Conceptual change in mathematics

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In traditional educational contexts, mathematics is considered a hierarchical structure in which new concepts logically follow from prior ones. From the viewpoint of the theories of conceptual change, however, the learning of mathematics is characterized more by discontinuity than gradual and continuous enrichment. These theories stress the crucial role of prior knowledge in learning. According to these theories, prior knowledge does promote learning, but it can also restrict it and lead to misconceptions. This is the case especially with those kinds of concepts where learning demands a radical change in prior knowledge, which is typical of mathematics and science. One example of these kinds of changes in mathematics is the enlargement of number concept from natural to rational numbers. In this article, three different theories of conceptual change are presented and the perspectives of these theories on the difficulty of the above-mentioned enlargement are discussed. Results of empirical research and some implications for teaching mathematics from the viewpoint of theories of conceptual change are also dealt with.

In traditional educational contexts, mathematics is considered a hierarchical structure in which new concepts logically follow from prior ones, thus allowing students to enrich their knowledge step by step (Dantzig, 1954). These presumptions are based on thinking how new information is connected to prior knowledge, and learning is seen as a process of enrichment or addition of knowledge. Moreover, the learning of mathematics is seen as accumulation of knowledge so that prior knowledge is not to be forsaken or reorganised. Thus, the changes are mainly like additions, where the previous knowledge becomes more detailed, more exact. This

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kind of thinking might be based on presumptions or beliefs of the historical development of mathematical knowledge.

In the research into difficulties in learning mathematics, it has been seen that although students are capable of producing correct answers in tasks where they need to implement the concepts, it is obvious that they have not understood them after all (Lehtinen and Repo, 1996). Part of the explanation is that, on advanced levels, mathematics begins to be both abstract and complex. But the complexity of the concepts is only a partial explanation; there is also the relationship between students' prior knowledge and the new information to be learned to be considered. The content of textbooks on mathematics has usually been organised according to the concepts' mathematical hierarchy, which is logical and consistent. But we want to argue that this construction is logical and whole only from the points of view of experts in mathematics who are already familiar with the structure (Lehtinen, 1998). For the students, however, it looks fragmented and inconsistent because, at this phase of their learning, they cannot yet possess enough structural knowledge to recognize the logic. Enough attention has not yet been laid on this crucial difference between the experts and novices in the traditional teaching of mathematics. Thus, because of the different perspective, students are prone to learn tricks and algorithms, which they are not capable of correctly inserting into their prior knowledge structure, instead fragmented pieces of knowledge and various misconceptions are formed. In recent research (Merenluoto, 2001; Merenluoto & Lehtinen, 2002), the results indicated a very low level of conceptual understanding not only of irrational numbers but also of rational numbers, by students of advanced mathematics at upper secondary schools in Finland. They also indicated that the level of mathematical thinking about numbers was quite low. For example, the names or qualities of different number domains possibly seem so obvious to teachers, that they have not been explicitly taught; teachers have trusted that students will learn them easily.

The perspective of theories of conceptual change

The researchers (Vosniadou and Brewer, 1987; Vosniadou, 1994; 1999; Schnotz, Vosniadou and Carretero, 1999) make a distinction between two levels of difficulty in the learning process targeted at conceptual change: a continuous growth and discontinuous change. The easier level of conceptual change is learning by enrichment, suggesting continuous growth or improving the existing knowledge structure. The more difficult conceptual change is needed when the prior knowledge is incompatible with the new information and so needs *revision*. This kind of knowledge acquisition is typical in specific domains of science, and it requires significant reorganisation of existing knowledge structures (Vosniadou 1999).

Thus, theories of conceptual change (e.g. Carey, 1985; Carey, 1991; Carey and Spelke, 1994; Chi, Slotta, and de Leeuw, 1994; Chi and Slotta, 1993; Vosniadou, 1994; 1999) focus on the role of prior knowledge in learning. According to these theories, the nature of prior knowledge crucially regulates the learning of new concepts. These theories analyse the relationship between the prior knowledge and the information to be learned, in order to find explanations for the misconceptions. Moreover, in this research, a large amount of empirical data has been collected and it has been found that the misconceptions of students are very similar, independently of the nationality or group of students (e.g. McCloskey 1983).

