Limits of functions

Traces of students' concept images

Kristina Juter

Students at a Swedish university were subjects in a study about learning limits of functions. The students' perceptions were investigated in terms of traces of concept images through interviews and problem solving. The results imply that most students' foundations were not sufficiently strong for them to understand the concept of limit well enough to be able to form coherent concept images. The traces of the students' concept images reveal confusion about different features of the limit concept.

Students studying basic university courses in mathematics perceive the concept of limits differently in different situations (Juter, 2003). Students are in a situation where they have to integrate a vast amount of new information with their mental representations of what they already know. During a period of such implementation there will probably be times of confusion and ambiguity in the representations. Different representations can be incompatible without the students trying to resolve the contradictions. It can be that a student acts as if one rule applies in theoretical discussions and a different one in problem solving situations.

The concept of limit of a function has been shown to be difficult to understand due to various characteristics of the concept. One example is the dual nature of the concept, which can be thought of as an object and an infinite process at the same time, another is the formal definition which takes time and effort to understand (Cornu, 1991; Juter, 2003; Williams, 2001). The influence of the abstract concept of infinity is also something students have to work with (Tall, 2001). There are consequently several possible conflict situations in the ways students can perceive limits.

Kristina Juter
Kristianstad University College

I have done a study to find out how Swedish university students learn limits of functions during their first mathematics semester. This article focuses on one stage in the students' development of the concept of limits of functions. The aim is to find out how they are reasoning and if they show inconsistencies in their representations at that specific time. The research questions addressed are the following: How do students perceive limits of functions? How are the critical features of limits of functions, which students describe, connected, that is, do they form a coherent picture? Do students with high grades reason differently from students with average grades?

Theoretical framework

Theories with particular relevance to the topic of this paper are presented in this section.

Traces of concept images

When a process or an object is mentioned it is referred to by mental representations (Dreyfus, 1991). A *concept image* (Tall & Vinner, 1981) can be used as a model of this mental representation of a concept. The concept image consists of everything the individual connects with the concept. This includes, for example, experiences and relations to other concepts. It is not possible to see another person's concept image, but his or her actions are derived from the person's concept image and can therefore be regarded as a *trace* of the concept image. What the students do is regulated by their abilities, among other factors (Star, 2000). The trace of a concept image is influenced by the situation in which the students are at the time when the trace is being considered. Influences can come from peers or teachers, but also from the students' own minds when they think and argue for their conjectures.

Sfard (1991) and Cottrill, Dubinsky, Nichols, Schwingendorf, Thomas and Vidakovic (1996) have described theories of object formation of processes. These objects and processes form mental schemas to represent the concept and connections to it. Such schemas are part of individuals' concept images. Cottrill et al. (1996) conclude that the formal concept of limit is an intricate dynamical schema and not a static one. It is important to have a strong dynamical conception of the notion before it is possible to fully embrace a more formal interpretation. In the development of refining the schema of limits there are leaps between seeing a limit as a process and as an object. Sfard (1991) defines *reification* as an "ontological shift" (page 19) which is achieved when a process can be seen as an object, something familiar becomes clear from a completely different point of view. It is often cognitively demanding to reach this stage of perception (Blomhøj, 2002). There is a connection between interiorisation at a higher level with reification at a lower level in the development of mathematics which sometimes leads to attempts to reach the reification stage prematurely. In this dual situation there is a danger of confusion. If a student has a set of conceptions of limits and wants to solve a problem, it is not certain that what is consulted in the mind is the most appropriate conception (Davis & Vinner, 1986). The most dominant part of a concept image can cast a shadow over other parts for different reasons. It can be familiarity, difficulty or how long the conception has been available to the individual that influence what will be drawn from the conceptions. High achieving students create more complex representations of concepts than other students do, according to Chinnappan (1998), with the effect that the high achieving students can find new solutions to problems that the low achieving students are unable to find. The high achieving students are more likely to draw the most suitable schemas from their representations to solve the problems they are dealing with.

