A logical model for interventions for students in mathematics difficulties

Improving professionalism and mathematical confidence

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This article describes elements in a Danish model for interventions in mathematics for students in mathematics difficulties. The authors have been core members of the intervention development team for the last decade. The aim of the interventions is to support the students and teachers involved and to be an instrument for municipalities and schools to improve mathematics culture. In the article, we start by sketching a couple of political level incentives and by outlining the pilot study and two large-scale experiments in which the model was implemented and expanded. We then present the research question that guides the article. In the main section, we present the logical model, which consists of nine boxes of inputs, processes and outputs. In order to illustrate viewpoints and ideas behind the boxes and their implementation, we have chosen to include some extracts from identification and teaching materials and some data collected through the pilot study and the experiments. We only address the many other existing intervention models to the extent that comparing characteristics helps to clarity characteristics in our own model. We conclude the article by claiming that an open standard to deal with students' mathematics difficulties, and which is based on high expectations for students, teachers and schools, has been developed.

Setting the Scene

In recent decades, there has been growing interest in a wide variety of issues regarding mathematics difficulties within both the research and practice of mathematics education. The most recent sign of this interest is the publication of three handbooks: Fritz et al. (2019), Kadosh and Dowker (2015), and Chinn (2015). The former handbook is entitled "From the lab to the classroom", which underlines the double interest in

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research and practice. This double interest is also characteristic of the Nordic research network on special needs education in mathematics, NORSMA, with its nine conferences to date. As part of this research network, we share this double interest.

The growing interest in mathematics difficulties includes attempts to design and explore interventions in schools for specific groups of students. For example, ten out of the 46 chapters in Fritz et al. (2019) are entitled ”Approaches to recognition and intervention”. We share the interest in designing and exploring interventions, and we aim to keep up-to-date with existing intervention models. However, our ambition is to broaden the scope of target groups and content compared with most other intervention programmes we know, which focus on numbers, numerals, number sense and arithmetic within pre-primary and primary education. We acknowledge that evidence shows that early number sense and arithmetic are essential for later mathematics learning in school and important for later educational and career success (see, among others, Aunio & Räsänen, 2016; Aunio, 2019); however, we suggest that focusing on broader areas of mathematics is also relevant to uplift for students in mathematics difficulties in order to become successful citizens in the current and future world. In addition, we suggest that focusing on broader areas of mathematics could provide students experiencing mathematics difficulties with access to learn mathematics and to develop mathematical motivation. It is also important to note that some intervention programmes use tutors who do not have formal teaching qualifications but who are trained in the specific programme. We prioritise the use of educated mathematics teachers with extra training, particularly in the programme, because it is our ambition to develop an open standard with room for teachers to activate and develop their professionalism.

In our opinion, it is important that ideas from programmes developed in other contexts and school cultures are adapted and further developed to suit the Danish school mathematics culture. For decades, we have seen promising approaches for teaching and learning mathematics develop in mathematics education, not only in relation to intervention. Our motivation has grown stronger to learn from such approaches and from intervention approaches in order to select, adapt and create approaches for interventions in a Danish model that can touch the target students' lives on both a systematic and (hopefully) sustainable level in mathematics classrooms across Denmark. In addition, we acknowledge Dowker's warning that, even when systematic research on programmes is conducted, it is ”likely that there will prove to be no single best intervention programme, and that different programmes would be suitable for different groups of children” (Dowker, 2009).

