an eternal world. Von Glasersfeld stresses that giving up the view that our representations corresponds to objectively existing objects in the world does not entail that we have to give up the existence of an external world alltogether:

> For believers in a representation [...] they immediately assume that giving up the representational view is tantamount to denying reality, which would indeed be a foolish thing to do. The world of our experience, after all, is hardly quite as we would like it to be (von Glasersfeld, 1995, pp. 14-15).

Secondly, as knowledge, as well as meaning, is actively constructed by the individual, no two individuals can share the same knowledge and therefore there can be no shared meanings. Von Glasersfeld's point is that when we learn new words our experiences are always different, and we will therefore necessarily associate different meanings to words.

> [T]he notion of "shared meaning" is strictly speaking an illusion. This is so because we associate the sounds we come to isolate as "words" not with things but with our subjective experience of things – and although subjective experiences may be similar for different subjects, they are never quite the same (von Glasersfeld, 1996, p. 311).

Thirdly, von Glaserfeld's position entails that the notion of truth as correspondence must be given up. Instead von Glasersfeld replaces a theory of truth by a notion of viability.

> Constructivism drops the requirement that knowledge be "true" in the sense that it should match an objective reality. All it requires of knowledge is that it be viable, in that it fits into the world of the knower's experience, the only "reality" accessible to human reason (von Glasersfeld, 1996, p. 310).

Thus Radical Constructivism has been characterised as a theory rejecting "the triplet of mutually supporting concepts of 'ontology', 'reality' and 'truth'" (von Foerster, 2000, p. xii).

Although von Glasersfeld stresses that Radical Constructivism is a theory of knowledge, the position also includes claims about ontology, truth and meaning. It is therefore fair to evaluate it as a general philosophy. Although there is no general definition of what constitutes a philosophy, it is widely acknowledged that it should at least have the following properties. Firstly, a philosophy should be internally consistent, and secondly, claims should be supported by arguments. If the philosophy is a 'philosophy of some domain' such as a 'philosophy of how people learn', the philosophy should also agree with certain acknowledged features of this domain.

The notion of 'meaning' has been widely discussed in the philosophical literature. Although von Glasersfeld does not give a strict definition of the concept of meaning, the following description seems to pick out what he means: 'The meaning of a word is actively constructed by the cognizing subject on the basis of her personal experience'. With this description it is clear, as experiences differ from person to person, that no two people can share the meaning of a word. Von Glasersfeld provides the word 'rhinoceros' as an example. Some people may have seen a real rhinoceros either on safari or at the zoo. Others may have seen only a picture in a book. Yet others may associate the word with a play.

> When [...] you read the word 'rhinoceros', you had no idea what game I was playing, nor what use I was making of the word. Yet, you produced your re-presentation of the word. I emphasize that it was yours, because it was you who had at some earlier point in time extracted or abstracted it from your own experience. It was this re-presentation which, at that moment breathed life into the word for you (von Glasersfeld, 1995, p. 135).

Later von Glasersfeld discusses the notion of meaning in terms of reference. Here he uses the example of children learning a language:

> By the time human beings are 6 or 7-years old, they have developed a considerable mastery of the language spoken in the social group in which they grow up. They can use words and be understood by others and they understand a great deal of what others are saying. They are not yet at an age where they ponder *how* such an understanding might be possible. Nor do they have reason to suspect that the things they have associated with words are elements of their own experience rather than things that exist in themselves in an environment that is the same for everyone. Hence it seems quite natural that words should refer to independent objects, and that their meaning therefore is universal, in that it is 'shared' by all individual speakers. Every day these apparent facts are confirmed innumerable times, and if at some later stage reflections about our language are entertained they will almost inevitably be grounded in this conviction as a premise (von Glasersfeld, 1995, pp. 136-137, italics in original).

In this passage, it seems that the notion of 'meaning' and the notion of 'reference' are mixed up. Following Frege, it would make perfect sense to distinguish the notion of meaning from that of reference. Meaning could have the sense that von Glasersfeld has already explained, whereas the reference of a word denotes the object that is supposed to be picked out by the word. Note that this would also make sense for von Glasersfeld as he is not denying the existence of an external reality. It would be possible to claim that our words refer to certain objects even though we do not have direct access to them, in von Glasersfeld's sense that we can not have a true picture of what these objects are like independently of our experiences of them. Something similar to these considerations was pointed out by Thomas.