Understanding and knowledge

A concept can be represented in more than one way and there can even be conflicting representations that are evoked at different times depending on the context. If the representations are not contradictory they can merge into one when the individual is able to see the connections (Dreyfus, 1991). If they are incoherent in any aspect, a conflict may arise. The incoherency can, for example, concern interpretations of rules or perceptions of definitions in different contexts. The more appropriate the connections between the mental representations are, the better the individual understands the concept (Dreyfus, 1991; Hiebert & Carpenter, 1992). Hiebert & Carpenter (1992) define understanding of a mathematical concept to be something an individual has achieved when he or she can handle the concept as a part of a mental network.

Knowledge of a concept is, according to Dubinsky (1991), the individual's tendency to bring to mind a scheme in order to be able to handle, organize or make sense of a problem situation.

In this paper I will regard understanding and knowledge of a concept as Hiebert and Carpenter (1992), and Dubinsky (1991) do.

Perceptions of limits of functions

In the beginning students can meet the notion of limit in an informal intuitive way where the tasks are from situations where they can easily see the outcome. This creates a feeling of control and the students think they know what the concept is about, even if they could not solve a more demanding task where they would have to master the full meaning of the definition (Cornu, 1991; Juter, 2003). This feeling of confidence conceals the need for further sophistication of the concept image. Limits of functions can be thought of in very different ways depending on what situation the limits occur in. Critical features of a concept depend on which perspective the students have on their approach to it. It takes different efforts to be able to understand or describe the concept of limits than it takes to be able to calculate limit values.

Everyday language can have a slightly different meaning compared to the language used in mathematics, which can have an effect on students' perception of the concept (Monaghan, 1991). We use words or phrases as convergence, border, arbitrarily close, tend to, and limit when we work with limits of functions. The everyday meanings of the words can influence students' perceptions of the words in a mathematical context. There is an ambiguity in the way the concept of limits can be perceived. One can focus on the process of approaching the limit and hence consider it a never ending procedure. But one can also think of the limit as a static entity to which functions can be compared. This kind of ambiguity is exemplified in a study by Williams (2001) where two students' conceptions of limits were examined. Both of the students understood that continuous functions reach their limit values, but they were not sure if it was correct to say that the limit was reached in the limit process. The students appear, as I see it, to be confused by the context of the task. They know that continuous functions can reach their limit values but their concept images of limit processes interfere with their concept images of continuous functions, causing this hesitation. Williams mentions the complexity of the notion of infinity as a source for the problem in this situation, but it can also be that the students do not accept the definition of limits. If they do not see any use for the definition, or if the definition is too hard to implement to the students' concept images, the students can chose not to accept the definition. Instead they can create an alternative definition more suitable to their concept images. If such students have problems understanding the definition, it is most likely that the alternative definition has focus on processes rather than objects (Blomhøj, 2002; Cottrill et al. 1996; Sfard, 1991). Przenioslo (2004) has done a study about limits of functions with university students in years three to five. She used tasks similar to Task 1 and Task 2 presented later on in this study. She found that students' concept images differing from the formal concept definition could have their origin in an intuition that limits do not exist if the function is not continuous. The intuitive view comes from secondary school work, according to Przenioslo. Opposed to this intuitive view, earlier results from the study presented in this paper show students stating that a value is not a limit value if the function can attain it (Juter, 2003).

Traces of concept images from 15 students are described and analysed in order to find out more about their learning of limits of functions.

The study

The aim of the whole study, as well as the part presented in the present paper, is stated in this section. The student sample and the course are described, followed by a presentation of the methods and instruments used to answer the questions posed. The methods of the whole study are outlined briefly to frame this part of it.

Aim

The general aim of the study was to find out as much as possible about the students' development in their learning of limits of functions. No such investigation had been done in Sweden before. The aim of this paper in particular is to look at the students' conceptions at one specific time. The questions investigated are:

1. How do students perceive limits of functions?
2. How are the critical features of limits of functions, which students describe, connected, that is, do they form a coherent picture?
3. Do students with high grades reason differently from students with average grades?