On a political level, we were motivated by the OECD's review of the Danish municipal primary and lower secondary school (Mortimore et al., 2004). This review identifies both weak and strong characteristics and presents a range of recommendations. One recommendation is to develop initiatives for improving basic understanding of numbers, where these initiatives

> do more than repeat the learning methods that have already shown to be inefficient for the target learners. We recommend that the National association of local authorities [in Danish: Kommunernes landsforening] reviews the programme for teacher in-service education in order to ensure that a sufficient number of teachers receive continuing education so that they are equipped to cater for pupils with special needs for ordinary special education.
>
> (Mortimore et al., 2004, p. 145; trans. by authors)

We were also motivated by a 2012 law called "Inclusion of students with special needs in ordinary teaching", which fundamentally changed education for students with disabilities and those facing learning difficulties. This law built on the view that, whenever possible, inclusive education should take place in regular classes in ordinary schools. *Ministry of children and education* in Denmark (2019) shows that the aim is

> [...] to retain the students in the children's community so that we do not separate children with special needs into special education services, but let them be taught in the regular class with the necessary support and aids. The goal of inclusion means that the students are part of the academic and social community, that there is a professional progression and that the well-being of the students is preserved. (trans. by authors)

One issue for school systems is to set up organisational support for inclusion and to clarify what constitutes inclusive teaching in general. In our view, it is quite another issue to clarify what constitutes inclusive subject-matter teaching. Scherer (2019) demonstrates the needs in Germany to clarify inclusive subject-matter-specific themes for mathematics, and we believe this need also applies to Denmark. Especially, the Mathematics recovery work since the 1990:s of R. J. Wright and colleagues (Wright & Ellemor-Collins, 2018) inspires us.

In 2007 Frederiksberg municipality became the first municipality in Denmark to finance the education of mathematics teachers to become so-called mathematics counsellors. The University College in Frederiksberg, now University College Copenhagen, provided education for two teachers from each school (Fokusaftale for 2010–2012). A year later, the

Frederiksberg municipality collaborated with Copenhagen University College and Aarhus University to start a two-year project for school interventions connected to teacher in-service training in order to support students in mathematics difficulties in grade 2 (8 years old), similar to the municipality's successful early interventions in reading. The project involved 35 intervention second grade students from 14 classes at 8 out of 9 schools in Frederiksberg. The 35 students were the two or three lowest performing students in the class, according to their teachers and municipality test results. The experiment did not include a formalised randomised controlled trial (RCT) design. Instead, the previous year's second grade students counted as a reference group. The experiment compared test result distribution from the experiment with the previous year's test result distribution. In addition, the experiment compared the individual student's standardised grade 2 test mark with his/her mark in grade 1. According to these two measures the students in this small-scale experiment at Frederiksberg showed a significant improvement in mathematical test performance.

Once the project received extra financial support[1], it turned into a genuine pilot study for developing what we term a Danish model for mathematics interventions (Lindenskov & Weng, 2013a).

Owing to promising results from the pilot study, the Egmont foundation decided to support further research into and communication about the model, and we established an RCT experiment. Other Danish studies had shown that high performing students might also experience mathematics difficulties, described as "falling into a mathematics hole", whereby they lose motivation and give up on mathematics from grade 4 onwards (Weng & Jankvist, 2018).

For this reason, the first large-scale RCT experiment (TMTM2014) included both low and high performing students in grade 2. The TMTM2014 experiment ran for two years with 39 voluntarily participating schools from 28 municipalities around the country with 2363 students in total. It is our impression that the teachers at each school also participated voluntarily. Among the 2363 students, 281 students participated in interventions, half of them low performing and half of them high performing. Students were randomly selected among their own school's 20 % lowest and 20 % highest performing students according to a special test developed for the intervention model. The students from the school's 20 % lowest and 20 % highest performing students who were not selected served as control groups. Results from pre- and post-tests showed significantly better results for intervention students than for control group students among the high performers, and better, although not significantly better, results for low performers.

