Perceiving the derivative: the case of Susanna

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This paper is a report of a study on how a less successful student perceives the derivative from the graph of a function. A task-based interview of a grade 11 student was analyzed to find how she perceived the derivative from a graph of a function and what kind of representations she used for this. The results show how she used representations of the increase, the steepness, and the horizontalness of the graph to perceive the derivative. Gestures were an integral part of her thinking. This case shows that with appropriate representations students can perceive essential aspects of the derivative from the graph of the function, and that students can consider the derivative as an object at the very beginning of the acquisition process.

The paper focuses on the issue of perceiving the derivative from a graph of a function. This kind of perceptual activity is proposed to be beneficial for learning calculus (Tall, 2003, 2004, 2005; Berry & Nyman, 2003; Heid, 1988; Speiser et al., 2003; Repo, 1996). It is also argued that perceptions and motoric actions are part of thinking and understanding (Nemirovsky, 2003). In an approach to calculus which builds on perceptual activity, many other than only formal-symbolic representations are emphasized. In this study, representations are considered as tools for thinking (Davis & Maher, 1997), and the dichotomy of 'external' versus 'internal' representations is broken. It is recognized that cognition is not only inside the individual's head but is also distributed to the social-cultural environment that includes tools (cf. Sfard & McClain, 2002; Rasmussen et al., 2004). This allows analyzing the role of inscriptions and gestures as visible sides of representations in thinking. A detailed characterization of the representation concept follows later.

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In this study, one student, Susanna, was interviewed after a teaching sequence that emphasized perceptual activity, gesturing, and open problem solving. The study is not a teaching experiment. It does not try to show the effectiveness of teaching. On the contrary, it resembles a *learning experiment* (Speiser et al., 2003) under particular conditions. Effective tools that students may use to think about the derivative are explored. The analysis of the interview focuses on Susanna's use of representations in perceiving the derivative from the graph of the function. Susanna was selected because she was not a very successful student in mathematics. Many studies have reported students' errors and misconceptions in graphing (see Leinhardt et al., 1990). However, in this study, a similar view to that in Speiser et al. (2003) is adopted, focusing on Susanna's potentials instead of her mistakes. According to Davis and Maher (1997), a teacher has to recognize the representations that students use and to provide experiences that will promote further development. Therefore, this study gives important information on how less successful students may reason and use different representations in this kind of perceptual approach to calculus. The results suggest that Susanna may have good potential for learning calculus if she is allowed to use her own thinking tools.

Perceptual activity in doing and learning calculus

Following Piaget, there are at least two ways how concept acquisition may begin in mathematics. One way is to perform a symbolic *action on* an object and from this action to construct a new concept. For example, in the APOS (Action–Process–Object–Schema) theory an action is internalized as a process which is encapsulated as an object (Asiala et al., 1997). Similarly, in Sfard's (1991) theory a person may reify an operation to a static entity, which means a shift from an operational to a structural conception. The other way, according to Gray and Tall (2001), is that concept acquisition begins from the perception of an object and *acting with* the object. Gray and Tall call this kind of perceived object an *embodied object*. Embodied objects are mental constructs of the perceived reality, and through reflection and discourse they can become more abstract constructs, which do not anymore refer to specific objects in the real world (ibid.). In Tall's (2003, 2004, 2005) theory of three worlds of mathematics, the *(conceptual-)embodied world* includes real-world objects and internal conceptions that involve visual-spatial imagery. The other two worlds are the *(proceptual-)symbolic world* in which symbols allow people to shift from processes to do mathematics to concepts to think about; and the *formal-axiomatic world* which is based on axioms, definitions, theorems and deductive reasoning (ibid.).

There are many studies on students' reasoning in the embodied world of calculus. Berry and Nyman (2003) found that university students moved from an instrumental understanding of calculus towards a relational understanding when engaged in tasks where they sketched the graph of a function from the graph of the derived function, and then created the corresponding movement and compared that to the graph given by the motion detector. They recommend that, before entering formal symbolic calculus, students should understand the underlying concepts which can be enhanced with tasks like those in their study. In the studies of Heid (1988) and Repo (1996), students in experimental classes attended a calculus course in which, among other things, reasoning with graphs was used intensively. These students showed more conceptual understanding than the students in the control classes. Also Speiser et al. (2003) report how a group of third-grade high school students made sense of the motion of a cat while they worked with several graphical representations. There are also studies that have shown how even younger students may construct calculus-related ideas. In a study of Schorr (2003), middle school students investigated motion especially with graphic representations and computer software, and according to the results, the students build powerful ideas of related concepts. Thus, she concludes that meaningful mathematical experiences in the mathematics of motion are possible even at grades 7 and 8. In Wright's research (2001), even a third/fourth grade student was able to build mathematical ideas of motion when she was allowed to use her kinesthetic experience.

