The role of decision making and resources in group solutions of a problem involving Bayes' formula

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This study was concerned with the decision making behaviour of students at the upper secondary level of schooling. Transcripts of verbal reports produced by student pairs solving a probability problem involving Bayes' Formula were analysed using Schoenfeld's protocol parsing scheme. The students' monitoring and decision making were significant contributors to how solutions evolved but so too were particular words in the problem statement. Both the quality of control behaviour and students' resources contributed to the success or otherwise of their solution. Resource-related factors included misinterpretation of the problem, misreading of the problem, failure to make the connections between probability rules and their conditions of use, and confusion between compound and conditional probability.

Decision making and problem solving

In order to research students' problem solving behaviour Schoenfeld (1985b) developed a framework for exploring mathematical cognition based on his argument that "four categories of knowledge and behaviour [namely, resources, heuristics, control, and belief systems] ... must be dealt with, if one wishes to 'explain' human problem-solving behaviour" (p. 12). He has since broadened this framework by adding another category, the practices of schooling and classroom environments (Schoenfeld, 1992). However, he argues that metacognition (control), beliefs and practices are the critical components of this framework and "domain specific knowledge plays an altered and diminished role, even when it is expanded to include problem-solving strategies" (p. 363). Barkatsas and Hunting (1996) concur. In their review of mathematical problem solving

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research, they concluded "that metacognitive decisions could be considered the 'driving forces' in mathematical problem solving" (p. 26).

The investigation reported in this paper was concerned with gaining an insight into the problem solving behaviours of students at the upper secondary level of schooling as few problem solving studies, particularly those dealing with metacognition, (e.g., Geiger and Galbraith, 1998; Goss and Galbraith, 1996; Stillman and Galbraith, 1998) have targeted this area of schooling concentrating instead on college students (Schoenfeld, 1989) and upper primary students (e.g., Adibnia and Putt, 1998; Lester, Garofalo and Kroll, 1989; Wilson and Clarke, 2004). The study focussed on students' decision making and resource use whilst problem solving was occurring through an analysis of transcripts of verbal reports (protocols) produced by pairs of students in "speak-aloud" problem solving sessions. The protocols were analysed using Schoenfeld's parsing scheme (1985a). This scheme

> is appropriate for documenting the presence or absence of executive decisions. However, it is likely to be useful only on ... 'perplexing or difficult' problems, in which individuals must make difficult choices about resource allocation. (Schoenfeld, 1992, p. 364)

The problem chosen was expected to meet these conditions as it involved the use of compound and conditional probabilities and Bayes' Formula.

Conditional Probability and Bayes' Formula

Many students in upper secondary school and in tertiary courses have great difficulty with conditional probabilities (Falk, 1988; Peard, 1996; Pollatsck, Well, Konold, Hardiman and Cobb, 1987) and applying Bayes' Formula (Ichikawa, 1989). For example, Smith, Wood, Gillies, and Perret (1994) reported that a conditional probability question involving a relatively straightforward application of Bayes' Theorem was answered correctly by only 6% of 186 first-year university statistics students and another 75% did not attempt it or scored no marks. Various reasons have been given for this difficulty but the problem remains unresolved.

Language Difficulties

Smith et al. (1994) suspected facility with coding/encoding from the English language to mathematics played an important role in students' success with these problems but their preliminary investigation showed this coding/encoding skill was independent of linguistic difficulty of questions as measured by lexical density. However, other authors have suggested and investigated other possible sources of difficulties with language which support the notion that coding and encoding from English to mathematics plays a major role in success with such problems.

In trying to understand how students interpret contextualised word problems, Einhorn and Hogarth (1986) suggested "temporal order is an important cue that is difficult to ignore even when it is salient to do so" (p. 9). Temporal order has no role in formal probability theory but affects causal judgments in everyday life. Students can bring this into their interpretation of a probability question. In interpreting language the conjunction "and" frequently implies temporal order but in formal probability it implies conjunction of two events, irrespective of temporal order. A question like, "What is the probability of driving the Commodore to work and getting home by 5.30 pm?" is seen by a statistician as being a simple conjunction whereas it is not uncommon for students to perceive the question as requiring conditional probability. The sequence of events is seen as being important and conditions the uncertainty of the later event, "getting home by 5.30 pm", on the earlier one, "driving the Commodore to work". Einhorn and Hogarth (1986) found when the wording of such questions reversed the implied temporal order, that is, "What is the probability of getting home by 5.30 pm and driving the Commodore to work?" students used the conjunction more frequently.

Another source of difficulty is interference from causal reasoning. This can be due to poorly worded questions that can be interpreted as requiring a judgment about causality rather than conditional probability (Pollatsek et al., 1987). This is further compounded by the wording used to express conditional probability and causal relationships involving the same key words of "if" and "given that". This interference also comes from a belief that causal relationships, that is, p(effect|cause), are necessarily stronger than diagnostic ones, that is, p(cause|effect). Tversky and Kahneman (1980) and Kahneman and Tversky (1996) have argued a causal bias exists in judging conditional probabilities and have presented supporting evidence although their work has been criticised by Gigerenzer (1996) and Gigerenzer and Hoffrage (1995). Pollatsek et al. (1987) reported their results did not support the notion of a strong causal bias in judgments of conditional probabilities per se but in particular questions the use of probability wording (e.g., "the probability that a woman is a teacher") appeared to elicit causal reasoning whilst percentage wording (e.g., "the percentage of women who are teachers") did not.