Most of the empirical research from a conceptual change perspective has been done in the field of biology (see Carey 1985; Ferrari and Chi 1998; Hatano and Inagaki, 1998; Mikkilä-Erdmann 2002) and physics (see Vosniadou, 1994; Vosniadou and Ioannides, 1998; Slotta, Chi and Joram, 1995; Reiner, Slotta, Chi and Resnick, 2000; Ioannides and Vosniadou, 2001). There is not so much research from this perspective in the field of mathematics. Most of the empirical research in mathematics from this point of view has focused on problems in the number concept (Lehtinen, Merenluoto and Kasanen, 1997; Stafilidou and Vosniadou, 1999; Merenluoto and Lehtinen, 2000; Merenluoto, 2001; Merenluoto and Lehtinen, 2002; Vamvakoussi and Vosniadou, 2002 a and b). Empirical results from these studies refer to mistaken transfer from natural numbers to the domains of more advanced numbers. They also suggest the powerful and often restrictive nature of thinking based on natural numbers.

Theories of conceptual change and the case of numbers

Roots of the discussion on conceptual change

The discussion about problems of conceptual change in individual learning has its roots in at least two research traditions (Vosniadou, 1999): the science education tradition and the cognitive development tradition. The discussion in the realm of the science education tradition has been inspired by the discussion about scientific revolutions (Kuhn, 1970). In their widely known article, Posner et al. (Posner, Strike, Hewson and Gertzog, 1982) indicated how this discussion invited a large number of studies on misconceptions in science learning. They also presented the essential presumptions for conceptual change in science as: ”dissatisfaction with current conceptions, intelligible nature of new conceptions, initially plausible character of new ideas, and fruitful promises for future research” (see also Duit, 1999). The discussion in the realm of the cognitive development tradition has been inspired by cognitive psychologists to provide an alternative to the Piagetian explanation of accommodation (e.g. Carey 1985; Karmiloff-Smith, 1995). According to Vosniadou, developmental psychology has for years been dominated by Piaget's ideas of cognitive development by a general tendency to free itself from cognitive conflicts. There, conceptual change was described as a domain-general modification of cognitive structures, while the theories of conceptual change focus on domain-specific processes of conceptual change caused by acquisition of domain-specific knowledge rather than increase in general logical capabilities (Vosniadou 1999).

Thus, the initial theory of these researchers was pursued to provide answers to the question: how do learners make the transition from one conception to the following conception? It focuses on what is called major restructuring of prior knowledge structure. Thus, it is based on what Kuhn has called ”a paradigm shift” and on Piaget's notion of ”accommodation” (Duit, 1999). While these theories stress the importance of the role of prior knowledge in learning, at the same time, they explain both the quality of prior knowledge, and the changes in learning, differently. In this article, we briefly present the viewpoints of three different theories on conceptual change and briefly explain how the mistaken transfer from natural numbers to the domains of more advanced numbers could be explained from their points of view.

Theory of different levels of understanding

Vosniadou (e.g. Vosniadou, 1994; 1999) explains the problems of conceptual change with the *naïve frame theory*. According to this theory children very early construct their own theoretical framework of the world. According to this theory, they have very fundamental ontological and epistemological beliefs about the world. These beliefs are based on their intuitive observations about their environment. Some of these observations and experiences are made a long time before the children are able to speak, verbally explain or cognitively analyse their observations. The basic epistemological assumption of this naïve framework theory is that the world is what it seems to be. According to Vosniadou, the child's knowledge structure and beliefs system seems to be a naïve but coherent theory with the aspects of explanation and prediction. The problem of this early knowledge is that its structure is so unconscious that it seems impossible to consciously evaluate or test it (Vosniadou, 1994, p. 47). The second problem in this naïve frame theory is that the beliefs are based on a child's intuitive assumptions that have been verified in an everyday context. Some of these beliefs and assumptions are very fundamental and resistant to teaching.