Besides the potentials of the embodied world in learning the derivative, there are also some documented difficulties in working in the embodied world. According to Nemirovsky and Rubin (1992), students' tendency to assume resemblance in change of a function and change of its derivative is quite a general phenomenon. This is also well documented in the context of kinematics (McDermott et al., 1987; Trowbridge & McDermott, 1980; Beichner, 1994). Nemirovsky and Rubin (1992) argue that assuming resemblance does not necessarily mean that a student doing so cannot discriminate between the function and its derivative. According to them, one reason for assuming the resemblance may be the *linguistics cues*. These are ambiguities of the language that support the resemblance. For example, a position-time graph of a moving object that is slowing down does not necessarily go down (ibid.). Students may also use *semantic cues* which mean experiences that in some cases a function and its derivative actually do vary in the same way (ibid.). For example, often going faster implies going further. There are also *syntactic cues* which are based on graphical features, unrelated to the phenomenon under consideration (ibid.). According to Nemirovsky and Rubin, students may overcome the assumption of resemblance by focusing on how one function describes the local variation in the other. Students may use different mathematical notions, such as steepness and slope, for this (ibid.). In Hauger's (1997) study, four pre-calculus students made a similar error to that described by Nemirovsky and Rubin. The students draw distance-time graphs which represented constant speed instead of varying speed. The students used graphical slope, steepness, shape of the graph and changes over intervals to correct their error. Thus, Hauger concludes that these are powerful ways for pre-calculus students in thinking about the rate of change.

According to Tall (2003, 2004, 2005), students may learn in the symbolic world by internalizing procedures to processes and encapsulating these to procepts. Procept-conception means that the same symbol can act as a process and as an object. In the embodied world, students may learn by shifting their focus from actions to *effects* of those actions (Tall, 2005; Poynter, 2004). For example, in the case of the vector concept students may shift their focus from translations of a hand to the effects of the translations (Tall, 2005; Poynter, 2004). I propose that this is similar to the concept of *transparency*. The transparency of a tool means that the tool is visible for acquiring detailed information of the tool, but invisible for getting access to a phenomenon that can be seen through the tool (Meira, 1998; Roth, 2003). Similarly, eyeglasses are visible to a person so that he/she may notice when it is time to clean the glasses. But the eyeglasses are invisible so that the person sees the world through the glasses, and trashes and frames do not disturb him/her. Transparency is not a property of a tool but an emerging relation between the user and the tool (Meira, 1998; Roth, 2003). For example, a graph may become transparent to the user so that he/she sees the phenomenon behind the graph and does not only focus on the physical appearance of the graph (Ainley, 2000; Roth, 2003). Roth (2003) points out that in his study the graph affected what scientists were able to see of the phenomenon. He uses the eyeglasses metaphor that the graph was like glasses which allowed seeing the world clearer when the user was accustomed to the glasses. In mathematics, learning to use a graph means beginning to see essential aspects of mathematical objects that are represented in the graph. In the above mentioned example from Tall (2005) and Poynter (2004), the transparency of the hand movement embodiment would mean that one sees the vector as an effect through the hand movement. According to Noble et al. (2004), this kind of a disciplined way of seeing may evolve from *not seeing a whole* to *recognizing in* and to *seeing as*. Not seeing a whole means that one may be able to see the parts of an image without being able to see the whole. However, one may recognize in the image something he/she is familiar with. The experiences of recognizing in may cause one to see the image as something that he/she was not able to see before.

Also gesturing as part of students' reasoning, especially together with perceptual activity, has gained much attention in the literature. McNeill (1992) has argued that gestures together with speech are an essential part of thinking processes. According to him, there are different gestures, of which deictic, iconic, and metaphoric gestures are discussed here. *Deictic gestures* indicate something, *iconic gestures* resemble something and *metaphoric gestures* represent abstract ideas (McNeill, 1992). Roth and Welzel (2001) demonstrated by their case studies that gestures have an important role in constructing explanations in physics. They argue that gestures allow constructing complex explanations even in the absence of the scientific language and coordinating phenomenal and conceptual layers of the content. In their study, gestures seemed to make abstract entities visible. Similarly, Radford et al. (2003) report that gestures with words allowed the student to make sense of a distance-time graph of a moving object. Also, a study of Mosckovich (1996) highlights the importance of gestures, particularly, when describing graphical objects. She analyzed how nine-grade students used coordinated gestures and talk to negotiate a meaning for steeper in the context of linear graphs. Rasmussen et al. (2004) illustrate in their study how gestures are part of expressing, communicating and reorganizing one's thinking also in advanced mathematics of differential equations. They emphasize that meanings associated with gestures are both individually and socially constructed.

This study gives detailed information on one student's working in the embodied world. It is taken into account that cognition is distributed also into tools, inscriptions, and gestures. These are considered as visible parts of a person's representations. Because the focus is on the person's use of representations rather than the representations themselves, there have to be also some invisible parts of the representations. Therefore, what is meant by the representation needs to be clarified.