Another possible source of difficulty is confusion of inverse probabilities. This occurs when students confuse $p(A|B)$ with $p(B|A)$. Einhorn and Hogarth (1986) argue that the most likely cause is ambiguity of temporal cues in the task. Pollatsek et al. (1987) found little indication that subjects in their study reversed conditional probabilities in all but one of seven probability questions. In this question temporal cues were lacking. The probability version of the problems led to fewer reversals than the percentage version. Their study revealed another source of error could be a belief that pairs of reverse conditional probabilities expressed as percentages have to sum to 100%.

Conceptual Difficulties

Pollastek et al. (1987) raise the possibility that not all conjunction errors should be attributed to errors in translating the question statement to mathematics. There is also the possibility that students do not have separate concepts for conditional and joint probability but instead have an "amalgam of the two" (p. 269).

The Study

The specific research questions were:

1. Why does the solution to a problem evolve the way it does?
2. What accounts for the success or failure of the problem solving attempt?

In light of the literature survey it was expected that adequacy of control behaviour would be the reason the solution evolved the way it did and that control behaviour would contribute to the success or failure of the solution attempt but language and conceptual knowledge may be more important in this particular context than Schoenfeld (1992) indicated.

Participants

The study was conducted at a secondary girls' college where the researcher was a mathematics teacher at the time. The students were in their final year of high school (17 years old) and were studying a mathematics subject within the academic stream. This course was taken by students who intended undertaking further study at university. At the time of the study all students had completed three semester units of their mathematics course and the probability section of the final unit. All ten students were volunteers. Six of the students were of outstanding mathematical ability receiving a Very High Achievement[1] rating for mathematics at the end of the year. The other students received Sound Achievements overall but performed considerably better in the probability unit than in other units.

Task Variables

In light of the difficulties highlighted in the literature regarding the mathematical topic area, the problem was selected to meet the following criteria:

1. Use of percentage wording rather than probability wording should reduce the effects of causal bias although it may lead to more reversal errors.
2. Implied time order in the wording should be reversed in order to reduce the effects of temporal order cues.

On this basis, the following problem[2] was used:

> Lucy Jones owns two cars – a Holden Commodore and a new Toyota Corolla CS. About three quarters of the time she uses the Corolla to travel to work and about one quarter of the time she uses the Commodore to drive to work. When she drives the Commodore, she usually arrives home by 5.30 pm about 60% of the time; if she uses the Corolla she arrives home by 5.30 pm about 75% of the time. If, on a particular day, she gets home at 5.35 pm, what is the probability that she used the Corolla to drive to and from work?

It was to be worked with pen, paper, and calculator only.

Data Collection

The investigation involved videotaping five student pairs attempting to solve the problem. Pairs were used as two-person protocols are said to provide the richest data for observation of decision making behaviour (Goos and Galbraith, 1996; Schoenfeld, 1985a; Scott, 1994). The videotape provided a permanent record for coding the problem solving behaviour of the groups. Each videotape was transcribed and conversational turns (referred to as Items) were numbered consecutively. Students were allowed 20 minutes to solve the problem although groups 3, 4 and 5 were allowed more time to see if the time limit was restrictive but in all cases the extra time produced no better result. In the diagrams that follow only the first 20 minutes of a problem solving session will be used for comparison purposes.

Protocol Analysis

Protocols were parsed using Schoenfeld's framework (1985a) for analysing problem solving protocols. Protocols were divided into macroscopic chunks of consistent behaviour which Schoenfeld referred to as episodes. The episodes were then classified as: (a) reading, (b) analysis, (c) exploration, (d) planning, (e) implementation, or (f) verification (see Figure 1). Transitions between episodes were also noted. These may be marked by statements by students indicatong a change of direction such as, in protocol 1, "Okay, well" (Item 5) or "So" (Item 71) or these statements could be absent but the potential transition location was noted (e.g., the second transition in Figure 1) as these were points where managerial action should have taken place or at least been considered. Any new information or local/global assessments were also noted as these are additional potential loci for managerial action and decision making.

Results

Correct Solution with Little Overt Decision Making

Only group 1 (Kym and Jill) was successful in solving the problem. Figure 1 shows a graphical representation of the protocol parsing into four episodes totalling 124 items of dialogue. Initially, the girls held conflicting views on how they might approach the problem. They began with a brief reading episode followed by a seven minute episode of analysis and rereading of the problem statement. During this episode they identified and noted the required goal and all given conditions. A crucial factor in their success was their handling of new information during this episode. At Item 29 (see Figure 1), Kym asked, "Now, how do we involve this 5.35 p.m.?" Their interpretation of the statement "she gets home at 5:35 pm" as meaning "NOT by 5:30 pm" was critical. They now had all the necessary information to produce a satisfactory solution but if they had not made the connection between the stated condition and their encyclopaedic knowledge of the world, their solution would have been doomed. They had no difficulties agreeing that the required goal was a conditional probability, however, Jill wanted to interpret two of the given conditions as conjunctions but Kym was adamant they were conditional probabilities.