Vosniadou makes a distinction between changes that are *spontaneous* and changes that are created as a *result of teaching* (Vosniadou and Ioannides, 1998). Spontaneous changes are born in the enriching process of the child's knowledge structure as a result of observations in cultural and linguistic contexts. Both the mistaken and correct conceptions are included in these large theoretical interpretations from early on. As an example of those interpretations, Vosniadou presents the conception on the concept of force as a property of large and heavy objects (Vosniadou and Ioannides 1998, p. 1218).

When children learn to recite the number words they form specific theories in which they find the one to one correspondence of number words and the objects in the counting process. In the experiences of the everyday cultural context they develop beliefs that for every object there is a next object (figure 1).

The changes resulting from teaching scientific concepts create different kinds of changes, because teaching mostly focuses on conceptions

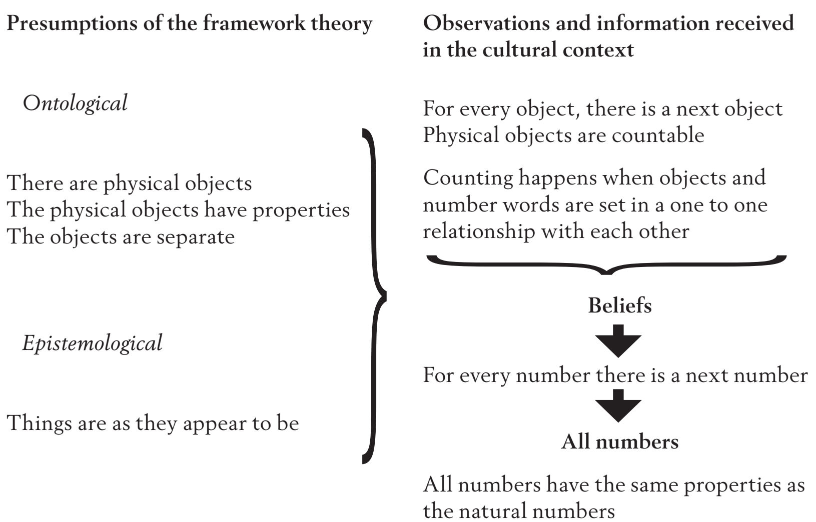

Figure 1. *Hypothetical conceptual structure for the interpretation of numbers as separate objects (adapted from Vosniadou & Ioannides*, 1998*)*

outside the frame theory, on, e.g. *specific theories*. A child learns scientific concepts and explanation models, while the learning is framed by the conceptions and beliefs of the frame theory. In formal mathematical instruction, some of the beliefs and prior conceptions are strengthened, and it is also necessary to strengthen them. One of these beliefs is thinking of numbers as discrete objects, because this is a fundamental property of natural numbers. This separate nature of small quantities seems to be one of the basic ontological presumptions of the naive framework theory of numbers By the terms discrete and separate we mean the instinctive feeling connected with numbers and quantities that there is always the 'next number', 'next quantity' (Hartnett and Gelman, 1998), and that there is some kind of space between them. This property of numbers is also found in the writings of Aristotle and, moreover, the idea of discreteness is also embedded in the rigorous definition of natural numbers explained by Giuseppe Peano from Italy (Boyer, 1959). There, it is formulated so that every number has a successor, and no two numbers have the same successor (cf. Landau, 1960; Kline, 1980; Russell 1993). Together with the principle of one-to-one correspondence with objects, they "are woven into the very fabric of our number system" (Dantzig, 1954, p. 9).