Characterization of the representation concept

Traditionally, a representation is conceived as something which stands for something else, and representations are divided into internal and external ones (cf. Janvier, 1987). Internal representation refers to the mental construction and external representation to the physical construction. This view of representation has been criticized lately. For example, there is a danger that representations may be thought to be mere representations of some objects and separated from meaning (Sfard, 2000). According to Sfard (2000), this position implies that objects and meanings are more important than representations, and these should be learnt before signs. The traditional view of a representation implies that representations are only used to store information and that the role of signs and symbolic tools is only to support and aid students (Sfard & McClain, 2002; Radford, 2000; Meira, 1998). Thus, this view does not use all the potential power of representations and other tools. Also, the dichotomy of internal versus external representations has been found artificial. According to Radford (2000) and Sfard and McClain (2002), traditional views often take a standpoint that external representations reflect the mental structures of an individual and that learning is the growth of mental structures. Meira (1998) and Cobb et al. (1992) point out that even when the decisive role of the student is acknowledged, representations are often analyzed from the expert's point of view as if external representations would include meanings. Thus, these analyses do not address the use or construction of the representations.

Meira (1998) has emphasized that the focus of studies in representations should move towards students' use and construction of representations. This focus can be noticed in the studies of Davis and Maher (1997) as they describe how students use representations as "tools to think with". According to them, the key attribute of effective tools is that they can be used to carry out thought experiments and to test hypothetical scenarios. Research has to focus on students' ideas and not just on testing their compatibility with experts' ideas (ibid.). In line with this, Speiser et al. (2003) emphasize capabilities of students rather than their errors. This move towards conceiving representations as tools has been made also by Radford (2000). In his study there is "a theoretical shift from what signs *represent* to what they *enable* us to do" (p. 241). Compatible with the view of a representation as a tool, Sfard (2000) has argued that representations are not born as such but they *may* become to stand for something else later. Several authors have also emphasized that meanings are constructed through the use of signs (e.g., Sfard, 2000; Radford, 2000).

Building on the criticism of the traditional view of the representations and the new views reviewed above, the representation is characterized in this study as follows:

> A representation is a tool to think of something which is constructed through the use of the tool. A representation has the potential to stand for something else but this is not necessary. A representation consists of external and internal sides which are equally important and do not necessarily stand for each other but are inseparable. The external side is visible to other humans through the senses but the internal side is not.

For example, a student may use the steepness of a graph of a function as a representation of a derivative of the function (see the results). This means that the steepness tool allows the student to perceive some aspects of the derivative, for example, the maximum point of the derivative. The student's conception of the derivative may have been constructed (and is being constructed) through the use of the steepness and other representation tools. There may be an external side of steepness, for example, the mere graph on the paper, speech or some gestures. Obviously, there must be some internal side, because for some people the graph would not allow to perceive the derivative. It is not the case that the external side only reflects the internal side, but it is the interplay between them that allows the student to use this tool efficiently. External sides are important for research because all the interpretations are based on these.

Following Goldin (1998), it is pointed out that one representation (or representational system) may be thought to consist, and usually it does, of other representations (or representational systems), and it is a matter of convention if we want to think of a single representation or its constituents. Often a graph of a function is considered as one representation. Instead, in this study the focus is on more specific representations which are used within the graph.

Several classifications of representations can be found in the literature. For example, according to Goldin (1998), internal representational systems can be a) verbal/syntactic, b) imagistic, c) formal notational, d) strategic and heuristic, and e) affective. Imagistic representations are the main ingredient in the embodied world and formal notational representations in the symbolic world. In other literature, representations have been classified also as enactive, iconic, graphical, formal, symbolic, algebraic, numerical, verbal, etc. In this study, these and other visible actions (e.g., gestures) or inscriptions are considered as external sides of representations. However, no beforehand classification of representations was used in the analysis although afterwards it is discussed which representations are used in the embodied world and which in the symbolic world.

Methodology

This case study focuses on Susanna's use of representations when perceiving the derivative in a *task-based interview*. Susanna was selected to this study because her success in mathematics was thought to be defective and it would be interesting to investigate how this kind of student is reasoning. Analyses of four other interviewed students are presented in Hähkiöniemi (in press, 2005, 2004). In a pretest before the teaching sequence, Susanna could read values from a distance-time graph and state when the distance is at its greatest. On the contrary, she could not state when the velocity is positive, negative or zero. She also calculated the instant velocity at a particular point as a total distance over a total time which gives an average velocity. Given the algebraic expression and the graph of a function, she reasoned the maximum value and the domain of negative values of the function incorrectly and did not notice that in the graph these do not make sense. Besides, she could not draw a tangent to the graph.

In this study it is tried to find effective tools that students may use in thinking about the derivative, particularly, in a similar approach to the derivative as reported here. I taught the teaching sequence in the autumn of 2003 as a part of a Finnish grade 11 course. The teaching sequence consisted of the first five lessons on the subject of the derivative. Working in the embodied and the symbolic worlds were emphasized in teaching. The teaching sequence applied the *open-approach* method as students were given open problems which had multiple correct ways of solving them (Nohda, 2000). Openness allowed students to use different representations in their solutions. Different solutions strategies were reflected upon in the whole class discussion. The teaching sequence began by examining motion graphs and by perceiving the rate of change of a function from its graph. Moving a hand along the curve, placing a pencil as a tangent, looking how steep the graph is, and the local straightness of the graph were used as representations. It was discussed how the above mentioned representations can be used to see the sign and the magnitude of the rate of change. The problem of the value of the instant rate of change was discussed and the derivative was defined through the solution of this problem.