55. Jill: I think.

56. Kym: You think.

57. Jill: ... that when she arrives home by 5:30 that should be an "AND" [pointing to P(5:30|Com)]. She doesn't. But we definitely agree that that is supposed to be "given" [pointing to RT find P(used Corolla | gets home 5:35 pm)].

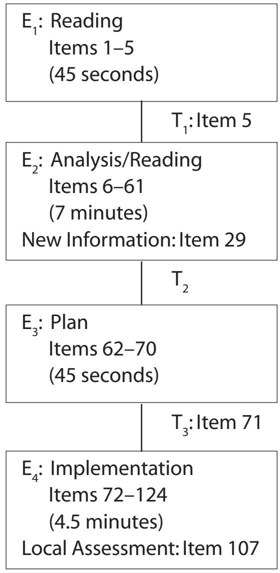

Figure 1. *Parsing of protocol for group 1's problem solving session into episodes (E_n) of consistent behaviour. Transitions between episodes are denoted by T_n.*

Jill's insistence that they use conjunctions rather than conditional probabilities appeared to be procedurally based as she could not recall a procedure which would enable her to use conditional probabilities. The conflict was resolved, however, when Kym revealed her plan (Items 62-70) showing how it was possible to derive the required conjunction from the information given.

59. Jill: Well, do you want to try using "AND" and if it's wrong, I'll admit defeat and then we can do it the other way because the way it's like that I can't see any way how you do it.
60. Kym: No, look. We did one of these the other day. They were "judged guilty" ones and they were "GIVEN's".
61. Jill: No, I used "AND's". I got it wrong anyway.
62. Kym: Umm, no, what you do is you say the probability of this here [points to P(used Corolla | gets home 5.35 pm)]
64. Kym: ... uhmm, is equal to the probability of the Corolla AND 5.35 pm.
66. Kym: ... divided by the probability of 5, I think it's 5.35. I think it is the bottom one or it might have been the top one. That's "AND" [Corrects an error in her writing out of the formula].

67. Jill: Yeah, that's right, because if you have that ... you see what I'm saying - we don't have any "AND".
68. Kym: No, but then what you do...
69. Jill: Oh! From there you can find it.
70. Kym: Yeah, you can.

The girls spent a further four and a half minutes in an implementation episode calculating the correct result but there was no verification episode.

Figure 2 shows a time-line representation of the protocol where overt signs of management activity are denoted by triangles (Items 29 and 107, respectively). As Figure 2 shows, there was very little overt evidence of management behaviour. In addition to their decision making concerning the new information mentioned above (Item 29) in the second episode, there was the need for only one local assessment at Item 107 to correct a substitution error in a calculation during the implementation episode. Their success appeared to be related to Kym already having a schema for

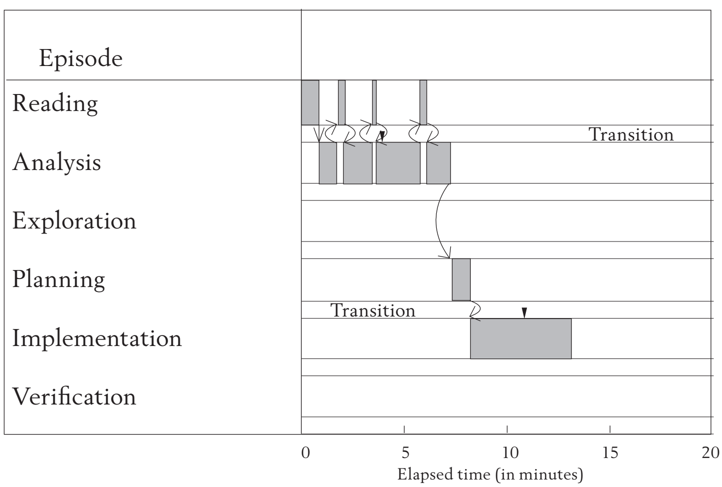

Figure 2. *A time-line representation of parsing of the protocol from group 1's problem solving session. Arrows show the sequencing of episodes. Overt signs of management activity are denoted by triangles.*

this type of problem despite receiving no direct instruction in class. She related the problem to one she had encountered previously and solved successfully. She retrieved this schema from memory and simply carried out the necessary procedures. As nothing occurred which did not mesh with her problem schema, there was no need for further assessments during the solution implementation. It would, perhaps, have been advisable to check the final solution for mechanical errors but the girls omitted to do this. In addition to these overt examples of decision making, there were only three transitions between episodes (T_1 to T_3 in Figure 1) where covert decision making may have occurred.