> The relation of knowledge to the world, misunderstood as a mirror image's relation to the object reflected must be replaced by the relation of reference. When one says 'cat' one refers to cats; one does not as von Glasersfeld agrees, 'capture' or reflect anything of or about cats [...] Knowledge and even lies and fiction refer; he seems to have forgotten this important but unique relationship, the existence and importance of which is widely acknowledged (Thomas, 1994, p. 34).

As Radical Constructivism rejects the idea that we may have objective knowledge about the world, it has been argued that it leads to solipsism. There are (at least) two versions of solipsism. The metaphysical version claims that the only thing that exists is the subject and its consciousness, whereas the epistemological version states that the only thing a person can know is that person's own experiences. It seems fair to say that Radical Constructivism embraces the epistemological strand of solipsism, whereas it is debatable whether this leads to the metaphysical claim. In many places von Glasersfeld rejects the idea that Radical Constructivism leads to solipsism (although he does not declare what he means by solipsism):

> The statement that the construction of the experimental world is irrevocably subjective has been interpreted as a declaration of solipsism and as the denial of any "real" world. This is unwarranted. Constructivism has never denied an ulterior reality; it merely says that this reality is unknowable and that it makes no sense to speak of a representation of something that is inherently accessible (von Glasersfeld, 1996, p. 309).

What von Glasersfeld rejects here is the metaphysical claim. One could ask: If the only thing we know for sure is our own re-presentations of the world or our experiences, then how can we be sure that anything but those exist? Could it be that we are only 'brains in a vat' and are only deceived by a mad scientist to believe that we have certain experiences (as is suggested by Hilary Putnam in 'Brains in a vat', 1981)?

The Construction of Numbers

According to von Glasersfeld, the basic concepts of arithmetic are unit, plurality and number[2]. In 'Radical Constructivism' (1995), von Glasersfeld describes how numbers are constructed using these concepts. Von Glasersfeld provides two things. First he presents an analysis of how the concepts of unit and plurality may be structured and second how these concepts may be obtained from perceptions followed by ”a succession of reflective abstractions”. *Reflective abstraction* is a term borrowed from Piaget and denotes any conceptual construction that does not depend on any particular sensory material (von Glasersfeld, 1995, p. 69).

Von Glasersfeld finds that numbers are not inherent in things but are somehow abstracted from our perception of things. Furthermore, numbers can be associated to things in any number of ways. For example, hearing four clock strokes can be associated with the number 4 if the strokes are associated with one clock and the strokes are taken to represent the time, or equally the four strokes could be taken to represent the number 1 if there were four clocks striking at the same time. From this von Glasersfeld concludes that ”Units, then, are the result of an operation carried out by a perceiving subject, not a property inherent in objects” (ibid, p. 165).

Having considered the notion of unit, von Glasersfeld notes that Euclid’s definition of number as ”number is the aggregate of several units” is inadequate.

Von Glasersfeld argues that before one arrives at the conclusion that there are a certain number of entities, one has to realise that there is a plurality of these entities. Plurality is arrived at by the awareness of repeating a specific categorisation. It can thus be said that a plurality is also an aggregate of several units.

To explain how the concept of unit is formed, von Glasersfeld uses a model which he denotes ’the attentional model’. Von Glasersfeld employs the fact that human beings are able to shift the attention of their perceptions without moving their eyes. In other words, we can deliberately shift focus from, for example, seeing a crowd of people as a crowd, to focus on one particular person. Using this model, von Glasersfeld explains how through two abstractions, we are able to view, for example, an apple as a unit. The first abstraction concerns ’uniting’ our different perceptions of an apple to the concept of an apple. The second abstraction abstracts the particular object, leaving the concept of a unit. Von Glasersfeld’s conclusion is that the result ”represent a wholly abstract entity, because it no longer matters what the central moment of attention was focused on or whether these was one or several” (ibid., p. 169).