After the teaching sequence, Susanna attended a 60-minute task-based interview. She was directed to think aloud in the interviews. The interview was recorded by one video camera focused on the papers and Susanna's hands. This focus was chosen because for the purposes of this study it was more important to capture hand gestures and notation than, for example, facial expressions. All the written documents were also collected. In this paper, the following interview tasks are discussed:

Task 2. The graph of a function f is given in the figure (Fig. 1). What observations can you make of the derivative of the function f in different points?

Task 3. Estimate as accurately as possible the value of the derivative of the function at the point $x = 1$.

Task 5. A car starts at the time $t = 0$ from the starting point. The figure (Fig. 2) represents the velocity v (m/s) of the car as a function of time t (s).

b. When does the distance traveled by the car increase and when does it decrease?

c. Sketch the graph of the distance traveled s (m) by the car as a function of time t (s) in the given (t, s)-coordinates.

Task 2 was designed to give information on how students can see the derivate from the graph of a function. The equation of the function was not given so that all the conclusions would be made from the graph. Task 3 was chosen to get information on how students estimate the derivative of a function for which they do not know the differentiation rule, and the use of the limit of the difference quotient is too difficult. Even estimations for the derivatives of exponent functions were not discussed at this stage in the course. A graph was not given to avoid restricting possible estimation methods to those involving a graph. Task 5 was planned to be similar to Task 2 but in a different context. The difference is that this task corresponds to the situation where the graph of the derivative function of a function is given, and students were asked how the values of the function are changing. In this task they were also asked to draw a graph.

The inductive analysis of the interview was based on the original video and the transcript. From the whole interview, the situations where Susanna used some representation were located. From each situation it was analyzed how she used these representations and what they allowed her to do. Then all these situations were compared to each other. In this way, an analysis of one representation was reflected against the analysis of the other representations. Susanna's uses of representations were also compared to those of the other four students. This allowed noticing common and distinct features in the students' use of representations and seeing some aspects of Susanna's behaviour in a new light.

Susanna's perceptions of the derivative

In this section, Susanna's perceptual activity in the three tasks is described. The excerpts from the interview are translated from Finnish, and [...] in the transcripts means that the text is snipped. Gestures are described in brackets [ ] and the points which indicate the use of a specific representation are <u>underlined</u>.

Task 2. Perceiving the derivative from a graph of a function

In Task 2, Susanna first focused on the graph going upward, horizontal, or downward. She seemed to use these to infer how the increase of the function changes:

Interviewer: The graph of the function *f* is given in the figure [Fig. 1]. What kind of observations can you make of the derivative of a function *f* at different points?

Susanna: Well, if you start to look from here [points to the graph at -3], then here the increase speeds up, we go upward [traces the graph with finger from -3 to -1.6]. Then at the top [points to the graph at -1.5] it is zero, it goes horizontally [draws a horizontal line in the air]. Then again it slows down here [turns pencil a bit, traces the graph with pencil from -1.5 to 0.8] and here again it is zero when it goes horizontally [points to the graph at 0.8]. And then again upward from here the rate increases [traces the graph with pencil from 0.8 to 2]. [...] Here the decrease is constant [traces the graph with pencil from 2 to 4].

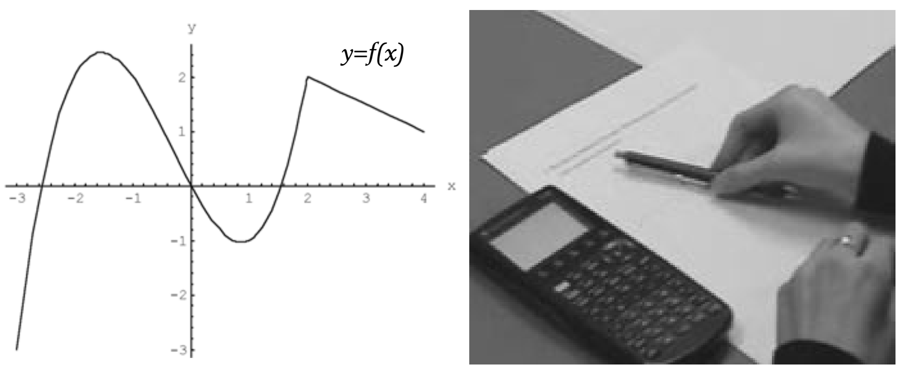

Figure 1. *The graph of a function f in Task 2 and Susanna's gesture with the pencil to see the steepness of the graph at the point 0.7.*

Above, Susanna seemed to speak of the increase or the rate when she mentioned "the increase speeds up", "it is zero", "it slows down", "it is zero", "rate increases" and "decrease is constant". The first and partly the third utterances are incorrect. Thus, Susanna probably mixed up the change of the function with the change of the derivative. After this Susanna wondered what the derivative would be at the point 2.