As many schools and teachers expressed a need for similar initiatives for older students, we were invited to develop intervention materials for students and curricula in the final years of compulsory school (Lindenskov et al., 2016), which were structured in the same way as those for the first grades[2]. This provided additional opportunities for the second large-scale RCT experiment (TMTM 2017) that followed. Unlike in the previous experiment, the 103 participating schools in the second large-scale experiment did not request to take part but were appointed by the research team in order to ensure generalisability. At each school, four high performing and four low performing grade 2 students and four low performing grade 8 students were chosen for interventions with corresponding class peers as control groups. 247 mathematics teachers and 129 mathematics counsellors were involved together with each school's leader, 23 municipality consultants, and more than 1100 intervention students. Based on previous experience, this project introduced two supplementary features to the model implementation. The first was a two-layer coaching of teachers, and the second was that some interventions took place with small groups of students while others continued to use the one-teacher-one-student design. Results on students' learning in the experiment are published in Harder et al. (2020).

Besides data from the students' pre- and post-test, data was collected from a wide array of sources, from notes taken at teacher meetings, observations, interviews and a survey questionnaire. Only some of this data has already been analysed. It is our ambition to continue to analyse all this data with realistic evaluation studies inspired by Pawson and Tilley (1997). We aim to acquire a deeper understanding of not only the intervention outcomes but also for whom and in which circumstances the interventions produce certain outcomes. Tilley uses the formulation "how the (outcomes) are produced, and what is significant about the varying conditions in which the interventions take place" (Tilley, 2000).

It is against the background presented above that we pose the following research question in this article:

> Can a standard be developed in order to teach students in mathematics difficulties in a way that strengthens teacher professionalism, student participation and mathematics confidence, and which considerations are decisive?

We answer this question by presenting a logical model and by illustrating our theoretical thinking on the model's elements with excerpts from materials and collected data.

The logical model

Figure 1 shows a logical model[3] that illustrates our interpretation of the project's elements and mechanisms. The nine boxes represent the intervention theory, which consists of nine elements. The two arrows represent the programme theory, and symbolise the postulated causal mechanism for effect. In this article, we focus on the nine elements and illustrate them with various kinds of examples from materials and data. The nine elements are divided into three levels: inputs for the intervention in schools, intervention and follow-up in schools, and the outputs aimed for.

The *input level* includes three boxes. Box I1 is the framework and materials to be used by the teachers before and during the intervention sessions. Box I2 is teacher education, which primarily consists of teacher in-service courses but may also involve modules at teacher education. Box I3 is the identification of students.

The *process level* includes three boxes. Box P1 concerns the collaboration between teacher and student so that the teacher can explore the student's mathematics profile, including motivational and attitudinal aspects. Box P2 concerns their collaboration, which aims to enhance the student's learning and motivation. Box P3 concerns follow-up activities after the intervention.

The *output level* also includes three boxes. Box O1 concerns the student's development of learning strategies and motivation. Box O2 concerns the teacher's development of insight and motivation for

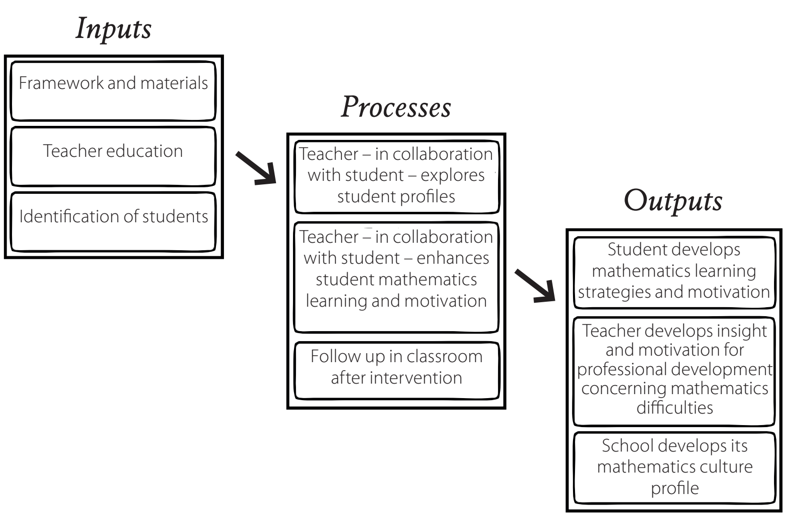

Figure 1. *The logical model*

professional development concerning mathematics difficulties. Box O3 concerns the school's development of its mathematics culture profile.