The sample

Of the 112 students that participated in the study 33 were female. The students were aged 19 and above. They were studying mathematics at the basic university level for twenty weeks, full time, divided in two ten week courses, called the α-course and the β-course respectively, both dealing with algebra and calculus. Not all students took both courses. The students had lectures together in the whole group with two 45 minutes lectures per day for three days per week. After the lectures they had two 45 minutes task solving sessions in smaller groups, with about 25 students in each. Both courses were assessed with an individual written exam. At the end of the second course the students also had an individual oral exam. The oral exams were mainly about theory and the written exams had a focus on task solving. I was neither teaching the students at any stage nor did I know any of them.

Q1		Q2	I1	Q3	I2
|α	limits		| β		|exams

Figure 1. *Timeline*

Methods

Different methods were used to collect different types of data. The timeline (Figure 1) shows when the sets of data were collected.

The students got a questionnaire, Q1, in the beginning of the α-course. It contained easy tasks about limits and some attitudinal queries. After limits had been taught in the α-course, the students got a second questionnaire, Q2, with more limit tasks at different levels of difficulty. The students were asked if they were willing to participate in two interviews later that semester. Thirty-eight students agreed to do so. Eighteen of these students were selected for two individual interviews each. The selection was done with respect to the students' responses to the questionnaires so that the sample would as much as possible resemble the whole group. The first session of interviews, I1, was held in the beginning of the β-course. The interviews were semi-structural with a set of questions and tasks, which were followed by questions based on the responses from the students. Each interview lasted about 45 minutes and was audio recorded. The students were asked about definitions of limits, both the formal one from their textbook and their individual ways to define a limit of a function. They also solved limit tasks of various types, with the purpose to reveal their perceptions of limits. This interview is the stage in the students' developments on which this paper is focused. The instruments used will be further described below. The students got a third questionnaire, Q3, at the end of the semester and a second interview, I2, was carried through after the exams. Field notes were taken during the students' task solving sessions and at the lectures when limits were treated to give a sense of how the concept was presented to the students and how the students reacted to it. Tasks and results from other parts of the study are described in more detail in other papers (Juter, 2003; Juter, 2004). In this paper I will discuss the results of the first set of interviews with respect to the students' conceptions of limits of functions.

Instruments

At the time of the first interview, the students were asked to comment on statements very similar to those used by Williams (1991) in a study about students' models of limits. The statements the students commented on are the following (translation from Swedish):

1. A limit value describes how a function moves as x tends to a certain point.
2. A limit value is a number or a point beyond which a function cannot attain values.
3. A limit value is a number which y-values of a function can get arbitrarily close to through restrictions on the x-values.
4. A limit value is a number or a point which the function approaches but never reaches.
5. A limit value is an approximation which can be as accurate as desired.
6. A limit value is decided by inserting numbers closer and closer to a given number until the limit value is reached.

The reason for having these statements was to get to know the students' perceptions about functions' abilities to attain limit values and other characteristics of limits. The students were given the statements to have something to compare to their own thoughts. There were other tasks designed to make the students consider the formal definition, to clarify what it really says, and tasks about attainability (Juter, 2003; Juter, 2004).

Two tasks dealt with attainability and what happens near and at a limit value (they will be referred to as Task 1 and Task 2). The aim was to see how the students were going to interpret the definition and attainability of limits. The students got two graphs of functions, Figure 2 corresponding to Task 1 and Figure 3 corresponding to Task 2. The graphs are presented, in the same way as they were to the students, in handmade pictures.

The tasks for the students were to determine right and left limit values and if the functions had a limit value, as x tended to the points, a and b respectively, marked in the graphs. They were also asked to determine the values of the functions at these points.