Having defined the operational concepts of unit and plurality, von Glaserfeld turns to the final concept, that of a number. In Glasersfeld’s model, the transformation of a plurality into a composite unit, that can be considered a number, requires two further operations, those of conceptual iteration and counting. Establishing the number of a collection of entities requires the realisation that there is a bounded plurality of these entities. Conceptual iteration is the mental activity taking place when a person stops considering one entity and moves on to the next. A counting procedure associates with each item "a number word of the conventional number word sequence". This counting procedure leads to the conception of associating a certain number to a certain collection. The last step consists of constructing the notion of number: "What constitutes the abstract concept of *number* is the attentional pattern abstracted from the counting procedure" (ibid., p. 172).

One of the key features in explaining the certainty of mathematical knowledge is the fact that "in the construction of the abstract concept of number all sensory material is eliminated"(p. 174). Thus von Glasersfeld arrives at the conclusion that 2+2=4 is not questionable as,

> the symbol '2' stands for a conceptual structure composed of two abstract units, to which the number words 'one' and 'two' were assigned respectively. The symbol '+' requires that the units on the left be lined up with the units on the right and subjected to a new count. Since the standard number word sequence is fixed, and the items in the count are not questionable sensory things but abstract units, there is no way it could not end with four (ibid., pp. 174-175).

There are a number of points worth discussing here. The first is a minor point concerning similarities between von Glasersfeld's definition of number and Frege's. I mention these similarities as von Glasersfeld dismisses Russell (and thus Frege) out of hand on the basis (of a misreading) of a very short quotation, not realizing that 'number' and 'number of' mean two different things for Russell (von Glasersfeld, 1995, p. 163).

In terms of ontology there is, of course, a major difference between the philosophy of Frege and that of von Glaserfeld. Frege, being a Platonist, believed in the independent existence of numbers and concepts (that are the referents of predicates), whereas von Glasersfeld rejects any talk of a fixed ontology. Yet, their respective descriptions of the construction of numbers are similar. Both agree that numbers are not in the things themselves, and they can both be construed to say that numbers are associated with concepts. Frege, as well as von Glasersfeld, starts out with associating (and thereby defining) the notion of 'number of a concept' to a concept that applies to a collection of things. Whereas Frege defines 'n is a number' by the definition 'n is a number if there is a concept that n is the number of', von Glasersfeld defines number as an abstractional pattern following the counting procedure. But to perform the counting procedure presupposes that there is given a concept that determines what we should count. Thus von Glasersfeld's definition is not dissimilar to Frege's.

In his discussion of numbers von Glasersfeld introduces the conception of numbers as 'abstract entities' (von Glasersfeld, 1995, p. 174) and since there is no discussion about this notion, one needs to question what von Glasersfeld thinks of these entities. Furthermore, it seems as if von Glasersfeld would claim that mathematical statements are objectively true, as their truths are certain and therefore can not be doubted. It may be that von Glasersfeld rejects the idea of a fixed ontology independent of our constructions, but at least now he has constructed a domain of entities about which our discourse is objective. This poses two questions. Firstly, if communication about mathematical statements is objective, are there then other parts of our discourse that is objective in the same sense, allowing that it is in fact possible to 'share meanings'? We shall return to this question in the next section and argue that the answer is yes.

The second question concerns the assumptions underlying von Glasersfeld's description of how numbers are formed. It seems that some of these assumptions contradict von Glasersfeld's general theory. One central assumption is our ability to perform reflective abstractions. The question is whether this ability is uniform to all human beings, or whether it could be imagined that different subjects perform this operation differently, resulting in different concepts of a unit. He also makes an assumption about how human beings perceive objects, but could it not be that our sensory apparatus works differently? Another assumption used is that we have an ability to regard certain objects as similar, for example, when determining that there is a plurality of a certain kind of object. This requires that we have the ability of 'seeing an entity as being the same as another entity'.