Good Control but a Flawed Solution

Figure 3 shows the time-line representation of the protocol parsing for Group 2 into five episodes consisting of a period of silent reading followed by 105 dialogue items. Ann and Kit's solution was promising despite being incorrect. The flaw was that they failed to realise that "she gets home at 5.35 p.m." means "NOT by 5.30 p.m." It was not that they did not notice this crucial piece of new information in the problem statement. Kit read it out at Item 13 (first triangle in Figure 3) during an exploration episode but Ann immediately dismissed it, making the erroneous comment "so she gets home by 5.30 ... same thing ... same language". Subsequently, when Kit mentioned it at Item 21 (still in exploration episode), Ann immediately dismissed it again saying, "It's just by 5.30". Kit did not mention it again but it appeared by her comment during their implementation episode, "What! She's already gotten home at 5.30!" (Item 57), that she had not realised Ann's interpretation until then. This interpretative blunder was pathological for their solution. In effect they interpreted the question as: *If, on a particular day, she gets home by*

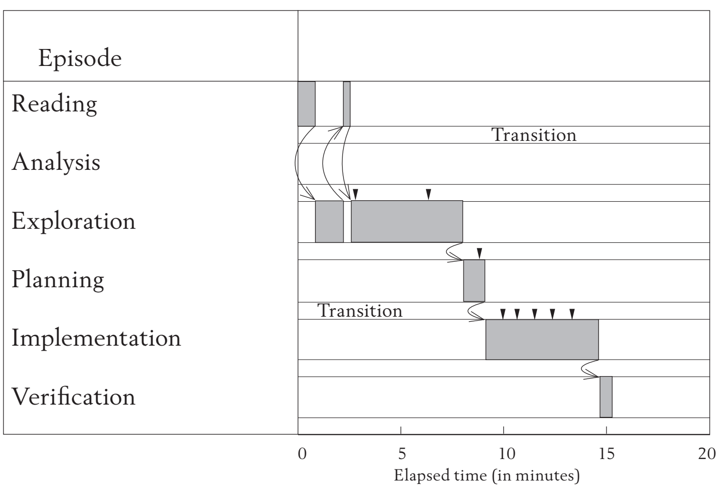

Figure 3. *A time-line representation of parsing of the protocol from group 2's problem solving session.*

5.30 p.m., what is the probability that she used the Corolla to drive to and from work? The solution submitted is correct for this interpretation.

As the students' control behaviour as the solution evolves is of interest, a brief analysis of Protocol 2 is in order as this pair of students appear to monitor their progress much more closely than did Kym and Jill. After briefly silently reading the problem statement, the students embarked on what was classified as an exploratory episode although, as shown in Figure 3, they re-read the problem during this episode. In contrast to Kym and Jill, Ann and Kit seemed less sure of what the problem entailed and appeared to conduct a broad tour through the problem space during this episode hoping something fruitful would show up. During this exploratory episode, Ann and Kit made explicit note of all problem conditions and their interpretation of the goal state but they did not explicitly use probability notation, symbolism or even conditional probability terminology. They made only one local assessment (Item 33, second triangle in Figure 3) of their understanding of the given probability information. An intuitive plan evolved from their explorations although their local assessment of its likely success (Item 48, third triangle in Figure 3) revealed they were unsure of its viability. Despite this, the implementation of this plan was quite orderly. On five occasions (shown as triangles in Figure 3) local assessments were undertaken, mainly by Ann, to check on the adequacy of the evolving model, the time remaining, and the adequacy of a local plan. When they finally reached a result, they checked to see if it appeared reasonable in light of their interpretation of the problem statement (verification episode in Figure 3).

A feature of this protocol is the number of points at which the path of the solution is overtly monitored. Neither girl appeared to have an inbuilt schema for this particular type of problem. They made no attempt to relate it to any other problems they had encountered. Ann took charge and used her conceptual probability knowledge rather than explicit rules of conditional probability. Kit found this particularly unsettling as her approach was to "use one of the rules" (Item 65) but what she gleaned from the problem did not mesh with her understanding of probability rules. She did not recognise Ann was intuitively using probability rules. She saw no correspondence between Ann's solution and formal probability theory, observing: "It makes sense logically. I don't know mathematically" (Item 83). As the girls appeared to be progressing through uncharted waters, they were very cautious and consequently monitored their solution often.

To Schoenfeld (1985b) "the defining characteristic of actions at the control level is that they have global consequences for the evolution of a solution" (p. 27). Ann's dismissal of the new information (regarding time of arrival) as unimportant guaranteed the solution attempt failed. The impact of this decision was certainly global. However, rather than see this as an indication of poor control, an alternative view in keeping with Nathan, Kintsch and Young's (1992) theory of algebra word problem comprehension would be that Ann's action resulted from an omission from the mathematical model she constructed caused by her failure to make the expected inferential elaboration regarding the time of arrival. According to Nathan et al., such omissions "match information that is unstated in the text [of a problem statement] but necessary for complete understanding of the situation" (p. 334). The statement of this problem is very sensitive to the differences in the English language implied by "at 5.30" and "by 5.30".

Flawed Decision Making

The students in group 3 had more than adequate factual, algorithmic and procedural knowledge but they failed to review unsuccessful paths in their solution attempt. Tina and Sue wasted both knowledge and temporal resources because potentially useful directions were not fully exploited or were abandoned without any attempt to check for mechanical or logical errors. The time-line representation of the first 20 minutes (204 items) of their protocol is shown in Figure 4.