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Interviewer: What would the derivative be at that point [points to the graph at 2]? At about two.

Susanna: It can't really be zero, because it doesn't actually go horizontally at any point. Hmm. Or in a way it can be zero, for example, if you look at it with the pencil [moves pencil as a tangent from ascending to descending at the point 2], then at some moment it can be horizontal [...] [pause] If it is at the point 2, then if you look, then in a way it would have to be 1 [...]

Interviewer: On what grounds?

Susanna: Mm. Well. x. [Holds pencil above the notion $y = f(x)$ in Fig. 1]. If you take the derivative at the point 2, then [pause] [writes $y'(2) =$]. There's only the x [points to the notion $y = f(x)$ in Fig. 1], so then it would be 1.

First Susanna looked at the horizontalness of the graph which showed that the derivative cannot be zero. Then she used a pencil as a tangent and got a contradictory result. She solved this contradiction by differentiating x from the symbolic expression $y = f(x)$ concluding that the derivative would be 1. After the interviewer corrected the mistake, she again used steepness to consider the derivative.

Interviewer: Anyway, is it [derivative at the point 2] positive or negative, can even that be said?

Susanna: Mm. There it rises quite steep upward [points to the graph little before the point 2]. And then over here [points to the graph little after the point 2] it starts to go not so steep downward [imitates graph with pencil in the air]. In my opinion it is positive somewhere there [at the point 2].

She used still another way of thinking when she reasoned that the graph raises steeper before the point 2 than falls after it, and concluded that therefore the derivative would be positive. It seems that Susanna's first two ways of thinking could have been fruitful but unfortunately she started to use the differentiation rule and make conclusions based on steepness on the other points. This could be partly because she was striving for an answer and did not consider that the answer could be that there is no answer.

After that she used the increase and the steepness of the graph to see the sign and the maximum and minimum points of the derivative.

Interviewer: When would the derivative be positive in general at the whole graph and when negative?

Susanna: It would be positive approximately from here to somewhere there [points to the graph at -2.6 and -1.5], when the <u>graph rises upward</u> [<u>moves pencil upward</u>]. And then from somewhere here to there [points to the graph at 0.8 and 2]. [...] Negative from somewhere here to about there [points to the graph at -1.5 and 0.8]. And from that on [points to the x-axis from 2 to 4]. [...]

Interviewer: Could it be said when the derivative is at its greatest and when at its smallest?

Susanna: At its greatest it is when the graph <u>rises most steepest upward</u> [<u>moves pencil upward</u>]. It could be [<u>puts pencil as a tangent</u> to the graph at points -3 and 1.9]. Hmm. Somewhere there [points to the graph at (-3, -2.6)]. Or here [points to the graph at 1.9]. Where it <u>falls most steepest downward</u> then, hmm [<u>holds pencil as a tangent</u> above the graph at about point 0] somewhere hmm. It's a bit hard to see, but somewhere there [points to the graph at 0.7]. [...]

Interviewer: How did you look that it goes most steepest downward there?

Susanna: Mm. Here [<u>puts pencil as a tangent</u> to the graph at (2, 4)] it clearly goes not as <u>steep</u> as there [<u>puts pencil as a tangent</u> to the graph at 0.7, the pencil's position is steeper than at (2, 4) and it is steeper than it should be, see Fig. 1].

Interviewer: Ok. What about in this point [points to the graph at -0.6] compared to that point [points to the graph at 0.7]?

Susanna: Yes. [<u>puts pencil as a tangent</u> to the graph at -0.6]. It <u>falls quite slowly</u>. It won't quite go. [<u>puts ruler as a tangent</u> to the graph at -0.6, its position is steeper than that of the pencil]. It could be <u>steepest</u> also there.

Susanna seemed to use the increase of the graph of the function as a tool to perceive how the function changes at some interval. For example, the sign of the derivative was easy to perceive when the derivative was represented as the increase of the graph. Along the representation of the increase she used the steepness of the graph. Steepness seemed to represent the magnitude of the change of the function. So it was an especially good tool for perceiving local properties of the derivative, such as the maximum point. Yet Susanna had difficulties to perceive the minimum point of the derivative and proposed an incorrect point. It seems that Susanna placed the pencil steeper than it should be at 0.7, and this misled her. Even when interviewer suggests the point -0.6, she still suspects that the graph is not steeper there saying "it falls quite slowly" and "it won't quite go". Finally, after replacing the pencil with the ruler she admitted that it could be steeper at -0.6. It might be that at this point she guessed that 0.7 is not the correct point because of the many questions and this affected her to accept also the point -0.6.

In perceiving the derivative Susanna made a lot of gestures. She made iconic gestures of moving her hand along the graph and moving it upward or downward in the air while considering the increase. These gestures, as well as the graph itself and her speech, seemed to be external sides of her representation of the increase. The iconic gesture of drawing a horizontal line in the air seemed to be an external side of her representation of the horizontalness of the graph. She also made a metaphoric gesture of placing a pencil as a tangent to the graph. This gesture seemed to be an external side of the steepness representation. Susanna also used deictic gestures to indicate some points in the graph.