Finally, von Glasersfeld claims that we have agreed on the counting procedure and the number word sequence. Could it not be that this is a seeming agreement where in fact each member of the community associates something different with their number sequences, a difference that has not yet surfaced? (See e.g. Kripke,1982, p. 8-12)

In von Glasersfeld's description of how we form numbers, it seems as if he needs to make certain basic assumptions that contradict his general philosophy, but nevertheless can be taken as fundamental when explaining why at least part of our communication is objective. Some of these assumptions are:

– We have the ability to forming abstractions.

– We have the ability to 'seeing things as the same'.

– Our sensory apparatus works in the same way.

Radical Constructivism and Mathematics Education

Radical Constructivism has been very well received by teachers and mathematics educators (Skott, 2004). Here I shall outline a few basic themes that are central to our discussion. The first theme is constructivism, according to which the students actively construct their knowledge. As knowledge is not passively received, teaching should somehow lead the students to actively construct their knowledge. Tools to make this possible are language and the use of perceptual material[3]. When using perceptual material, von Glasersfeld emphasizes that the mathematical objects are not inherent in these, ”but they must be seen as providing opportunities to reflect and abstract, not as evident manifestations of the desired concepts” (von Glasersfeld, 1995, p. 184).

A picture that illustrates the use of language is that of a dog helping a shepherd to steer his sheep. The first point to make here concerns the use of these tools while not accepting that language refers. Consider, for example, a situation where students are about to be taught the concept of a triangle. The teacher could be using a physical triangle to help the students form this concept. But how can the teacher tell the students to consider the triangle that he is holding, if his word ’triangle’ is not supposed to refer to it? According to von Glasersfeld, the meaning that a certain student would attribute to the word triangle depends on his or her previous experience with triangles. Perhaps, when the teacher talks about a triangle one student would start to think of music and others of all sorts of (irrelevant) features of triangles. The point is that, if the teacher refers to the triangle he is holding up, then it would be possible for the students somehow to abstract from their personal experiences with other triangles and concentrate on the actual triangle that is shown to them. Then the teacher could start telling the students that they should somehow abstract the shape of the figure to form the abstract mathematical concept of ’triangle’. Put differently, what the students are supposed to construct is the type ’triangle’ where the actual physical triangle can be regarded as a token.

In ’Moving beyond Constructivism’ Lesh and Doerr (2003) provide a number of reasons why (Radical) Constructivism is inadequate as a theory of knowledge. Firstly, Lesh and Doerr claim that, as a theory, general constructivism has proved not to be falsifiable. It seems as if any approach to learning can be formulated within the framework of constructivism. Secondly, they point out that forming constructions is only one of many ways by which knowledge is formed.

> [D]evelopment also typically involves sorting out, differentiating, reorganizing, refining, adapting, or reflectively abstracting conceptual systems that already exist at some concrete or intuitive level in students thinking (Lesh & Doerr, 2003, p. 532).

Furthermore, Lesh and Doerr refer to Clements (1997)[4] and Kamii (1982)[5] who claim that there are certain things that students learn that do not involve constructions, such as learning ”notational and procedural conventions that can be learned through demonstration, imitation and practice with feedback” (ibid., p. 532).

The second theme in Radical Constructivism as a theory of learning concerns the role of the teacher. This role has been extensively described in Steffe (1991). The teacher's role is to instruct the student to construct the appropriate knowledge and to be able to do this, the teacher somehow needs to know what the student knows. But according to Radical Constructivism, this is not possible, as there can not be any sharing of meanings. Instead the teacher forms a hypothetical model of the student's knowledge.

> Maya's language and actions were the experiences available to me in a learning environment as I strove to learn her schemes of operating and that my interpretation could be made only in terms of my knowledge. So from my point of view, my description of Maya's mathematical knowledge is unavoidably an expression of my own concepts and operations. Although I acknowledge that Maya had a mathematical reality separate from my own, I had no direct access to it (Steffe, 1991, p. 188).

This quotation makes it clear that there is no inter-subjectivity in the class room. A number of mathematics educators, see for example Confrey (1991) and Cobb (1999), have found this very problematic and have formulated versions of social constructivism, theories that combine constructivism with inter-subjectivity. Nevertheless Lerman (1996) has argued that it is not philosophically sound to combine theories that claim that knowledge is constructed by the individual with claims of inter-subjectivity. Lerman instead argues that inter-subjectivity is something that actually takes place in classrooms and therefore should not be explained away, leading to the conclusion that Radical Constructivism must be false.