After a short episode of silent reading and a transitional verbal exchange about how to place the recording sheet, the girls began a two minute analysis and reading episode during which they chunked the problem into subgoals with their first being the compound probability that Lucy used the Corolla and arrived home at 5.30. Their first plan was to work the problem through using the Commodore and the Corolla and then compare these solutions as a check. After a lack of progress implementing this plan, they decided to stay with the Corolla and not waste time. They spent some time evaluating the utility of their approach but were hampered by a misreading of the question. They misread the phrase "by 5.30" as "at 5.30". They then made the legitimate interpretation (in line with this reading of the question) that arriving at 5.35 was not the same as not arriving at 5.30. This prevented their using the complement of the given conditions to calculate the conditional probability of being late with a particular car. Eventually, the inference the problem writer intended was taken after Sue introduced new information from her rereading of the problem that "she gets home at 5.35" meant Lucy definitely wasn't home by 5.30 (Items 95-97, shown as first triangle in Figure 4). However, they still persisted with the notion that the complement of not arriving at 5.30 was to be late or early. No attempt was made to resolve this

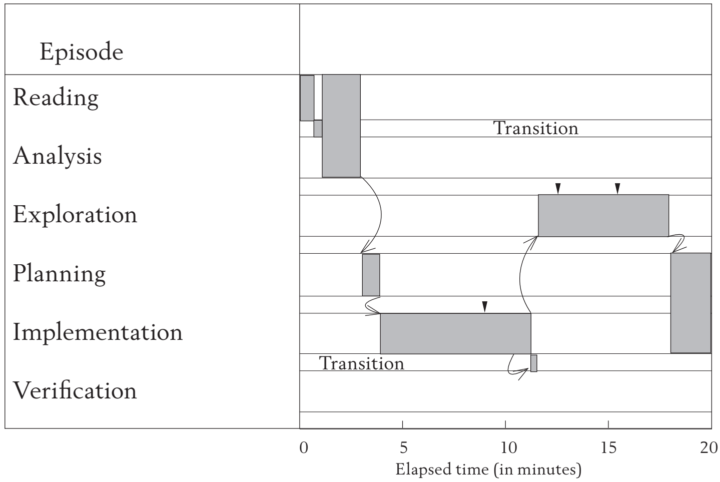

Figure 4. *A time-line representation of parsing of the protocol from group 3's problem solving session.*

conflict. After nearly 11 minutes, a result was produced for the subgoal but it appeared that they had been prepared to invest this time and effort when they did not know how the subgoal related to the solution path.

112 Tina: That was really pointless.

As they did not know where they were headed, it was decided to embark on an exploratory episode using Tina's suggestion (Item 113, second triangle in Figure 4) that they try the conditional probability theorem. Up until this point they had been prepared to use their conceptual understanding of probability rather than formal probability theory. Their reluctance to use formal theory appeared to be motivated by a belief that the problem had a simple solution. Two overt instances of the need to make decisions (shown as triangles in Figure 4) occurred during this episode. Firstly, they had to decide whether Sue's new information (Item 127) about the independence of events affecting the calculation of a conditional probability was relevant or even true. Secondly, during a local assessment (Items 164-172), they decided finding the probability Lucy is late and using the Corolla was of "some relevance but it is not on the main stream."

The subsequent short plannning/implementation episode involved finding the subgoal – the probability of being late. They did not just discard the value they had calculated for the previous episode, labelled "the time she uses Cor, and arrives home late (or early)", but they erroneously used its complement for "the probability of being late with the Corolla".

Inexplicably, they then carried out a parallel calculation to compute "the probability of being late with the Commodore" but did not take its complement. They calculated their result just as they ran out of the allotted time, continuing on for another 12 minutes. A quick local assessment revealed that the probability was much lower than expected. This plan was abandoned and more exploration and another plan and its implementation followed but they were very confused by then. In their final implementation episode they confused inverse probabilities thinking they were calculating P(Cor|late) but actually using P(late|Cor).

Although flawed control decisions were the main reason for the solution failing, initial misreading of the problem was costly in terms of time and this could have pressured them into bad control behaviour. A strategic review of their second plan could have given them the success they needed at that stage in the solution to implement their eventual plan of using the conditional probability theorem but it did not occur.

Group 4's attempt (Figure 5) is an example of "a wild goose chase" (Schoenfeld, 1985b). One might hypothesise from the meandering solution that Kay and Jo were just grasping at straws aimlessly trying whatever came into their heads in the hope that they might be lucky and somehow solve the problem. This supposes that neither student had enough background knowledge in probability theory to solve the problem. As was apparent throughout their solution attempt, they did have the mathematical knowledge necessary to produce a correct solution but they failed to use

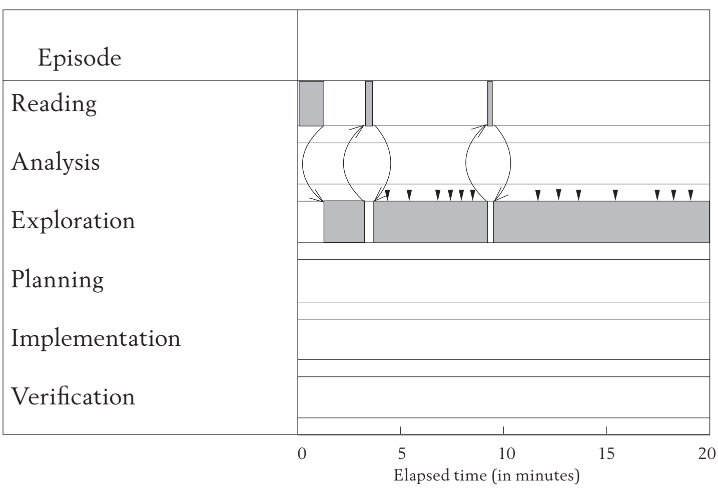

Figure 5. *A time-line representation of parsing of the protocol from group 4's problem solving session.*

it properly. The difficulty appeared to be that when they tried to apply the concepts of conditional probability and compound and independent events together with basic probability rules, they ignored the connections between the rules and the limiting conditions under which they operate.