Although innate principles can foster learning, they can also serve as barriers to learning. If what is to be learned does not share the same basic assumptions as the available knowledge, then the risk is high that the information meant to foster new learning will be assimilated to what is known and, therefore, will be misinterpreted (Gelman and Brenneman, 1994). The very fundamental idea of a successor is necessary for learning the notion of natural numbers. From the cognitive viewpoint, however, this seriously conflicts with the understanding of the very character of both rational and real numbers (Kieren, 1992). For example, Hartnett and Gelman (1998) found that by relying on the intuition of the next number, even 5-7-year-old children were able to say that it is not possible to write the largest natural number. The children had considerable difficulties, however, when the prior knowledge structure did not support the new information to be learned. This was the case when children were asked to sort fractions: they tended to use the same logic they had learned to use in sorting natural numbers (Hartnett and Gelman, 1998; Stafilidou and Vosniadou, 1999). These procedures were difficult because of the constraining nature of the intuition of the "next" number.

If a learner does not see or understand the need for reconstruction of his/her prior knowledge structures, then he/she tries to synthesise the new information with his/her prior knowledge. These kinds of constructions are presented by Vosniadou as "synthetic models", thus referring not to misconceptions but to something that is still in the process of change. The knowledge stored in specific theories or mental models has characteristics different from the knowledge in the naïve frame theory. The specific knowledge is built up from a sample of related beliefs or presumptions, which are used to describe physical objects or their behaviour. These conceptions describe the inner organisation of the concepts and they are acquired by making observations in the world outside systematic teaching. Changes in the specific theories are easier to make than changes in the naïve frame theory, because within these specific theories, changes can take place happen without any connection to the frame theory (Vosniadou, 1994; 1999).

Thus, a student's number concept could be called synthetic, if he/she is working reasonably well with rational numbers on the operational level (see Sfard 1991), while her/his thoughts of numbers are still based on discrete natural numbers (e.g. Merenluoto and Lehtinen, 2002). According to the results from large empirical studies, it is possible to categorise students' synthetic models into clearly different levels of understanding, representing the development of understanding. Synthetic models have been found in the development of concepts of the physical world (Vosnidou, 1994), in the concept of force (Vosniadou and Ioannides, 1998), and in the concept of numbers (Merenluoto and Lehtinen, 2002; Vamvakoussi and Vosniadou, 2002).

On the other hand, Vosniadou also explains how it is possible to keep conflicting pieces of information in the mind as fragmented microstructures to be used in special situations (Vosniadou 1994, pp. 49-50). One example of such fragmented structures of knowledge is pieces of mathematical knowledge, which do not build a whole logical structure, but are stored as pieces of microstructures to be used in solving particular problems. This seems to be the case in situations referring to narrow conceptions of school mathematics in solving real world problems (e.g. Verschaffel, Greer and De Corte, 2000).

Theories based on ontological differences in categorization

On the basis of ontological categories, the objects in the physical world are classified into different categories. Chi, Slotta and de Leeuw (1994) present a theory of conceptual change which is based on philosophical analysis of problems of categorising. According to this theory, humans have a tendency to classify the objects in the physical world into different categories, such as: matter, processes or mental states. In this frame of reference, the objects in the ”matter” category have qualities which make it possible to store and touch them, whereas the objects in the category of ”processes” have qualities that are related to time. They have a beginning and an end and they are related to change. The objects in the ”process” category are further divided into two subcategories: in one group are objects that have a beginning and an end, and in the other no beginning or end is defined (Ferrari and Chi, 1998).

What constitutes the more difficult conceptual change in this frame of reference is that some object is going to be reassigned to a different category from the one it originally was placed in. Then misconceptions are also most likable to occur. One example of a misconception about electrical current, which was found in students' answers, was ”electrical current is stored in batteries” (Chi and Roscoe, 2002). Chi explains this problem as a misclassification of electricity into the category of ”substance”, that can be stored. For a deeper understanding of electrical current there should be a conceptual change (as a category change) into the category of ”processes”.