Way out

We have pointed out that it is problematic for mathematics education if there is no inter-subjectivity, but we have also found that Radical

Constructivism seems to embrace a kind of objectivity concerning statements in mathematics. It is therefore tempting to propose, following Quine[6]:

> This type of argument stems of course from Quine, who has for years stressed both the indispensability of quantification over mathematical entities and the intellectual dishonesty of denying the existence of what one daily presupposes (Putnam, 1972, p. 57).

The question becomes whether there are reasons other than pragmatic ones, to accept that some knowledge is objective. I claim there is. In the dialogue Phaedo, Plato argues that certain concepts are independent of all experience. One concept is that of equality:

> So it necessarily follows that we knew the equal at a time previous to that first sight of equal objects which led us to conceive all these as striving to be like *the* equal, but defectively succeeding (Plato, 1955, p. 70).

Plato uses this to argue that our soul is immortal, but we can equally accept his conclusion that the concept of equality is a faculty inherent to our mind and is not obtained through experience. This leads to a Kantian inspired perception of the mind as consisting of certain faculties that are independent of our sensory experience and which form our experiences. It also leads to the conclusion that there are some concepts on which there seem to be universal agreement, and that are independent of experience.

I shall now present an alternative position, where mathematical objects are not claimed to exist independently of human beings, but where knowledge of them is nevertheless objective. This entails that communication is possible and that some meanings can be shared.

Constructive Realism

The general philosophy from which my position Constructive Realism has been developed is the conviction that a philosophy of mathematics ought to agree with mathematical practice. The position is the outcome of 1) a case study in contemporary mathematics and 2) a discussion of contemporary realist positions[7] in the light of this case study. These considerations will just be summarized here, for further details refer to Carter (2002, 2004).

The topic of the case study was the origin of K-theory, which today is a very successful tool used in many different branches of mathematics. K-theory started when the mathematician Grothendieck introduced a group, which he denoted the 'K-group'. The main question addressed was: what happens when new objects are introduced in mathematics? The case study showed that the K-group was constructed from previously existing entities (so-called sheaves). The construction methods were those of 1) forming a free abelian group and 2) forming a quotient. The case study provided further examples of objects that were introduced by constructions on previously existing mathematical objects. A simpler example of a mathematical object that is constructed from another mathematical object by forming a quotient is the set of rational numbers. (Note that I do not claim this is how the rational numbers were first introduced!) The rational numbers can be formed as a quotient of the set $Z \times Z$ under the relation R where $(a, b) R (c, d)$ if and only if $ad = bc$.

The main points of Constructive Realism can be summarised as follows:

1. Mathematical objects are introduced or constructed by human beings.
2. After a mathematical object has been introduced, it exists as an objectively accessible conceptual object.

To avoid confusion, I denoted mathematical objects as 'conceptual objects' rather than 'abstract objects', since the latter notion often is associated with that of not existing in time and space, which again is taken as implying that these objects exist independently of human beings. In what follows, we shall clarify the notions of 'conceptual object' and 'objective accessibility'.

In describing what a mathematical object is, I drew on a description of mathematical objects in Hilbert's writings. In his 'Grundlagen der Geometrie' (1968)[8] first published in 1899, he refers to mathematical objects as Gedankendinge, 'thought-things': "Wir denken drei verschiedene Systeme von Dingen: die Dingen der ersten Systems nennen wir Punkte und bezeichnen sie mit A, B, C,..." (p. 2). Thus mathematical objects are described as 'objects of the mind' or objects that we can think of. This does not lead to the conclusion that mathematical objects merely exist in the individuals mind. Rather a mathematical object should be regarded as the *type* where the object, that the individual mathematician thinks about, is a *token* of this type. Something like this can be derived from Hilbert's statement:

> [...] the objects of number theory are for me – in direct contrast to Dedekind and Frege –the signs themselves whose shape can be generally and certainly recognized by us – independently of space and time, of the special conditions of the production of the sign,

> and of insignificant differences in the finished product (Hilbert, 1922/1966).