Management behaviour (see Figure 5) was not lacking as there were three instances of new information to make decisions about, nine local assessments, and even a global assessment acknowledging they were "going around in circles" but flawed decisions resulted in wastage of resources, particularly time, and potentially useful directions were not fully exploited. Ultimately, the solution degenerated into confused explorations which were inconsistent and contrary to previously established facts.

A Controlled Efficient Attempt

Group 5's protocol totalling 164 items of dialogue showed signs of control (Figure 6) and the solution evolved in an efficient manner. Lea and Kate spent three and a half minutes reading and analysing the problem statement during which they elicited the given conditions but did not explicitly identify the required goal state. A brief exploratory episode followed revealing two important pieces of new information (Items 28 and 37). Firstly, they realised that the time was important and, secondly, Lea pointed out that the probabilities which they had called "P(Com – 5.30)" and "P(Cor – 5.30)" and thought were compound, were conditional. Although they used this latter piece of information in the subsequent planning/implementation episode, they did not make the connection that the required goal was not a conjunction but was also conditional. The exploration episode ended with the identification of their first subgoal as "the probability of not being home by 5.30". There was no assessment of the utility or the relevance of this subgoal. In their first planning/implementation episode the students embarked on this new direction which was relevant to the correct solution but was a diversion from what they decided was the required goal. After some unsuccessful attempts at finding this subgoal, they abruptly abandoned this path as they could not see its relevance.

73. Lea: I don't know why we are doing this. It's just that plus that, isn't it, the probability she can either get home by the Commodore or the Corolla?

They immediately pursued a new goal - the probability of using the Corolla and not being on time - during a second planning/implementation episode. They quickly calculated this correctly and then spent a few min-

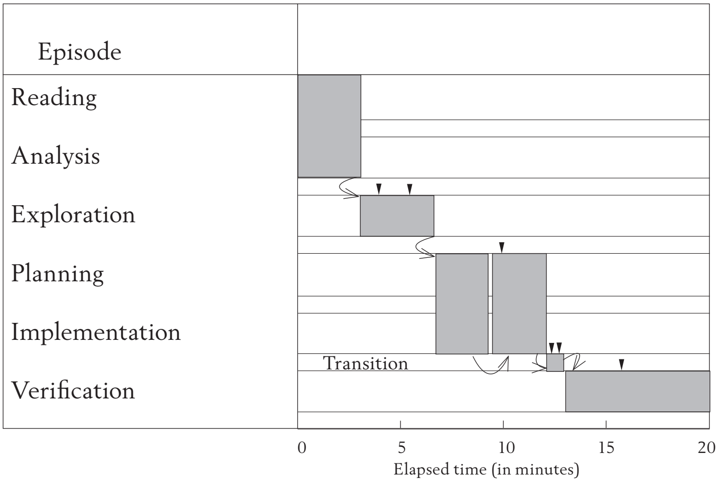

Figure 6. *A time-line representation of parsing of the protocol from group 5's problem solving session.*

utes accounting for all Lucy's time ensuring that all the probabilities involved summed to one using, at one point, Lea's new information that "using the Commodore AND not being on time" was not a conditional statement (Item 95, third triangle in Figure 6). Two local assessments (Items 105 and 111) followed during a transition conversation. As Lea and Kate could not think of another alternative, they set about trying to prove that this was the correct answer in a long verification episode. After Kate introduced the new information (Item 144) that "you've got to look at whether they're independent", some time was expended trying to show that P(on time|Cor) and P(on time) were independent which they discovered was not true but this had not affected their solution. They spent a lot of time trying to justify their answer and all their arguments were sound, however, they were pursuing the wrong goal as they had used a compound probability model rather than the necessary conditional probability model. The error in this solution is an incorrect problem representation rather than inadequate control behaviour.