Chi explains this category change also as a process of ”reassigning”, which a student is not capable of before he or she has learned what the new category is and how it is defined. According to Chi (1992), the students do not even have a possibility to understand a concept before they have understood something about the new category and a conceptual change has occurred in their prior knowledge. Then, they are able to reassign the concept according to the new ontology of the concept to be learned (Chi et al., 1994, p. 34). The basic problem of misconception according to Chi (e.g. Chi and Roscoe, 2002) is that students are not aware of their categorisations and, thus, are blind to their lack of understanding. In her article, Chi (Chi et al., 1994) especially mentions natural and rational numbers belonging to different ontological categories. According to Russell (1993) natural numbers are defined as a ”class” representing cardinal or ordinal numbers, whereas rational numbers are defined by a relation, thus belonging to ontologically different categories. This kind of conceptual change demanding a radical change of the ontological category (Chi 1992) seems to explain the difficulty in the change because it seems to be impossible through any acquisition mechanism such as deletion or addition, discrimination or generalization. This is because the knowledge from concrete objects cannot be directly transformed to the abstract kind of knowledge typical of advanced mathematics (cf. Ohlsson and Lehtinen, 1997).

New nativistic theories

There is evidence that mathematics is a distinct domain already in the innate cognitive mechanism (Gallistel and Gelman, 1992). Gallistel and Gelman (1992) argue that infants' numerically relevant responses to sets of inputs are supported by a skeleton of nonverbal counting and relate to numeral reasoning principles. The term ”domain-specific” refers to a domain that consists of a given set of principles, the rules of their application and the entities to which they apply (Gelman and Brenneman, 1994). Many researchers (Starkey, Spelke, and Gelman, 1990; Spelke 1991; Carey and Spelke, 1994; Gelman, and Brenneman, 1994, Karmilof-Smith, 1995) argue that human reasoning is guided by a collection of innate domain-specific systems of knowledge. According to this hypothesis, each system is characterised by a set of core principles that define the entities covered by the domain and support the reasoning about those entities. According to Carey and Spelke (1994), a conception of numbers as separate objects, and preliminary intuition about operations between these objects, are two of the core concepts of mathematics. In several empirical studies, it has been found that even infants have an intuitive conception of numbers and small cardinalities (e.g. Starkey, Spelke and Gelman 1990, Starkey, 1992). In these studies, it has been found that linguistic development of cultural influence is not necessary for early conceptions of small cardinalities (from one to three or four). Starkey (1992), for example, has shown that very small children are able to recognise different cardinalities and also the increasing and decreasing of quantities long before they have the cultural experiences of numbers. Gallistel and Gelman (1992) speak about a preverbal counting mechanism, which children use when making differences between small quantities. This mechanism operates as a basis for operations leading to verbal counting. The preverbal system is then operating as a frame of reference when learning verbal counting. Learning to count then means the ability to map the preverbal counting mechanism and later learned verbal and written symbols. According to these researchers, this early mathematical system guides the formation of mathematical knowledge even into adulthood.

Carey and Spelke (1994) explain that this domain-specific system of knowledge has been found in at least three different scientific domains: physics, psychology and mathematics. According to this point of view, learning means enriching these kinds of basic reservoirs, which serve as a ’skeleton’ for new information to be learned. The more difficult conceptual change in this frame of reference means, for example, that the principles outside the core principles become more important than the core concept, or forsaking the prior theory when it becomes incommensurable with the new theory.

In the number concept, the principles of the core concept (as discrete numbers) do not become less important, but in the enlargement process the principles outside the core concept create a new frame of reference where mapping from rational numbers to natural numbers is formed, and natural numbers are treated as a subset of rational numbers. Thus, the notion of a successor is still always valid whenever operating with natural numbers and integers. From the cognitive point of view, however, this enlargement demands the abstraction of a parallel mental model for numbers and the flexible ability to move between these different models, according to the problem at hand. The problems resulting from the ontological shifts (Chi, Slotta and de Leeuw, 1994) in learning begin very early, because fractions are learned very early in the course of mathematics learning (Kieren, 1992). Moreover although the more advanced properties of rational numbers are understood very much later, these features are embedded in the rules of operations with these numbers which are very different compared to the rules of operations on natural numbers.