Data analysis

The interviews were transcribed and the students' answers and statements were categorized with the aid of the computer program NUD*IST (N6, 2003). The program makes it possible to categorize parts of a text such as a transcript of an interview. Parts of the text containing traces of a certain perception were selected and saved in a category. The categorization is thereby somewhat subjective in character since traces of perceptions occur in various forms and with varying clarity. The categorization was done several times to make the judgements as accurate as possible with respect to the issues chosen for the study, for example attainability or limits as approximations. The key issues coded were the headings of table 1. The program makes the categorization easy to survey and change. The categories at this part of the study were primarily chosen to reveal the students' traces of concept images at critical parts. The six statements from the interview served as a basis in the category selection process. Attainability in theoretical contexts and problem solving contexts became two different categories since there were interesting findings related to this separation. One aim with the categories was to discern conceptions of limits as objects from conceptions of limits as processes. The complexity of the concept made it hard to do this discernment without inferring too much. The results are therefore presented with all the last four categories in table 1 instead of just two categories in terms of objects

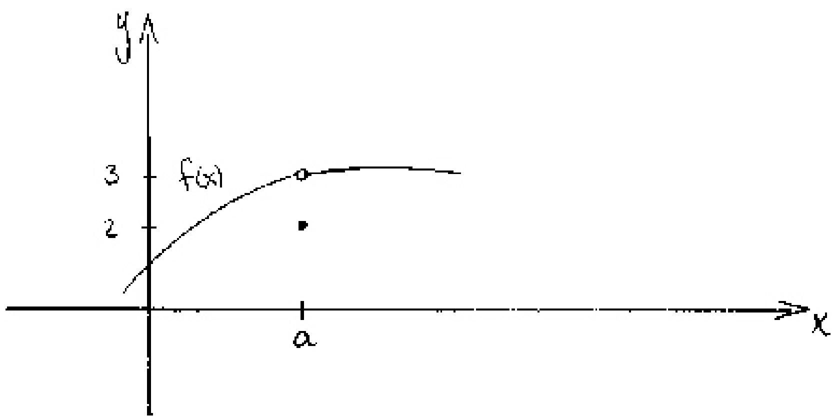

Figure 2. *Graph corresponding to Task 1*

and processes. The numbers in table 1 indicate how many times each perception was confirmed during the interview.

Results

The results of the interviews are presented in table 1 and table 2. Table 1 contains seven categories, each of which is identified with a phrase or a word. Some student examples from each category are provided. The students are listed in order of their grades from the lowest to the highest grades. The letters after the names indicate the results of the examinations. An "a" means the highest grade on both exams at the first attempt, "b" is one exam passed with the highest grade and the other passed, both at the first attempt, "c" means both passed at the first attempt, not with the highest grade, "d" means one passed at the first attempt and one at the second, and "e" means one passed and one not passed. The oral exam and the second written exam are here together considered to be the second exam.

The findings displayed in table 1 reveal that there is a variety of combinations of the perceptions in the categories. Several of them are incoherent at different points. There is no obvious distinction between high and average achievers.