We underline, that the arrows in many practice situations will go both ways. Experienced teachers will give powerful feedback to the teacher-in-service course providers and adapt the input materials to their own students' needs and motivation. Schools will build upon previous experiences in repeated interventions and report problems and insights to course providers and developers.

BOX I1. Framework and materials

Framework

The framework broadly describes how the programme characterises mathematics, mathematics learning and mathematics difficulties.

Mathematics is characterised as the study of patterns with problem posing and problem solving as core elements. Mathematics competence is seen as highly significant for everybody both now and in the future. Mathematics learning is characterised as moving around in mathematical landscapes and participating in mathematical activities. Mathematics difficulties are characterised as being complex and including cognitive, affective and social aspects.

In Denmark, no official or professional consensus exists on how to delimit and conceptually understand mathematics difficulties. It seems that three theoretical dimensions are simultaneously at play, concerning individuals, mathematics, and how specific the difficulties are. Firstly, it is possible to view mathematics difficulties on a spectrum from being attached to specific individuals to being attached to specific relations between individuals and systems. Secondly, it is possible to view mathematics difficulties on a spectrum from covering all mathematics to covering specific mathematical concepts, procedures, or competences. Thirdly, it is possible to view mathematics difficulties on a spectrum from involving only mathematics learning to involving general learning difficulties or disabilities.

Different terms for students facing mathematics difficulties may have different connotations of medical and social justice perspectives, as stated in Scherer et al. (2017). These authors argue that understanding difficulties as deriving from failures inside the individual, as some medical models tend to, may result in deficit views. They argue that more relational views on mathematics difficulties may view all students as having the potential and motivation to learn maths. Among teachers in Denmark, the terms ”students with mathematics difficulties” and ”students in mathematics difficulties” seem to be used most frequently. The preposition ”with” may have a medical connotation. The preposition "in" may have a social justice connotation, identifying environmental factors in- and outside school as background factors. Often the "in" connotation concerns how didactical principles and tools delimit students' possibilities to learn. Some scholars suggest the views are conflicting and incompatible; for instance, Schmidt (2016) found that a context-oriented rationale is currently competing with a rationale that emphasises individual causal explanations (though the former is dominant).

We question whether understanding these views as incompatible limits the range of initiatives available to help students' everyday life in school, since we intend to contribute to the support of all students, irrespective of whether or not they are diagnosed. We are therefore keen to find a term that includes both the "in" and the "with" connotations and to avoid choosing one of them. In line with Gervasoni and Lindenskov (2011), we could use the term "students who are vulnerable in mathematics" to cover students who lack sufficient opportunities to thrive in mathematics and therefore have special rights to quality mathematics education. With this expression, it is access to quality mathematics education that is seen as the way to solve or cope with the mathematics difficulties, irrespective of their cause.

However, in Danish, it is not common to use the term vulnerability in this context. For this reason, in this article, we use the term "students in difficulties" for the model's target students, and we extend the term to include what is normally meant by "students with difficulties". In addition, we use the metaphor of mathematics hole to capture the idea that experiencing mathematics difficulties is like falling into a mathematics hole whilst moving around in mathematical landscapes and participating in mathematical activities. The mathematics hole metaphor does not identify specific causes but it identifies school mathematics as the main medium to cause and to cope with the difficulties (Lindenskov, 2006). The metaphor of the mathematics hole is optimistic, since it also provides three different approaches to cope with mathematics difficulties. The first approach is to fill up the hole. This equates to a traditional approach to mathematics difficulties based on the idea that students should learn what they have not previously learned in order to progress. The second approach is to build a bridge over the hole – for example, by using another, non-traditional tool or approach – in order to compensate for mathematics difficulties. The third approach is to temporarily move to other areas of the mathematical landscape so that the mathematics hole does not limit the students' further mathematical investigations.