Task 5: Perceiving the distance from velocity-time graph

In Task 5 b Susanna could also consider how the distance is changing when given the velocity-time graph of the car (Fig. 2). First Susanna assumed the distance to change in the same way as the velocity (like in Task 2, she mixed up the change of the function to the change of the derivative).

Susanna: It starts from here [points to the origin], then its distance from the starting point <u>increases</u> [<u>traces the graph</u> with pencil] until that top [<u>draws a horizontal line in the air</u> above the point 10]. Hmm. And then. [pause] Hmm. Here, <u>at the top it is not moving</u> [<u>draws a horizontal line in the air</u> above the point 10].

When the interviewer asked what the velocity would be at point 10, she realized how the distance is changing. In this task she used the same kind of representations as in Task 2 but now to consider how the velocity (derivative) is changing and to interpret from that how the distance (function) changes.

Interviewer: What would the velocity be there [t = 10]?
Susanna: Hm. It would have to be zero, but. Hm. No, the velocity has to be something [points to the v-coordinate with a ruler], 14 metres per second. Hm. Yeah. And here it goes at a <u>constant speed</u> a very little time [<u>draws a horizontal line in the air</u> above the point 10]. Then the velocity starts to <u>slow down</u> [moves pencil in the air <u>imitating the graph</u>].

[...] [Traces the graph with pencil from the point 10 to 20] it still goes forward although the velocity decreases. [...] The velocity changes negative there [points to the point 20 and continues to move pencil forward]. [...]

Interviewer: How would you reason when its distance increases and when it decreases?

Susanna: Hm. When the velocity is positive, the distance increases, because it goes forward. When the velocity is negative, it backs up, that is, the distance decreases.

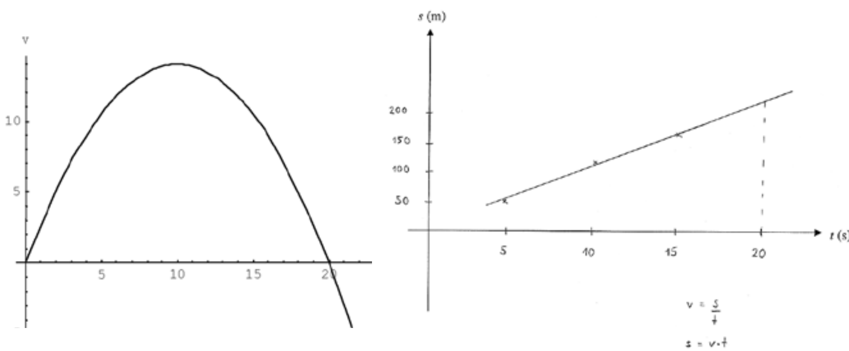

Figure 2. *The given velocity-time graph of the car and the distance-time graph drawn by Susanna in Task 5.*

It seems that the constant velocity was a starting point for Susanna's reasoning. She drew a horizontal line in the air, which suggests that she probably used her representation of the horizontalness of the graph for this observation. Her gestures seemed to be integrated to her thinking and were not just for focusing on particular points of the graph. In addition to horizontalness, this can be well noticed of the way how she perceived that the velocity is decreasing while imitating the graph.

Obviously, Susanna also thought of the movement of the car, because she stated that "it is not moving", "it moves at constant velocity", "it goes forward" and "it backs up". Thus, she coordinated the physical appearance of the velocity-time graph, change of the velocity, movement of the car and change of the distance using her representation of the increase and horizontalness including gestures as external sides of these representations. Despite considering also the movement of the car, her gestures seemed to be related to the graph and not to the movement of the car. As in Task 2, also in this task she did not spontaneously consider how the rate of change of the distance changes, for example, when the distance increases fastest. Nevertheless, she interpreted well when the distance is increasing and when decreasing.

Instead, when she actually had to draw the graph of the distance, she relied on an inappropriate rule of calculating the distance with formula "$s = v \cdot t$", where t is the time from the beginning and v is the velocity at time t. This produced a graph (Fig. 2) which was in contradiction with her previous perceptions of the change of the distance. She did not notice this conflict even when she stated that the distance is greatest at $t = 20$ and the interviewer asked whether it would not be even greater after $t = 20$ according to her graph. She just replied that then the velocity is negative and the distance is greatest at $t = 20$. Thus, she considered the distance graph apart from the perceptions of the velocity–time graph.

Task 3: Estimating the value of the derivative

In Task 3, Susanna used the differentiation rule incorrectly to calculate the derivative of the function at a point:

Susanna: [writes $f'(1) =$] Actually the derivative of 2 would be 0.
Interviewer: Hm.
Susanna: Would this be then 0 [writes $f'(1) = 0^1 = 0$].

She used the differentiation rule similarly in Task 2. When the interviewer asked about this result, she could convince herself that she had calculated the derivative incorrectly.