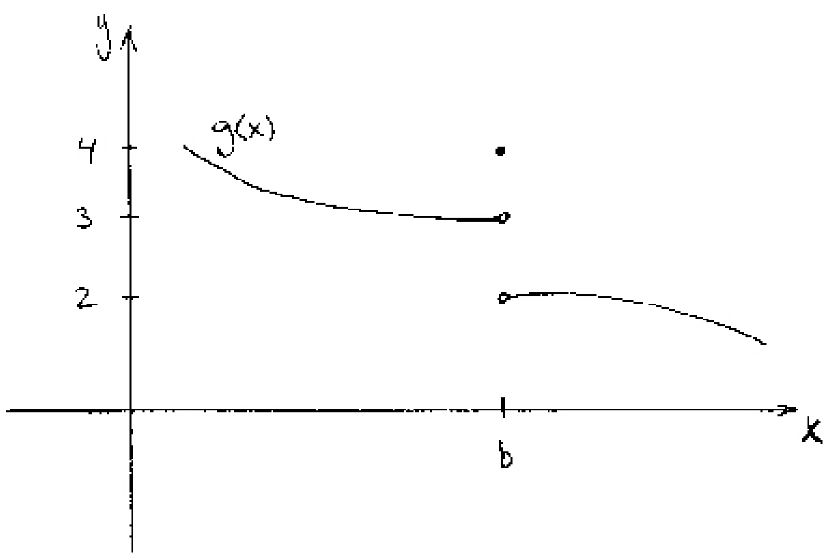

Figure 3. *Graph corresponding to Task 2*

Table 1. *Students' conceptions of limits of functions**

	Attain limit value, T	Attain limit value, P	Do not attain limit value	Distance between function and limit	Border	Approximation	Exact
Filip, e	1	1	4	2	1		1
Martin, d		2	3				1
Tommy, d	1		3	2		1	
Anna, d		1	3	1			
John, c	2	1	1	1	5		
Frank, c		1	6				1
Louise, c	1		2	2	1	1	
Leo, b		1	3				
Dan, b	1	1	2	3			1
Mikael, b		1	6	2	1	1	
David, b	1		1	2	1		1
Julia, a	3			1		1	
Dennis, a	1				1	1	
Emma, a	4	1	3	3	1		1
Oliver, a		3	4		1		1

Note. * The numbers show how many times each view was confirmed during the interviews

Examples of students' statements in the different categories of table 1.

Attain limit value, T: Functions can attain limit values in theory.

Emma: *... the definition of limit values is like that it is possible, most or it depends, there are limit values which are attainable.*

Julia: *... the function can attain values beyond the limit value if it ... if x is not, well for other x so to speak.*
Yes it is okay that it reaches its limit value, that is no problem.

Attain limit value, P: Functions can attain limit values in problem solving situations.

Int.: *... If it* [the variable] *had tended to minus one instead and the same function, what would that be?*
Leo: *Mm, it becomes 0.*
Int.: *Mm, can it attain that value?*

Leo: *Yes.*
Int.: *Can this function ever be equal to two?*
Martin: *Yes when it is one, then it becomes two.*

Do not attain limit value: Functions cannot attain limit values.

Martin: *... you never reach the limit value, you approach the limit value.*

Emma: *Visually it feels like* [...] *if one continues long enough, it will reach its limit value, but it does not since it does not reach the limit value, it is part of the definition, but ...*

Distance between function and limit: Speaks of distance between limit value and function values.

David: *Yes, the difference between f(x) and A is less or equal to epsilon.*

Border: Limits as borders.

Oliver: *You get close to another value as some kind of border, yes.* [...] *As a principle when it comes to limit values there cannot be anything beyond.*

John: *You think of some border of some kind, that you come to a border and beyond it you have decided something, or I think a lot about the body what you can do physically or something like that.*

Approximation: Limits as approximations of function values.

Mikael: *We can do it as accurate as we want, but we can never get it exact.*

Exact: Limits as exact values.

Dan: *... approximations can be done in several ways but a limit value is uniquely determined.*

Attainability is clearly an issue in need of attention since a clear majority of the students had conceptions from both one of the two first categories and the third category. The last two categories, on the other hand, neatly complement each other. There are no cases of inconsistencies regarding limits as approximations or as exact values. Not all students thought of limits as either an exact or an approximate value.

Six of the students stated that limits are unattainable and that limits are attainable in problem solving situations, but not in theoretical contexts. Theory and problem solving seem to be separate issues for these students. Three students revealed an opposite trace to the six students, when they claimed that limits are not attainable and that limits are attainable in theory, but they did not say that limits are attainable in problem solving. The theory used appears to be unclear for the students. Emma and Martin, as two examples, contradict themselves in the examples given previously. The definition is causing Emma's divided perception, and Martin seems to evoke different parts of his concept image in the different situations.