According to the framework, irrespective of the cause of students' mathematics difficulties, every student has the right to participate in quality mathematics education. We actually have no documentation to confirm the number of participating students with diagnoses, but we know that some students had official diagnoses, such as memory and awareness problems. According to the framework, despite how it may seem in the everyday classroom, all children can learn and enjoy mathematical phenomena and ideas.

Materials

Generally speaking, intervention materials present a structured approach to support students in mathematics difficulties, as underscored in the intervention material (Lindenskov et al., 2016, p. 6). The structured approach emerges at three levels: the level of mathematical content, the level of communication, and the level of system collaboration.

At the level of mathematical content is a structure with a number of areas. Instead of using mathematical disciplines, content and processes as topics, each title points towards specific aims for students' learning by using the terms knowledge, perception, strategies and understanding. The ten chapter titles in the intervention materials for the first school years include (Lindenskov & Weng, 2013b):

- Knowledge of numbers in their interrelationship.
- Basic strategies with numbers in multiplication and division.
- Basic descriptions and terms related to geometrical forms and figures.

The materials includes one chapter for each area. Each chapter follows the same structure. Page 1 includes the goal of exploring the student's profile. For instance, "The goal is that the teacher finds the student's prerequisites and motivation for developing strategies with numbers in multiplication and division." This expands into six questions that focus the teacher's exploration on key aspects of the student's knowledge, skills and attitudes. Page 2 (approximately 400 words) presents arguments for how the six key aspects of the area are significant for the student's further learning and life. Some hints are given to theoretical and empirical backgrounds for the six questions and how they can be used.

At the level of communication, on pages 3–14, there are two pages for each focus question. These pages include structured questions and tasks, which are sufficiently open problems to provide the teacher with relevant insight into the student's conceptual and processual prerequisites, mathematics holes and motivation. This is an example task:

> You would like to bake chocolate cake for the whole school. Please give me some ideas on how to find out how much cake you need to bake.

Such tasks have the potential to allow the teacher to see the strategies the students employ when they choose which measuring units to use, when they are invited to investigative work in what Ole Skovsmose calls landscapes of investigation with references to a so-called semi-reality (Skovsmose, 2001).

At the level of system collaboration, the material recommends collaboration with people around the students in order to review resource contexts and resource individuals that can contribute to the student's development. The preliminary conversations aim to provide insights into which neurological, psychological, sociological and didactical elements may play a role for the student in mathematics difficulties (Engström, 2000). In the material, it is recognised as essential that, during the intervention, the collaboration around the student is viewed as a social whole in which the strengthening of the student's motivation to use mathematics in investigative activities is central.

BOX I2. The teacher training course

The teacher training course comprises five full days. In experiments TMTM2014 and TMTM2017, these five days took place during one working week in Copenhagen. All the participating teachers attended this course. The course introduces teachers to the framework and to core literature from mathematics education that underpins the framework. This is organised in a way that respects the teachers' experiences and perceptions of what kind of theoretical underpinnings will be relevant to discuss (Lindenskov & Kirsted, 2017).

Specific parts of the materials are in focus during the course, where the teachers try out and discuss parts of the intervention material. Experiences from earlier interventions are presented and discussed among the teachers. In addition, the teachers' opportunities to exchange experiences, ideas and worries are prioritised. The teachers engage in discussions on excerpts from the material. For instance:

> How should you interpret the question "Should I multiply or divide?" from a lower secondary student working with contextualised tasks? This question does not necessarily result from a lack of motivation, concentration or subject reading skills. A mathematics education interpretation urges you to explore the student's knowledge and interest in questions and situations suitable for applying multiplication and division.

The teachers discuss the relevance of focus questions, and they are presented with experiences from previous interventions. For example:

- How does the student divide a set of maximum twenty objects into equally sized groups?
- How does the student decide the total number of objects in a number of equally sized groups?
- What is the student's auditive understanding of division like? (meaning when the student hears spoken words, numerals and symbols)
- Which number patterns is the student aware of in multiplication table matrixes?