Interviewer: How could you figure out whether it could be 0?
Susanna: [pause] If you draw a figure and. [draws a coordinate system and marks the points (1, 2) and (2, 4)] At least, it can't be zero there because. It never seems to go horizontal. [...] [First tries to draw the graph by paper and pencil but then uses a graphic calculator and copies the graph on paper] [...] Well, at the point 1 the derivative can't actually be 0, because it increases anyway at the point 1 [moves pencil upward imitating the graph].

Finally she also obtained an estimate for the derivative.

Susanna: You could also draw the tangent to the point 1. [...] [draws a tangent to the point 1] [...] Then the slope would be two.
Interviewer: Hm. Ok. How did you see that from it?
Susanna: You take two from there [writes 2], one [writes 1], and then two divided by one [writes 2/1 = 2].

Susanna again used the horizontalness of the graph to think of the zero point of the derivative. She decided by herself to compare the obtained result to the graph. The representation of the increase allowed her to perceive the derivative qualitatively and to confirm that the derivative is not zero. This was opposite to Task 5, where she did not relate her perceptions of the distance to the graph produced by "$s = v \cdot t$" rule. After noticing that the derivative is not zero, she could even estimate the derivative quantitatively with the slope of the tangent. In Hähkiöniemi (in press) it is discussed how Susanna tried to find a better estimate by drawing secants approaching the tangent and by using her version of the limit of the difference quotient: "$\lim_{x \longrightarrow 1} 2^1 = 2$ ".

Discussion and conclusions

Above it is shown how Susanna could perceive and reason many essential properties of the derivative from the graph of the function: the sign of the derivative, the zero point and the maximum point of the derivative, and the interval when the derivative is constant. For these perceptions, she used her representations of the increase, the steepness and the horizontalness of the graph. She could also use the increase and the horizontalness to correct the result obtained by using the differentiation rule inappropriately. In the context of the movement of the car, she used the same representations and could make sense of the velocity-time-graph and describe how the distance is changing. It seems that to Susanna the increase was a property of an interval which concerned the quality of the derivative, whereas the steepness was a point wise property and gave the magnitude of the derivative.

Susanna had also some difficulties. She used the differentiation rule inappropriately in Tasks 2 and 3 and she determined the minimum point of the derivative incorrectly in Task 2. In Tasks 2 and 5 Susanna first perceived that the function and its derivative (distance and velocity) are changing in the same way. This is quite a general phenomenon (Nemirovsky & Rubin, 1992; McDermott et al., 1987; Trowbridge & McDermott, 1980; Beichner, 1994). The word "up" in expressions "graph going up" and "speeding up" as a *linguistic cue* (Nemirovsky & Rubin, 1992) could have affected Susanna to assume a resemblance between the function and the derivative in Task 2. In Task 5 it was true that at the beginning both the velocity and the distance were increasing. This *semantic cue* (ibid.) might have influenced Susanna to assume a resemblance between the distance and the velocity. It seems that in the both tasks the interviewer's questions helped Susanna to focus on local aspects. After that she used her representations of the increase, the steepness and the horizontalness of the given graph to describe aspects of the derivative or the distance. In this way, she was able to overcome the assumption of the resemblance. However, in Task 5 she finally abandoned her good perceptions of how the distance is changing and begun to use the inappropriate symbolic rule to calculate points to draw the graph of the distance.

In the following, the most important aspects of Susanna's working in the embodied world (Tall, 2003, 2004, 2005) are discussed. First, the role of gestures and other external sides of representations in thinking are discussed. Then, the transparent use of representations and the perceptions of the derivative as an object are discussed. The paper is completed by a discussion on the potentials of the embodied world.

Interplay between internal and external sides of representations

I have tried to emphasize the distributed nature of knowledge and break the classical external versus internal dichotomy by focusing on external and internal sides of representations. Susanna's case shows that with appropriate representations students can perceive essential aspects of the derivative from the graph of the function. Especially, the representations of the increase, the steepness, and the horizontalness seemed to embody important relations between a function and its derivative. Gestures of imitating the graph, tracing or pointing to the graph, drawing lines in the air and placing the pencil tangent-like to the graph as external sides of representations were an essential part of Susanna's thinking. These visible parts of her reasoning were not just reflecting her internal images or processes.

Susanna's case supports arguments that gestures are important for thinking and part of expressing, communicating and reorganizing one's thinking (McNeill, 1992; Radford et al., 2003; Rasmussen et al., 2004; Roth & Welzel, 2001; Moschkovich, 1996). For an advanced person in mathematics, these gestures may seem meaningless or useless. But for a novice like Susanna, they may be in great aid and help to focus attention on particular aspects, such as increase and steepness. Actually, the gestures did not just help but they were an integral part of Susanna's thinking. As in the study of Roth and Welzel (2001), the gestures seemed to help Susanna to make an abstract concept visible and concrete.