Some students thought of limits as objects, as something a function's values can approach. Such statements belong to the category *Exact*. Some students viewed limits as processes with a focus on the function's values

Table 2. *Students' solutions to Task 1 and 2*

	Task 1				Task 2			
	Right limit	Left limit	Limit	Function value	Right limit	Left limit	Limit	Function value
Filip, e	C	C	C	C	C	C	C	C
Martin, d	C	C	FV	C	C	C	FV	C
Tommy, d	C	C	FV	ND	C	C	2	ND
Anna, d	C	C	CW	C	C	C	CW	C
John, c	C	C	FV	C	C	C	FV	C
Frank, c	C	C	C	C	C	C	C	C
Louise, c	C	C	C	C	C	C	3	C
Leo, b	C	C	CW	C	C	C	C	C
Dan, b	C	C	C	C	C	C	CW	C
Mikael, b	FV	FV	FV	LV	C	C	C	C
David, b	C	C	C	C	C	C	C	C
Julia, a	C	C	C	C	C	C	C	C
Dennis, a	C	C	FV	C	C	C	FV	C
Emma, a	C	C	C	C	C	C	C	C
Oliver, a	C	C	ND	C	C	C	C	C

Notes.

Correct answers

C: Correct answer

CW: Wrong answer followed by an adjustment to a correct one. The error was that the students pointed out the function value in all cases

Wrong answers

FV: Chose the function value at the point instead of the limit value

LV: Chose the limit value at the point instead of the function value

ND: Answered "Not defined"

2: The student wrote 2

3: The student wrote 3

getting closer to the limit value. The category *Approximation* contains statements of this kind. The categories *Distance between function and limit* and *Border* can be perceived both as process categories and object categories since it is not clear whether the students considered the distance and motion or the limit as the main object. Table 1 shows that most students saw limits as both objects and processes. Four students had traces from only one or less of the last four categories.

Only two students, Julia and Dennis, showed traces of concept images which are coherent. They showed no traces of problems with functions attaining limit values and they both explicitly stated that functions can attain limit values. The rest of their conceptions did not match each other, but there were no contradictions either.

Students with high grades did not have a very different structure in their perceptions compared to the other students. The one difference is that the two students with coherent traces of their concept images were among the students with the highest grades.

Table 2 shows the outcome of the students' solutions to the tasks with the graphs (Task 1 and Task 2). There is an uncertainty about the limit value in the results of both tasks.

Five students gave all answers correctly. Almost all students were able to determine right and left limit values. The point of discontinuity caused difficulties for the students when they were asked to determine the limit value. Half of the group of students managed to give an accurate answer at their first attempt. Most errors were connected to the value of the function at the point in both tasks. The results in table 2 show that the number of errors is somewhat higher among the low and average performing students at Task 2.

Discussion

The results of the study are discussed in this section and compared with the literature presented earlier. The three research questions are addressed in the order of the prior presentation.

How the students perceived limits of functions

The current study shows that students perceive limits as objects and as processes. None of the perceptions dominate and they seem intertwined in the students' responses to tasks and questions. Almost all students strongly stated that limits are unattainable for functions, but in another context, they are attainable. A majority of the students connected limits to distance, which the formal definition in their textbook also does, but the hesitation about the discontinuous points in Task 1 and Task 2 and the results in table 1 show that the definition in most cases is not really clear to the students.

Eight of the 15 students considered a limit to be a border. This view can constrain their thoughts about functions and prevent them from understanding that functions can, for example, oscillate over and under the limit value and still tend to that limit. A common misinterpretation of the limit definition among these students is that it says that limits are not attainable (Juter, 2003) and that would mean that a function cannot oscillate over and under the limit value. The impact of everyday language (Monaghan, 1991) is particularly obvious from the examples below table 1 for the category "Border". The Swedish word for limit is a synonym for the word border and there is hence a natural transition between the words. Limits become connected to boundaries, for example physical limitations, which are not supposed to be crossed. It is crucial that students develop their mathematical language well enough to be able to separate it from everyday language. Mathematical language development has been proven to take more time than most students have in a course at the basic level (Grevholm, 2004).

Limits of functions have both procedural and static features (Cottrill et al, 1996; Sfard, 1991). The dual nature of the concept makes it harder for the students to embrace it. It is necessary to pass a stage where limits are seen as processes and objects at the same time to reach the reification stage (Sfard, 1991). Most of the students in this study had, at the time, not yet come so far in their development that they clearly could comprehend the facets of the notion and go between limits as objects and limits as processes. Several of the students confused the process of approaching a limit value with the actual limit value as the object to approach.