The last sentence can be read as: we recognise a sign independently of its instantiations, and thus may interpret the sign as a type, whereas the productions of it can be regarded as tokens of the sign. It is a general feature of mathematics that objects may have multiple representations. For example, the number π can be considered as the relation between the circumference of a circle and its diameter or as the sum of an infinite series. The imaginary unit i can be considered as the solution to the equation $x^2 = -1$ or as the pair (0, 1) in the real plane.

Accessibility of Mathematical Objects

It seems to be possible for mathematicians to provide rigorous descriptions of mathematical objects so that other mathematicians can access them. An example is the K-group which is defined as a quotient. However, it is by no means clear what constitutes a rigorous description (see for example, Panza, 2004). In some cases, when a mathematical object is first introduced, it does not have a rigorous description. An example is the infinitesimals, the fundamental objects of calculus. They were employed in the first methods used by mathematicians to solve problems of finding tangents and extrema (minimum and maximum) of curves[9] and continued to haunt Leibniz's and Newton's formulations of calculus.

Fermat was one of the mathematicians who found methods to solve the above mentioned problems. In part these methods consisted of first dividing an expression by a magnitude E and later removing all remaining occurrences of E.

The problem was, of course, that these magnitudes, later denoted infinitesimals, were on the one hand considered to be different from zero, and on the other hand, equal to zero as they could be removed. The problem with infinitesimals did not disappear until the 19th century when a rigorous definition of a limit was introduced, which actually got rid of the infinitesimals. Even later, the infinitesimals were reintroduced in the so-called non-standard analysis.

An example from more advanced mathematics is the surface introduced by Riemann, later named a Riemann surface. Riemann's contemporaries found it very hard to grasp his description of this kind of surface:

> For many questions, such as the study of algebraic and Abelian functions, it is advantageous to represent the mode of branching of a multiple-valued function geometrically as follows. Imagine a surface in the (x, y)-plane, coinciding with it (or an infinitely thin sheet

> spread over it), which extends as far, and only as far, as the function is defined. By continuation of the function, this surface therefore will also be extended further. In a region of the plane where the function has two or more continuations, the surface will [...] consist of two or more sheets, each of which represents one branch[10] (Zweige) of the function. Around a branch point[11] (Verzweigungsstelle) of the function, each sheet of the surface will join onto another one, so that in the neighborhood of such a point the surface may be regarded as a helicoid whose axis is perpendicular to the (x, y)-plane at this point, having an infinitesimal pitch. If, after z has made several circuits around the branch point, the function again attains its former value after n circuits of z about a (like $(z-a)^{m/n}$ with m and n relatively prime), then one must indeed assume the top sheet of the surface continues through the others into the bottom one.
>
> The multiple valued function has only *one* value defined at each point of such a surface representing its branching, and can therefore be regarded as a single-valued function of position on this surface (Riemann, translation from Birkhoff, 1973, p. 52).

This can be classified as an intuitive description of this mathematical object. Later, in 1955, Weyl in 'Die Idee der Riemannschen Fläche' provided a rigorous definition of a Riemann surface:

> (a) There is given a set of objects called "points of the manifold $\mathfrak{I}$". For each point p of the manifold $\mathfrak{I}$, certain subsets of $\mathfrak{I}$ are defined to be neighborhoods of p on $\mathfrak{I}$ [...] (Weyl 1955, p. 17).

According to Hilbert, what makes us recognize a mathematical object is the sign that represents it. Above, we took this claim as a stepping stone to distinguish between types and tokens. The type is the mathematical object and tokens the different representations of the object. In mathematics we use 'symbols' or notations to represent objects all the time, for example, '2' represent the natural number 2. In some (simple) cases these symbols may also tell us how to access the objects. But this generally is not true. For example, the symbol of the K-group is 'K(X)', but this does not tell us how to access the group. Thus in general, a symbol is merely a useful notation for a mathematical object and does not tell us how to access the object.