Discussion

It is emphasised that this investigation was purely exploratory. When only a small number of detailed analyses are performed, it is possible to obtain a reasonably clear insight into the decision making and resource use of those students working in the particular circumstances described but it is difficult to get a sense of the typicality of this. The present investigation involved only female students as it was convenient for the teacher-researcher to conduct research in her own school. Furthermore, all the students performed well in the mathematical topic which was the basis for the problem. Findings may be different for boys and less well performing students. However, the study has largely supported the studies conducted by Schoenfeld with American college students using geometry problems. In terms of the research questions there appears to be support for the notion that students' control behaviour is a significant contributor to how the solution evolves but it is by no means the only significant contributor. Good quality monitoring and decision making afford students the opportunity of being successful and developing an efficient solution path. Although poor control behaviour and flawed decision making contributed to the failure of some solutions, other factors that were identified as contributors in the particular circumstances of the investigation were resource-related. These included (a) misinterpretation of the problem statement, (b) misreading of the problem statement, (c) failure to make the appropriate connections between probability rules and the conditions under which they operate, and (d) confusion of compound and conditional probability concepts. In keeping with the view of several authors in the literature review coding and encoding from English to mathematics played a pivotal role in success with this problem. However, critical interpretations like those required in this problem are, as noted by Nathan et al. (1992), not peculiar to conditional probability problems but are more indicative of the type of interpretative elaborations which draw on students' prior experiences that are necessary for successful solutions to highly contextualised applications problems (Stillman, 1998). Problem solvers separating the conditions of use of a rule or technique from its application has been found in a number of studies (e.g., Galbraith, Pemberton and Haines, 1996). Hiebert and Lefevre (1986) point out that "procedures...may or may not be learned with meaning" (p. 8). They propose that procedures that are learned meaningfully are those that are linked to their conceptual underpinnings. Furthermore, "building relationships between conceptual knowledge and the procedures of mathematics contributes to memory (storage and retrieval) of procedures and to their effective use" (p. 10). Thus, from a mathematics viewpoint it is important for related declarative (conceptual) knowledge to be activated at the same time as procedural knowledge. The confusion of compound and conditional probabilities is quite a common error in probability problems of this type (Pollatsek et al., 1987) and could be the result of either interpretative or conceptual difficulties. There was also some evidence of confusion of inverse probabilities despite wording the problem to reduce the effects of temporal order cues.

Problem solving behaviours that were revealed by the detailed analyses included the following:

1. All groups began by analysing the problem statement thoroughly in order to establish the given conditions and the required goal. Groups 2 and 5 misinterpreted the goal whilst group 4 did not close on the correct goal until about half way through the allocated time. It appears that all students in the study felt it was necessary to establish an accurate mental representation of the problem before beginning to select tactics.
2. Most groups made a rapid choice of tactics embarking on a particular solution path without delay. Some groups evaluated the potential of a solution path before beginning and wasting precious resources whilst others pushed ahead totally ignorant of the relevance of the particular path to the solution.
3. Students rarely made global assessments of progress. Once a solution attempt produced a result, it was assessed at the local level only. There were no attempts to compare current solution attempts with previous attempts.
4. Attempts that were judged incorrect were discarded almost completely with little being retained for the next attempt.
5. Verification was rarely present at all. This may indicate that students in the study believed verification was not part of the solution process or, more likely, they felt it was not expected of them at the time. This is consistent with Schoenfeld's (1987) finding that part of students' mathematical world view is a belief that "justifications aren't necessary, unless you're specifically asked for them - and that's only because you have to play the rules of the game" (p. 35).
6. The overall solution strategy was very much "event driven". Students often abandoned potentially successful paths because they found them in error instead of checking for the cause of error. All that was important was the knowledge that the present path had lead to an error so there was little sense in persisting with an apparently unsuccessful path whether or not it was relevant to the solution.

Conclusion

The central purpose of this paper has been to report on an exploratory investigation of students' decision making and resource use during the solving of a probabilty problem involving Bayes' Formula. The present study has supported Schoenfeld's (1985b) and Barkatsas and Hunting's (1996) contention that control behaviour is a significant contributor to the way a particular problem solution evolves; however, the sensitivity of the particular problem statement to the distinction between "by 5:30" or "at 5:30" was a central influence on the problem-solving process. The quality of control behaviour contributed to the eventual success or failure of the solution, but resource-related factors were also significant particularly those related to language use and domain specific knowledge such as the concepts of conditional and compound probability and the probability rules that connect them.

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This research was carried out when the author was a postgraduate student at the University of Queensland.

References

Adibnia, A., & Putt, I. J. (1998). Teaching problem solving to year 6 students: A new approach. *Mathematics Education Research Journal*, 10(3), 42-58.

Barkatsas, A.N., & Hunting, R. (1996). A review of recent research on cognitive, metacognitive and affective aspects of problem solving. *Nordisk Matematik Didaktik (Nordic Studies in Mathematics Education)*, 4(4), 7-30.

Falk, R. (1988). Conditional probabilities: insights and difficulties. In R. Davidson & J. Swift (Eds.), *Proceedings of the Second International Conference on Teaching Statistics*. Victoria, BC: University of Victoria.

Galbraith, P., Pemberton, M., & Haines, C. (1996). Testing to a purpose: assessing the mathematical knowledge of entering undergraduates. In P. Clarkson (Ed.), *Technology in mathematics education, Proceedings of Nineteenth Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 215-220). Melbourne: MERGA.

Geiger, V., & Galbraith, P. (1998). Developing a diagnostic framework for evaluating student approaches to applied mathematics problems. *International Journal of Mathematical Education in Science and Technology*, 29 (4), 533-559.

Gigerenzer, G. (1996). On narrow norms and vague heuristics: A reply to Kahneman and Tversky (1996). *Psychological Review*, 103(3), 592-596.