BOX I3. Identification of students to be offered intervention

The intervention is designed to be suitable for every student who, according to their mathematics teacher's and mathematics counsellor's observations, is experiencing – or is at risk of developing – lasting mathematics difficulties concerning specific areas of the mathematics landscape. To assist teachers and counsellors to identify students, the materials include screening tests with some questions about the student's view on mathematics in school and some tasks to be answered in a conversation between student and teacher or counsellor.

BOX P1. Exploring student's profiles

The materials include guides (six- or seven pages) for teachers to interview the student in an uncovering conversation. Information drawn from the screening test and interview serves to qualify the teacher's decision on which parts of the materials to start using and how to adapt the material to the specific student's potentials and motivations.

During the student's work on the task, the teacher observes the student and communicates with the student in order to acquire some preliminary ideas about how and with which approaches the student approaches the problem. The teacher reflects on questions such as: How does the student bring forth more or less systematically the information needed in order to answer the question? Which data, facts, and guesses does the student provide as already acquired knowledge? How does the student provide parameters that he/she sees as necessary to answer the question but does not yet know?

As mentioned above, the materials include two pages for each of the six focus questions. The first of the two pages presents activities and tasks to retrieve in-depth information on the student's knowledge, skills and attitudes related to the focus question. It includes activities and tasks that lead to student actions and teacher-driven conversation. In addition, it includes activities and tasks that lead to the student handling a problem formulated as a real-world story and open teacher-student conversation, where the teacher asks about the student's view on mathematics-related phenomena. It may also involve the student creating something; for example, a pattern. The teacher adapts the material's context, numerals and complexity level to the individual student whenever he/she deems it relevant.

Through the teacher-student conversation, the teacher elicits information to answer the focus question and to choose among and adapt the ideas presented on the second of the two pages, which concentrate on how to support the student to develop knowledge, skills and attitudes.

BOX P2. Enhancing student's mathematical learning and motivation

Again, it is the teacher-student conversations based on the second of the two pages that potentially enhances the student's mathematical learning and motivation. To exemplify this point, let us consider the following excerpts from the student's auditive understanding of multiplication:

> The teacher talks to the student about different multiplication signs. The teacher asks the student to explain or examine what the sign looks like, for example, when writing on a calculator, on a computer and on a mobile phone.
>
> The teacher talks to the student about what makes a task easy and difficult and about when to multiply.
>
> Together, the teacher and student find as many everyday examples as possible.
>
> The teacher says: Sophie has walked to school and home again for 189 days. She lives 4 kilometres from the school. How far do you think Sophie walked – to the school and home again – in a year?
>
> The teacher asks the student to tell some stories that suit certain symbolic expressions.

Ideas to choose among and adapt for the sake of further learning include the following:

> In the conversation, the number sizes may expand, and the context may be more complex. For some stories, you may just talk about which operation to choose.

> The teacher can ask the student to pose three difficult multiplication tasks to be solved by the teacher. The student is not obliged to know the answer beforehand.
>
> The teacher can ask the student to construct a story that suits 17 x 14 = 238.
>
> If the student finds it difficult to work from auditive input alone, the teacher provides the opportunity to use other input types. You may let the student use manipulatives, draw on paper, or use a calculator or an IPAD. For instance, you can invite the student to draw specific rectangles on quadratic paper and to construct similar physical rectangles with centicubes. You may also invite the student to use calculator.

BOX P3. Follow-up in the classroom after the intervention

The student's ordinary mathematics teacher plays a key role in the decision to include a student in the intervention, so he/she is fully aware of its aims and processes. During the intervention period, the student also participates in ordinary mathematics classes. This gives the student's ordinary mathematics teacher the chance to observe and encourage the student. During and after the intervention period, the intervention teacher and the mathematics teacher communicate, and the intervention teacher provides information and recommendations on how the mathematics teacher can encourage the student to participate more fully than before in the classroom community. Particular attention is paid to the student's motivation and mathematics self-confidence. With this in mind, other people around the student may be invited to support the student.