To describe how we are able to access mathematical objects, I propose a view inspired by Kant on the formation of concepts. On this view an object has a form and content. The content of an object consists of the 'material' that the object is formed of. This material can be physical stuff or, when considering mathematical objects, abstract or conceptual stuff. The form of an object consists of its construction method or the way that an object is considered in our mind. Note that the form used to consider a physical object can be the same as the form used to construct a mathematical object. There are several examples of this. One is the concept of a set. We are able to consider both a collection of objects and a collection of abstract objects as forming a set. It seems reasonable that the concept of a set used in mathematics has been inspired by this innate ability of ours. Some of the properties that sets are supposed to have according to the axiom system of Zermelo-Fraenkel, also seem to have their origin in how we perceive collections of objects. For example, the axiom of pairing that says that it is possible from two collections to form a new collection that is again a set.

A quotient can be regarded as formed by a conceptual construction that we can also perform on physical objects. Consider, for example, a room full of people. Then it is possible to form the relation 'same sex as'. The quotient, that results from the set of people under this relation, consists of the two-membered set of women and men. Furthermore, when reading Riemann's description of his surface on the previous page, it is hard not to imagine the sheets as very thin physical sheets on which the functions can be defined. This means that as long as we have some abstract objects (or conceptual objects) on which to begin, we form new ones by forming conceptual constructions on them. This does not entail that the abstract objects on which to construct new objects are fixed once and for all. New abstract objects are extracted from our experience all the time and are embedded to the current mathematical universe.

Constructive Realism proposes that human beings introduce mathematical objects, but that knowledge of them is nevertheless objective. New objects can be introduced in a number of ways. They can be extracted from physical reality and then formulated in a mathematical system making discourse about them objective. Examples of this kind are the introduction of the natural numbers and the Riemann surface. Another method of introducing mathematical objects is to construct objects from previously existing objects by accepted construction methods. The construction of the K-group is an example. What is argued here is that the basic methods used in the introductions of mathematical objects are certain concepts or even forms of considering objects that are inherent in our minds and that are therefore independent of all sense experience. It is also precisely because these forms are inherent to our minds that mathematical statements are objective. The statements that can be formulated about a mathematical object concern our possible ways of considering this object. Note that on this view, it is still possible to formulate versions of some of the slogans of constructivism, for example, that each individual forms his or her version of 'mathematical reality'.

Some implications for mathematics education

Let us conclude by considering two themes relevant to mathematics education. The first theme concerns motivation. It is often the case that the formulation of a mathematical object is the result a long process starting with an unrigorous description of an object, such as an infinitesimal or a Riemann surface, and ending with a rigorous description in a certain framework or theory. It is usually this theory that is presented to the student. It may be accessible enough, but I want to emphasize that it could better motivate the student to learn in which context a particular object was introduced. For example, the differential quotient is often presented with examples of use that illustrate its origin and motivates the definition. But the definition of Riemann surfaces due to Weyl reveals little of the origins of these surfaces. I propose that it could be helpful for the student to learn, that they were introduced to turn Abelian functions into 'real' functions by defining these functions on the surfaces and to read Riemann's very intuitive description of the surfaces[12].

This leads us to the second theme. A reading of Riemann's description of the surface also provides a picture – or a representation – of the mathematical object.

When discussing how to access mathematical objects, I noted that looking at a symbol representing a mathematical object is not sufficient to gain access to the object. To gain access to a mathematical object, one needs to know about the objects it is constructed from *and* to understand its construction.

Furthermore, to be able to reason about an object, one needs somehow to have a picture or a representation of this object. In the example of the Riemann surface, a picture of the surface can be obtained from the former description of the object quoted above. However, this does not have to be the case. A simple example concerns reasoning about the multiplication of natural numbers. Multiplication of two numbers can be pictured as (the area of) a rectangle. By using this picture, one can see many things, for example, that multiplication is commutative. Reasoning using representations may not constitute an accepted proof (although it may lead to one), but it leads to new ideas as well as understanding of why theorems hold. I claim that many mathematicians think in terms of representations and that when teaching a subject, one should provide students with such representations.