Results from empirical research

The characteristics of natural numbers lead to very consistent beliefs which are often mistakenly generalized to the domains of other numbers. In her research Huhtala (2000) found several different ”mini theories” in the mathematical thinking of adults who had had severe learning problems in mathematics during their school years. Some of these ”mini theories” were based on the very consistent beliefs that have been formed in experiences of operating with natural numbers. Some of these beliefs are that subtraction is possible only if a smaller number is subtracted from a larger number; division is possible only when a larger number is divided by a smaller; it is impossible to subtract from zero; in multiplication the result is larger, in division it is smaller. These beliefs are valid in the domain of natural numbers and the problems students were having indicated problems of conceptual change, which suggests that these kinds of beliefs are very resistant to teaching.

The extensions of the number concept are, in traditional teaching, primarily treated as enlargements, which are justified by the possibilities of new kinds of operations. Although the profound properties of rational numbers, like the compact nature of the number line, are not discussed in teaching rational numbers at lower levels of education, these properties are, however, embedded in the operations calculated with those numbers. Large numbers of studies have been done on difficulties that students have with rational numbers and multiplicative thinking (e.g. Sowder, 1992; Kieren, 1992; English and Halford, 1995; Carraher 1996). In this article, we argue that these problems are not only due to learning difficulties or the increasing complexity of these concepts, but also to the quality of students' prior knowledge, which is based on natural numbers, thus referring to the problems of conceptual change.

At the advanced mathematics level, where students need to learn the concepts of continuity and limit, the question of the compact nature of rational numbers on the number line and the continuum of real numbers begin to have a very significant role. However, the early understanding of this discrete quality of numbers is so fundamental that it is possible that the question of density of numbers on the number line is overlooked in teaching mathematics. In fact it is possible that the student keeps his/her original conception, while at the same time learning the formal definition of more advanced numbers (e.g. rational and real numbers). It is also possible that he/she does not even notice the cognitive conflict between these concepts (Vinner, 1991). According to the study of Neuman (1998) an incorrect transfer from natural numbers led to the result that only a few seventh-graders understood that there are an infinite number of fractions between any two fractions. In a recent study (Merenluoto and Lehtinen, 2002) we found that a majority of even 17-18-year-old students at the upper secondary level had not restructured their prior system of beliefs to understand the density of fractions on the number line, even at the preliminary level. Their comprehension of the hierarchy of the number system was confused; many of the students spontaneously used the logic of natural numbers in the domain of rational numbers.

For example, the discrete nature of the students' comprehension of the number line was obvious in those answers where we presented them with a cognitive conflict with questions like: which fraction is the next after 3/5 or which real number is closest to 1.00? The majority of the students were not sensitive to the conflict, but based their answers on the fundamental intuition of whole numbers: they answered that 4/5 is the next fraction after 3/5 or that the number 0.999... is the closest to 1.00. Only one fifth of the students presented answers in which there were any references to the compact nature of rational numbers or the continuum of real numbers. Because the subjects were students of advanced mathematics courses in upper secondary school, they had a lot of experience with numbers and operations. It is therefore likely that most of them would have remembered that it is possible to divide fractions infinitely had they been reminded.

The results, however, indicate a situation where this knowledge of fractions constituted an isolated piece of knowledge. When the students read the word ”next” they spontaneously used the logic with which they had more experience. Using the theory of Vosniadou (1994, 1999), we could call this kind of number concept a ”synthetic model”. In order to arrive at the correct answer of ”it is not possible to define the next number” the students need to develop metaconceptual awareness of their conceptions of numbers and to have consciously pondered these questions at some earlier stage. These changes are difficult because of the constraining nature of the prior knowledge of natural numbers.