BOX O1. Developing mathematics learning strategies and motivation

Data (collected in Danish, translated by the authors) from interviews with 23 students who participated in TMTM2014 shows that the following citations are typical when the students are asked to compare the intervention with their ordinary mathematics teaching:

> I got help from the teacher.
>
> Mathematics is getting easier, because I have learnt several more ways to do mathematics.
>
> I don't really remember.

Another illustration of how the intervention may effect students comes from telephone interview (March, 2013), which was conducted after the pilot study. In this interview, the mother of one of the participating students claims:

> Before the intervention, my daughter, like her elder sibling, said she found mathematics difficult. Sometimes she felt she did not understand anything.
>
> She got her self-confidence boosted. There was no feeling of being different or special, and she perceived it as support.
>
> [When asked about possible ways to improve the intervention] We had a very positive experience. We did worry about mathematics, but my daughter is happy now. Maybe I could suggest having a meeting with the parents and teachers before and after the intervention.
>
> [When asked for a final remark]. Really, it was a success with one motivation after another. This is important, because, if you give up, you cannot learn. I hope the state primary and lower secondary school will be able to provide such offers in the future.

BOX O2. Teacher develops insight and motivation

It is at the core of the model that the teacher's systematic engagement with the student unfolds in both affective and cognitive ways. In addition, the model invites the teachers to be flexible and adapt to each student. This is because, with this intervention, we aim to offer the teachers circumstances in which they can develop their insight into the students experiencing mathematics difficulties today and into how various tasks and teacher behaviours affect student learning and motivation.

By setting up frames for collaboration between teacher and student and by letting the teachers experience their own use of the frames, we invite the teachers to develop. By encouraging the teacher to adapt materials to the specific student, we invite the teacher to supplement the student interview guide, rearrange the order of chapters, rearrange the order of tasks included in a chapter, re-formulate tasks and reflect upon what the adaptations meant to the students.

The Danish scholar Christa Amhøj has observed TMTM interventions and interviewed TMTM intervention teachers. She claims:

> The TMTM model offers standards that give the teacher power to positively affect the student and to be positively affected by the students (in Danish: magt til at berøre eleverne og til at lade sig berøre af eleverne). The model builds on teaching standards that mobilise the teachers' intuition, feelings and social sensibility with a relatively detailed standard for teaching that is based on the individual student and involves space, body and consciousness.
>
> (Personal communication; see more in Amhøj, 2019).

The model has also been implemented in a shorter form at University College Copenhagen for those wishing to qualify as teachers in primary and lower secondary school, which has given rise to further insights and motivations. In 2019, the module "TMTM – *Early mathematics intervention for marginal groups*" equated to 10 ECTS points[4]. The teacher students, individually or in pairs, conducted a four-lesson intervention with one student or a small group of students in primary or lower secondary school. The teacher students videotaped episodes that they later analysed in a written report.

In experiment TMTM2017 interviews with eleven intervention teachers were conducted, after carrying out interventions in grade 2. The eleven teachers were chosen to include teachers with low performing and high performing students, and to include schools where the measured effects on student performance were relatively high and relatively low.

The interviews illustrated Gibbons and Cobb's (2017) characteristics for the high-quality professional development of mathematics teachers. The first characteristic is sustained professional development. It is important that teachers continually explore particular aspects of the framework and the materials, and have the opportunity to try ideas in the classroom and reflect on the results.

The second characteristic is that high-quality professional learning activities focus on the problems that teachers encounter in their daily work. It is important that conversations with the students give the teachers opportunities to train skills that they themselves feel are required in their ordinary mathematics teaching.