The incoherence of the traces of concept images

The results of this study show the students' movement between different ways of perceiving limits of functions. The cognitive way to reach the notion of limit is very different from the mathematical way to reach it (Williams, 2001). The mathematical manner is strictly formal with deductive reasoning from axioms and definitions to prove theorems, while the cognitive way has various stages of abstraction for the individual to go through and back again as described by Sfard (1991) and Cottrill et al. (1996). The sometimes very different characteristics of the two ways to advance to a mathematical concept can be a reason for the incoherence in the results.

The students who considered limits to be unattainable except when they solved problems show two settings of conceptions, one for theory and one for problem solving. They were not evoked at the same time, which gives the incompatible results in table 1. Many problem solving situations were manageable for the students and they could explicitly see that limits are attainable in some cases. In theoretical discussions, however, they selected the dominant, and in this case not most suitable, part of their concept images (Davis & Vinner, 1986), which states that limits are not attainable. The impression that limits are unattainable often comes from the definition with strict inequalities (Juter, 2003). The limit definition is hard for many students to understand (Cornu, 1991; Juter, 2003; Juter, 2004). If it is not implemented in the students' mental networks, so that they understand the concept (Hiebert & Carpenter, 1992), it becomes almost impossible to connect theory and problem solving, leaving the students with an ever present feeling of uncertainty about their calculations. There are contradictory perceptions within several students' minds, but serious inconsistencies between different students' perceptions also occur.

The students' confusion about what happens at the approached points seems to be linked to the students' lack of clarity in their own perceptions of the definition. As we have seen in table 1, many students were not sure whether functions can attain limit values or not. The different results from Przenioslo (2004) and Juter (2003), where students in the former study had a perception that only continuous functions have limits and students in the latter argued that a value cannot be a limit value if the function can attain it, reveal quite opposite conceptions in students' concept images. Williams' (2001) investigation with two students also showed uncertainty about what happens at the value approached. Continuous functions reach their limit values, but the students were hesitant to say that the limit was reached in the limit process. These fundamentally different ways to comprehend the notion of limits indicate that students need more guidance, from lecturers or peers, in their work. Their concept images need to be confronted and tested in various situations to make it clear for the students if their mental representations are valid in general or just in special cases.

Comparison of students with different grades

The students are listed in order of their grades with the students with best grades last. Students from Julia to Oliver have the highest grades. The only two students with coherent traces of concept images were among these high achievers (Julia and Dennis), which confirms the results of Chinnappan's (1998) study that high achievers construct representations of higher quality. Other than this, there are no clear patterns suggesting relations between grades and the outcome of the study. Filip, the only one having the lowest grade (he did not pass the second course), solved Task 1 and Task 2 correctly. Oliver and Dennis, who both belonged to those having the highest grades, made mistakes solving the tasks. Even though Dennis had a coherent concept image, he was not able to solve Task 1 and Task 2. He confused the limit value with the function value. Julia solved both tasks correctly. The high achievers had a better result at solving Task 2. The different perceptions of limits that almost all students showed and the problems with the tasks in this study indicate that the students still had concept images with more than one representation for the same part of the concept without realising it. The merge of the representations (Dreyfus, 1991) had not yet come to pass.

Conclusions

Earlier results found in literature and the data presented in this paper suggest that students perceive limits in very different ways. The traces of the students' concept images were not coherent in most cases in this study, not individually nor as a group. The results imply that the students needed to work further to relate the concept of limit in problem solving situations to the theory they learn. If they had had more opportunities to become aware of the inconsistencies of their mental representations and experienced an urge to make them accurate, the results of this study would probably have been more positive. Most courses have a tight schedule and there are many concepts to handle, so creating such opportunities is not an easy thing to do, but even the smallest effort to provoke inconsistencies might get some students to start questioning their representations.

References

Blomhøj, M. (2002). Vanskligheder ved tilegnelse af matematik. In C. Bergsten (Ed.), *Dokumentation av 12:e matematikbiennalen* [Proceedings of 12:e Matematikbiennalen] (pp. 106-109). Linköping University.