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Notes

1 The quotation is a bit puzzling and reads "It is an attempt to explain a way of thinking and makes no claim to describe an independent reality. That is why I prefer to call it an approach to or a theory of *knowing*. Though I have used them in the past, I now try to avoid the terms 'epistemology' or 'theory of knowledge' for constructivism, because they tend to imply the traditional scenario according to which novice subjects are born into a ready-made world, which they must try to discover and 'represent' to themselves" (pp. 1-2). The view described here is what philosophers usually denote a common sense or naïve version of knowledge and it is strange that von Glasersfeld refers to it as 'the traditional scenario', especially as later in the book he describes a whole collection of philosophical positions distinct from this view.

2 Von Glasersfeld actually writes that these concepts are basic to mathematics. It may be argued that they are basic to arithmetic, but other thinkers

have advocated the claim that there are different concepts underlying different parts of mathematics. One common division is that between arithmetic and geometry as a division between the discrete and the continuous.

3 It is a common theme to constructivists of all sorts that communication is an important part of mathematics teaching and, for some, even an important aspect of knowledge of mathematics (see e.g. Björkqvist, 1993). It is thus vital that communication is possible.

4 Clements, D.H. (1997). (Mis?) Constructing constructivism. *Teaching Children Mathematics*, 4 (4), 198-200.

5 Kamii, C (1982). *Number in preschool and kindergarten*. Washington, DC: National Association for the Education of Young Children.

6 It is claimed by Lesh and Doerr (2003) that we constantly form and test models of physical reality. One way of testing models is to construct a physical version of a model that can be causally tested. Actually, this has been done for several centuries in science and now constitutes the basis for our constructions of buildings. We do not doubt that the models from which these buildings have been built are 'true'. If we did, we would need to think twice before crossing the bridge between Funen and Sealand!

7 These are the Set theoretical Realism of P. Maddy (1990), Structuralism of S. Shapiro (1997) and Realistic Rationalism of J. Katz (1998).

8 In the lecture 'Logische Principen des mathematischen Denkens' from 1905, Hilbert gives a similar description.

9 The history of this subject is too rich to tell here. Refer to V. Katz (1998) for a fuller exposition. Before calculus was introduced as a general discipline by Newton and Leibniz, a number of mathematicians worked on different methods to solve some of the problems that can be solved by calculus. Among these problems were: finding sub tangents and maxima of curves and finding areas of certain regions. Note that these problems were considered as isolated problems, in the sense that different methods were used to solve them and that the mathematicians could not see their relationship as we do today. Note also that the notion of function as used today was not used then.

10 The different continuations of a multiple-valued function are denoted the branches of the function by Riemann.

11 Riemann defines a branch point as the point about which a branch of the function continues into another branch.

12 An interesting mathematical notion from algebraic topology is that of a (simplicial) complex. Spanier (1966) defines it as a set K consisting of a set of vertices $\{v\}$ and as a set $\{s\}$ of finite nonempty subsets of $\{v\}$ called simplexes subject to some conditions. The point here is that this abstract notion of a complex is the result of generalizing the concept of a polyhedron to arbitrary dimensions. To grasp the notion of a complex, it is very useful to draw two- or three-dimensional pictures of triangular shapes.

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Sammendrag

Artiklen formulerer to problemer for von Glaersfeds Radikale Konstruktivisme. Det første problem vedrører forkastningen af ideen om mulighed for delagtiggørelse af mening. Det andet problem er, at den Radikale Konstruktivisme benægter eksistensen af en objektiv ydre virkelighed, som vi kan have objektiv viden om. Dog ser det ud til, at von Glasersfeld mener, at vi kan have objektivt gyldig viden i matematik. Vi præsenterer en alternativ position – Konstruktiv Realisme – som beskriver hvad matematikkens objekter er og redegør for hvorfor viden i matematik er objektiv. Desuden argumenterer vi for, at nogle af de antagelser som von Glasersfeld bruger i beskrivelsen af hvordan tallene dannes, støtter påstanden om at nogle meninger er objective og at kommunikation er mulig. Til sidst giver vi nogle forslag til hvilke implikationer denne position har for undervisning i matematik.