A third and related characteristic is to orient teachers to attend to their students' thinking in order to improve the teacher's ability to elicit and build on this thinking. It is important to recognise that this may very well be a challenge for the teachers.

BOX O3. School develops its mathematics culture profile

We can illustrate the challenges for school development with data from a reflection seminar that took place in May 2013 (the second in 2013), after the pilot study. The seven mathematics counsellors and two municipality consultants who took part in the seminar (all of whom were involved in the study) identified six challenges: sustainability, certification of trained teachers, avoiding the school's use of un-trained substitutes, the need for teacher networks, the need for material updates, and the resource competition with early reading intervention, which has already established legitimacy and support from the municipality level.

In our view, the six challenges are still evident. However, the reflections show the teachers are keen to transform – and not directly implement – when they use the model in their practice. The term transforming is advocated by Holen et al. (2019). We find that the teachers' ambitions to transform are in line with the ambitions in the framework and materials.

In addition, Gibbons and Cobb's (2017) fourth element of high-quality professional learning, which fosters the development of teacher communities with discourses that identify critical aspects and support teachers in taking necessary risks, is illuminated by the following data from teacher interviews:

> I wish to have colleagues who could say "I tried this and this. Let me show you".
>
> A mathematics counsellor said: "We wish to share the ideas with colleagues. However, we are not yet sufficiently competent in the framework and its implementation".
>
> A teacher said: "My colleagues know that I am participating in the project. However, I do not feel prepared to inform them in any detail".

Gibbons and Cobb's (2017) fifth characteristic is that high-quality professional learning provides opportunities to both investigate and enact specific pedagogical routines and practices. This is illustrated by teacher interviews:

> A teacher said: "I feel you can utilise your colleagues. It is an advantage that three teachers at our school are participating and have the opportunity to discuss and share experiences. You can say, 'Look, I have got this idea, can we elaborate it together?' ".
>
> A teacher said: "I feel we are lucky to have a school leader who listens to us and who prioritises mathematics when we present ideas that we believe are effective and that we have experience with".

Conclusion

We conclude that an open standard for coping with mathematics difficulties has been developed in order to strengthen student participation and mathematical confidence as a human right and to strengthen teachers' professionalism. However, this standard was not developed overnight but over a period of 10 years. It has emerged and been expanded through the active participation of students, teachers, schools and municipalities. The decisive considerations involve the framework thesis that irrespective of the cause of students' mathematics difficulties, every student has the right to participate in quality mathematics education; the structured approach of intervention materials concerning mathematical content, communication, and system collaboration; and teacher education with theories underpinning the model, trying outs and discussions. Also, schools must engage in addressing awareness and resources to mathematics difficulties; teachers must engage in exploring the students' difficulties and potentials in order to let them guide the teachers in selecting effective teaching content and methods; and students must allow themselves to be affected and inspired by the teacher and the mathematics.

There is much more is to be investigated and communicated about the model and its use in practice. Although many teachers have expressed great satisfaction with the standard, it is still relevant to critically explore its short- and long-term effects. Future scientific articles will present effects on students' mathematics learning from the conducted RCT studies. In addition, results from ongoing realistic evaluation studies will be published concerning change-inducing mechanisms that are triggered by the intervention and illustrated by arrows in the logical model. Results on effects and preliminary results on mechanisms are only communicated in reports (Lindenskov, 2014; Tonnesen et al., 2016; Harder et al., 2020).

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Notes

1 The private Egmont foundation gave financial support.

2 In addition, many schools and teachers expressed a need for similar initiatives for students in the middle years of compulsory school, so we developed a third material (Lindenskov & Weng, 2020).

3 We use the term ”logical model” to include what is often described as ”logic model” depicting the various resources, inputs, outputs, etc.. Our term also partly include what is often termed ”programmatic theory” mapping out causal chains, including mediating and moderating variables that link action to intended outcome See for instance Munter, et al. (2016).

4 www.phmetropol.dk/uddannelser/laerer/uddannelsen/studieordning

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