Gigerenzer, G., & Hoffrage, U. (1995). How to improve Bayesian reasoning without instruction: Frequency formats. *Psychological Review,* 102(4), 684-704.

Goos, M., & Galbraith, P. (1996). Do it this way! Metacognitive strategies in collaborative mathematical problem solving. *Educational Studies in Mathematics*, 30, 229-260.

Hiebert, J., & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: an introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics,* (pp. 1-27). Hillsdale, NJ: Lawrence Erlbaum.

Ichikawa, S. (1989). The role of isomorphic schematic representation in the comprehension of counter intuitive Bayesian problems. *Journal of Mathematical Behaviour,* 8, 269-281.

Kahneman, D., & Tversky, A. (1996). On the reality of cognitive illusions. *Psychological Review,* 103(3), 582-591.

Lester, F., Garofalo, J., & Kroll, D. (1989). *The role of metacognition in mathematical problem solving: A study of two grade seven classes*. Final report to the national Science Foundation of NSF project MDR85-50346.

Nathan, M. J., Kintsch, W. & Young, E. (1992). A theory of algebra-word-problem comprehension and its implications for the design of learning environments. *Cognition and Instruction*, 9(4), 329-389.

Peard, R. (1996). Problems with probability. In P. C. Clarkson (Ed.), *Proceedings of the 19th Annual Conference of the Mathematics Education Research Group of Australasia: Technology in Mathematics Education* (pp. 437-444). Melbourne: MERGA.

Pollatsek, A., Well, A. D., Konold, C., Hardiman, P., & Cobb, G. (1987). Understanding conditional probabilities. *Organizational Behaviour and Human Decision Processes*, 40, 255-269.

Schoenfeld, A. (1985a). Making sense of 'out loud' problem solving protocols. *The Journal of Mathematical Behaviour,* 4, 171-191.

Schoenfeld, A. (1985b). *Mathematical problem solving.* Orlando, FL: Academic Press.

Schoenfeld, A. (1987). Confessions of an accidental theorist. *For the Learning of Mathematics*, 7(1), 30-38.

Schoenfeld, A. (1989). Teaching mathematical thinking and problem solving. In L. B. Resnick & B. L. Klopfer (Eds.), *Toward the thinking curriculum: Current cognitive research* (pp. 83-103). Washington, DC: American Society for Curriculum Development.

Schoenfeld, A. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334-370). New York: Macmillan.

Scott, N. (1994). Profiles of some non-routine problem solving episodes. In G. Bell, B. Wright, N. Leeson & J. Geake (Eds.), *Challenges in Mathematics Education: Constraints on Construction, Proceedings of the Seventeenth Annual Conference of the Mathematics Education Research Group of Australasia, Vol. 2*, (pp. 531-538). Lismore: MERGA.

Smith, N., Wood, L. N., Gillies, R., & Perrett, G. (1994). Analysis of student performance in statistics. In G. Bell, B. Wright, N. Leeson, & J. Geake (Eds.), *Challenges in Mathematics Education: Constraints on Construction, Proceedings of the 17th Annual Conference of the Mathematics Education Research Group of Australasia, Vol. 2*, (pp. 539-543). Lismore: MERGA.

Stillman, G. (1998). Task context and applications at the senior secondary level. In C. Kanes, M. Goos, & E. Warren (Eds.), *Teaching Mathematics in New Times Proceedings of the Twenty First Annual Conference of the Mathematics Education Research Group of Australasia, Vol. 2*, (pp. 564-571). Gold Coast: MERGA.

Stillman, G. A., & Galbraith, P. L. (1998). Applying mathematics with real world connections: Metacognitive characteristics of secondary students. *Educational Studies in Mathematics*, 36(2), 157-195.

Tversky, A., & Kahneman, D. (1980). Causal schemes in judgements under uncertainty. In Fishbein, M. F. (Ed.), *Progress in social psychology.* Hillsdale, NJ: Erlbaum.

Walpole, R. E., & Myers, R.H. (1978). *Probability and statistics for engineers and scientists* (2nd ed.). New York: Collier Macmillan.

Wilson, J., & Clarke, D. (2004). Towards the modelling of mathematical metacognition. *Mathematics Education Research Journal*, 16(2), 25-48.

Notes

1 Students received ratings of Very High Achievement, High Achievement, Sound Achievement, Limited Achievement and Very Limited Achievement in Mathematics at the end of the two year senior program.

2 This problem is based on exercise 51 page 29 of Walpole and Meyers (1978).

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Sammanfattning

Denna studie behandlar beslutfattande hos elever på gymnasial nivå. Transkript från inspelningar där elever i par löser sannolikhetsproblem, som innefattar Bayes formel, har analyserats med hjälp av Schoenfelds *protocol parsing scheme*. Elevernas kontroll och beslutsfattande bidrog på ett signifikant sätt till hur lösningarna växte fram och det gjorde även specifika ord i problemformuleringen. Både kvaliteten på kontrollbeteendet och elevernas resurser bidrog till framgången eller misslyckandet med lösningen. De resursrelaterade faktorerna innefattade misstolkning av problemet, felläsning av problemet, misslyckande med att koppla ihop sannolikhetslagar och deras förutsättningar vid användningen, samt sammanblandning av sammansatt och betingad sannolikhet.