Implications for teaching from the point of view of theories of conceptual change

In previous studies, it has been found that conceptual change involves not only change in specific beliefs and presumptions, but also the development of metaconceptual awareness, and the construction of explanatory frameworks with greater systematising coherence and explanatory power (Vosniadou, et al., 2001). However, because of the difficulty of the process the students are prone to ”fall back” on their previous assumptions (Mikkilä-Erdmann, 2002). This kind of radical conceptual change is too difficult for many students, especially those having difficulties with mathematics, to achieve by themselves, so deliberate pedagogical interventions are needed (Chi, 1992; Lehtinen and Ohlsson, 1999). During the process, students need to be helped to become aware of their existing beliefs and presuppositions to understand their theoretical nature: thus, the development of metacognitive abilities is crucial. In this process, they need to actively make this change and they have to be able to explicitly discuss the difference between their prior knowledge and the properties of the new concepts.

Many different methods have been used to foster the conceptual change and help the students with these changes: by means of special text design, where the differences to prior knowledge are explicitly stated (Mikkilä-Erdmann, 2002); by using cognitive conflicts with anomalous data (Limón and Carretero, 1997; Limón 2001) or by analogies (Duit, Roth, Komorek and Wilbers, 2001); by teaching meta concepts (Wiser and Amin, 2001) and using self-explanations (Ferrari and Chi, 1998). While many of these methods have had good results, there are also severe problems, because the methods used to promote conceptual change in some students may fail for others (e.g. Duit et al., 2001; Limón, 2001). And, for example dissatisfaction with the use of cognitive conflicts as an instructional strategy has been found from observations where students often patch up local inconsistencies in a superficial way and do not undergo the more radical kinds of conceptual change (Vosniadou, 1999).

Conclusion

From a cognitive perspective, there are several difficult transitions, in which a radical change in prior thinking is needed in learning mathematics. Besides the enlargement of number concept from natural to rational numbers, we can think of the cognitive gap between arithmetic and algebra (Goodson-Espy, 1998) the incommensurable gap between rational and real numbers (Merenluoto and Lehtinen, 2002) and the transition from finite schemas to infinite ones (Kieren, 1992; Chi and Slotta, 1993; Cornu 1991, Tsamir and Dreyfus, 2002). In order to explain why these concepts are difficult to learn and why misconceptions occur, it is worthwhile to consider how the initial conceptual structures aresupported by a system of interrelated observations, beliefs and presumptions. These form a relatively coherent and systematic explanatory system, which works relatively well in everyday life but is resistant to teaching (Vosniadou, 1999).

We suggest that the theories of conceptual change provide the means to understand the viewpoints and difficulties of the students struggling to understand the concepts of mathematics and also to develop teaching methods to promote their understanding. In general, we assume that the difficulties students have in the acquisition of new areas of mathematical knowledge, like the extensions of number concept, are not only due to the increasing complexity of the knowledge, but also to situations where prior knowledge systematically supports the construction of misconceptions.

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Yhteenveto

Perinteisessä matematiikan opettamisen ajattelussa matematiikka näyttäytyy hierarkkisena käsitejärjestelmänä, jossa uudet käsitteet seuraavat johdonmukaisesti aikaisemmista. Käsitteellisen muutoksen teorioiden lähtökohdista matematiikan oppiminen näyttää kuitenkin edistyvän todennäköisemmin epäjatkuvana tapahtumasarjana kuin jatkuvana käsitteiden vähittäisenä rikastumisena. Käsitteellisen muutoksen teoreettisessa ajattelussa painotetaan aikaisemman tietämyksen keskeistä roolia uuden oppimisessa. Näistä teoreettisista lähtökohdista tehdyt empiiriset tutkimukset osoittavat, että vaikka aikaisempi tietämys edistää uuden oppimista, se voi myös rajoittaa sitä ja johtaa väärinkäsityksiin. Näin käy todennäköisesti sellaisten käsitteiden oppimisessa, jotka vaativat oppijalta radikaalia muutosta aikaisempaan ajatteluun. Yksi esimerkki tällaisesta muutosvaatimuksesta on lukualueen laajentaminen luonnollisten lukujen alueelta rationaalilukujen alueelle. Tässä artikkelissa esitellään kolme erilaista käsitteellisen muutoksen teoreettista suuntaa ja tarkastellaan empiirisen tutkimuksen valossa lukualueen laajennuksen problematiikkaa näistä näkökulmista.