In addition to the external sides of representations, there has to be something which is not visible. These internal sides of representations are important for the use of external sides. As Meira (1998) has pointed out, the expert-designed powerful external representations are not necessarily powerful for a student. For example, Susanna had difficulties to perceive the minimum point of the derivative although she used a "good" external side of a representation. To use a representation effectively the external side of the representation has to be coordinated with an appropriate internal side. The use of a representation is neither internal nor external but more like an interplay between these two sides.

Using representations transparently

Susanna's perceptions of the derivative focused largely on the graph as a physical object while she recognized some aspects of the derivative. Thus, she was still at the beginning of the process of learning how to see the derivative in a mathematical way . For example, she noticed such things as the graph going upward and the steepness of the graph, but she did not spontaneously perceive how the rate of change of the derivative was changing. The latter would have required perceiving aspects that demand a more disciplined way of seeing. She also used physical objects to see these aspects and had problems with the minimum point of the derivative in Task 2 (negative steepness). This proves that she is still focusing on very concrete aspects. As regards to the concept of transparency (Meira, 1998; Ainley, 2000; Roth, 2003), Susanna did not seem to use her representations of increase and steepness very transparently because she focused more on these tools than on the derivative which can be seen through them. It is emphasized that in this paper, the graph as a representation itself is not the focus but instead, the more specific representations which are used to perceive the derivative in the graph. These representations (increase, steepness and horizontalness) seemed to be aspects that Susanna could recognize in the graph. According to Noble et al. (2004), the experiences of recognizing something familiar in the picture may cause a person to see the picture in a new way. Susanna may still be on her way toward seeing the graph of a function as a representation of the derivative.

Perceiving the derivative as an object

This study suggests that with special kind of representations students can consider the derivative as an object that has some properties, such as sign and magnitude (cf. Sfard, 1991; Asiala et al., 1997; Tall, 2003, 2004, 2005). This is possible even at a very early stage in the learning process and even for students like Susanna, whose previous success in mathematics is not great. Although Susanna considered the derivative as an object, this object was presumably not encapsulated from the limiting process (cf. Asiala et al., 1997) because in a previous analysis it was found that Susanna was not very successful in using the limiting process inherent in the derivative (Hähkiöniemi, in press). Neither does the derivative as an object refer to the derivative as a function although

Susanna considered the derivative in many points. She considered the derivative at a point as an object and made some good perceptions but did them more or less intuitively without knowing why they can be made. This was also closely connected to certain representations.

Learning in the embodied world

Because Susanna perceived the derivative as an object, she is not constructing the derivative only from actions but also by acting with it. Thus, Susanna's learning does not seem to proceed (only) from operational through reification to structural (Sfard, 1991) or by encapsulating actions to an object (Asiala et al., 1997). This means that Susanna is learning the derivative also in the embodied world in addition to the symbolic world (Tall, 2003, 2004, 2005). It seems that, also in the embodied world, Susanna is at the beginning of her learning process because she is still focusing more on acting with the representation tools than on the effects of those actions. Learning towards focusing on effects could happen through many recognizing experiences and representations becoming transparent to the user. As representations become more and more transparent, the concept under construction becomes more and more detached from embodied objects which originally gave meaning to it. Thus, the concept becomes more abstract.

At many points of the interview, the symbolic world seemed rather procedural and confusing to Susanna. Instead, in the embodied world, she demonstrated some conceptual knowledge connecting some features of the graph of the function to its derivative. Thus, this study supports Tall's (2003, 2004, 2005) ideas that this kind of perceptual or embodied starting point may be fruitful for learning calculus. Also the studies of Berry and Nyman (2003), Heid (1988), Speiser et al. (2003) and Repo (1996) support this claim. Some studies have also shown how it may be possible to begin the learning of these concepts even before high school (Schorr, 2003; Radford, 2003; Wright, 2001). Yet, it can be noticed that Susanna has a lot to learn also in the embodied world and she should connect this world to the symbolic world.

In the conditions of this study, students were encouraged to reason aspects of the derivative from graphs and to use gestures. Gestures of moving a hand along a graph and placing a pencil as a tangent were also discussed in the class. Although this research is not a teaching experiment, the results suggest that teaching may have a positive influence on students' abilities to perceive mathematical aspects. At least, a teacher should not discourage students from perceptual activity but recognize the potentials of the embodied world.

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Yhteenveto

Tässä tutkimuksessa tarkastellaan miten hieman heikommin matematiikassa menestynyt opiskelija havaitsee derivaatan funktion kuvaajasta. Lukion toisen luokan opiskelijan tehtäväpohjaisesta haastattelusta analysoitiin miten hän havaitsi derivaatan funktion kuvaajasta ja millaisia representaatioita hän tähän käytti. Hänen huomattiin käyttävän kuvaajan kasvamisen, jyrkkyyden ja vaakasuoruuden representaatioita. Eleet olivat olennainen osa hänen ajatteluaan. Tulosten perusteella opiskelijat voivat tarkoituksenmukaisia representaatioita käyttäen tehdä olennaisia havaintoja derivaatasta ja tarkastella derivaattaa objektina jo oppimis prosessin alkuvaiheessa.