Chinnappan, M. (1998). Restructuring conceptual and procedural knowledge for problem representation. In A. Olivier (Ed.), *Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education* (Vol 2, pp. 184–191). University of Stellenbosch, South Africa.

Cornu, B. (1991). Limits. In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 153–166). Dordrecht: Kluwer Academic Publishers.

Cottrill, J., Dubinsky, E., Nichols D., Schwingendorf, K., Thomas, K. & Vidakovic, D. (1996). Understanding the limit concept: beginning with a coordinated process scheme. *Journal of Mathematical Behavior*, 15, 167-192.

Davis, R. & Vinner, S. (1986). The notion of limit: Some seemingly unavoidable misconception stages. *Journal of Mathematical Behavior*, 5, 281-303.

Dreyfus, T. (1991). Advanced mathematical thinking processes. In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 25-41). Dordrecht: Kluwer Academic Publishers.

Dubinsky, E. (1991). Reflective Abstraction in Advanced Mathematical Thinking. In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp 95–126). Dordrecht: Kluwer Academic Publishers.

Grevholm, B. (2004). Mathematics worth knowing for a prospective teacher. In B. Clarke, D. Clarke, G. Emanuelsson, B. Johansson, F. Lester, et al. (Eds.). *International Perspectives on Learning and Teaching Mathematics* (pp. 519-536) Göteborg: National Center for Mathematics Education.

Hiebert, J. & Carpenter, T. P. (1992). Learning and teaching with understanding. In D. A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning*. New York: Macmillan.

Juter, K. (2003). *Learning limits of function: university students' development during a basic course in mathematics* (Licentiate thesis). Luleå University of Technology, Dep. of Mathematics.

Juter, K. (2004). Limits of functions – How students solve tasks. In C. Bergsten & B. Grevholm (Eds.), *Mathematics and language. Proceedings of MADIF 4, the 4th Swedish Mathematics Education Research Seminar* (pp 146-156). Linköping: Svensk förening för MatematikDidaktisk Forskning.

Monaghan, J. (1991). Problems with the language of limits. *For the Learning of Mathematics*, 11(3), 20–24.

N6 (2003). NUD*IST, qualitative research software. Retrieved 2005-11-17 from www.qsrinternational.com

Przenioslo, M. (2004). Images of the limit of function formed in the course of mathematical studies at the university. *Educational Studies in Mathematics*, 55, 103-132.

Sfard, A. (1991). On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, 22, 1–36.

Star, J. (2000). On the relationship between knowing and doing in procedural learning. In B. Fishman & S. O'Connor-Divelbiss (Eds.), *Fourth International Conference of the Learning Sciences* (pp. 80–86). Mahwah, NJ: Erlbaum.

Tall, D. (2001). Natural and formal infinities. *Educational Studies in Mathematics*, 48, 199-238.

Tall, D. & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, 12, 151-169.

Williams, S. (1991). Models of limit held by college calculus students. *Journal for Research in Mathematics Education*, 22 (3), 219-236

Williams, S. (2001). Predications of the limit concept: an application of repertory grids. *Journal for Research in Mathematics Education*, 32 (4), 341-367.

Kristina Juter

Kristina Juter is a doctoral student in mathematics education at Luleå University of Technology, Sweden. Her working place is at Kristianstad University College, Sweden. Her main interests are students' concept developments at university level, particularly in calculus.

e-mail: kristina.juter@mna.hkr.se

Sammanfattning

Artikeln beskriver en del av en studie om studenters hantering av gränsvärden under sin första termin av matematikstudier. Femton studenter har löst problem och diskuterat gränsvärden i enskilda intervjuer. Resultaten visar att många studenters mentala representationer av gränsvärden är motsägelsefulla eller består av disjunkta delar som används i olika situationer. Studenterna behöver jobba mer med kopplingar mellan teori och problemlösning för att bli varse om gränsvärdesbegreppets fundamentala egenskaper.