From Galois to Riemann to Grothendieck

Dennis Eriksson

XXX

1 *Introduction*

In this overview we will focus on the theory of coverings of topological spaces and some extensions in algebraic geometry and number theory. Galois theory is in its essense the theory of correspondance between symmetry groups of field extensions and the field extensions, providing a link between group theory and field theory. Coverings of topological spaces are provided with the same type of interpretation. Here a covering of a topological space X is basically a topological space with a map $Y \to X$ such that Y and X "look similar" locally. The Galois theory of coverings will be a correspondance between symmetries of such covers and the fundamental group, the latter playing the role of the Galois group, and we recall this in the first section.

In the next two sections we consider a beautiful correspondance due to Riemann (in the case of curves) and Serre provides a strong link between these topological covers and covers which are "algebraic covers". One of its most potent applications is to the theory of algebraic curves, providing a concrete relationship between topological covers and field extensions of $\mathbb{C}(z)$.

If one considers in particular the case of coverings of the sphere with three critical points, this somehow incredibly sets up a correspondance between algebraic curves defined over number fields and topological covers. These covers can moreover be easily realized by simple drawings. So simple, that essentially any drawing without lifting the pen by a child gives an example, and Alexander Grothendieck baptized them "dessins d'enfants" (french: children's drawings). These provide a way to encode information on the Galois group of the rational numbers in terms of combinatorial data.

All in all, these notes provides some intriguing connections between more or less classical topology and complex analysis to much more modern developments in algebraic and arithmetic geometry, which provide new ways to look at the Galois group of $\mathbb{Q}$.

2 *Fundamental groups and topological Galois coverings*

Let $(X, \bullet)$ be any pointed topological space. Recall that the fundamental group of $(X, \bullet)$ is the group of loops starting and ending at $\bullet$, up to continuous deformation.

It is denoted $\pi_1(X, \bullet) = \pi_1(X)$. The group structure is given by the obvious composition of loops. In what follows we will only consider well-behaved topological spaces, to avoid pathologies, namely CW-complexes or topological manifolds.

2.1 Alternative description in terms of coverings

We will give a second standard characterization of the fundamental group in terms of coverings. Recall that, given a topological space X, a covering Y of X is a topological space Y, with a map $f : Y \to X$, such that for every point $p \in X$, there is a neighborhood U_p of p and a set T together with a commutative diagram

$$\begin{array}{ccc} f^{-1}(U_p) & \longrightarrow & T \times U_p \\ & {\scriptstyle f}\searrow \quad \swarrow & \\ & U_p & \end{array} .$$

Intuitively, locally around each point p, the inverse image of f are a number of copies of X indexed by the set T. The covering is said to be trivial if we can take $U_p = X$.

Example 1. Consider the circle $S^1 = \{z \in \mathbb{C}, |z| = 1\}$. The map $z \mapsto z^n$ is a covering map of the circle with itself, with the set T being the cyclic group $\mathbb{Z}/n$. Another covering is given by $f : \mathbb{R} \to S^1, f(t) = \exp(2\pi i t)$.

Example 2. If X is simply connected, i.e. $\pi_1(X, x) = 0$, then any covering of X is necessarily trivial.

Any topological space admits a universal covering space. This is a connected topological space $\widetilde{X}$ with a covering $p : \widetilde{X} \to X$, such that any other connected covering $f : Y \to X$, there is a covering $g : \widetilde{X} \to Y$ such that $gf = p$. It will be unique in the following sense. If we fix a point $x \in X$, and for every connected covering $f : Y \to X$ a point $y \in Y$ such that $f(y) = x$, then for the universal covering $p : (\widetilde{X}, \widetilde{x}) \to (X, x)$, and a covering $g : (Y, y) \to (X, x)$, the map $f : (\widetilde{X}, \widetilde{x}) \to (Y, y)$ is the unique one such that $f(\widetilde{x}) = y$.

Example 3. Consider the group of rotations in three-dimensional Euclidean space $\mathbb{R}^3$, $\mathrm{SO}(3)$. This group is $\mathbb{R}P^3$, which can be identified with the 3-sphere S^3 where antipodal points are identified. Since $\pi_1(S^3) = 0$, this realizes S^3 as a universal cover of $\mathrm{SO}(3)$, which is moreover a double cover. One concludes that $\pi_1(\mathrm{SO}(3)) = \mathbb{Z}/(2)$.

For any covering $f : (Y, y) \to (X, x)$, the group of deck transformations $\mathrm{Aut}(f)$ is the group of automorphisms of Y, preserving y and commuting with the map to X. Locally it means that the various covers are permuted. In particular, deck transformations induces automorphisms of the fiber $f^{-1}(x)$ and an inclusion $\mathrm{Aut}(f) \subseteq \mathrm{Aut}(f^{-1}(x))$.

Definition 2.0.1. We say that a covering $f : (Y, y) \to (X, x)$ is a Galois covering, if Y is connected and $G = \mathrm{Aut}(f)$ acts transitively on $f^{-1}(x)$ (and thus transitively on any fiber). Equivalently, if

$$Y \times_X Y = \{(z, z') \in Y \times Y, f(z) = f(z')\},$$

then the map

$$G \times Y \to Y \times_X Y$$

given by $(g, z) \mapsto (z, gz)$ is an homeomorphism.

The fundamental group also acts on $f^{-1}(x)$. For a path $\gamma : [0, 1] \to (X, x), \gamma(0) = \gamma(1) = x$, there is a unique path lifting $\gamma' : [0, 1] \to (Y, y), \gamma(0) = y$. Notice that we do not require $\gamma'(1) = y$, but that necessarily $\gamma'(1) \in f^{-1}(x)$. It turns out that $\gamma'(1)$ only depends on the homotopy class of γ, and we thus obtain two actions on $f^{-1}(x)$. One is by deck transformations, the other is by the fundamental group. For $f \in \mathrm{Aut}(f)$ and $\gamma \in \pi_1(X, x)$, they are related by $f.(c.\gamma) = (f.c).\gamma$. This means that there is a map $\pi_1(X, x) \to \mathrm{Aut}(f)$. **Note to editor: Innocuous lie, since the two groups act on the left and on the right, the identification is rather between the opposite group of π_1 and the group of deck transformations. But the opposite group G^{op} and G are naturally isomorphic through $x \mapsto x^{-1}$.**

Theorem 2.1 (Fundamental theorem of Galois theory for topological spaces)**.** *If the covering is moreover a universal covering space $p : (\widetilde{X}, \widetilde{x}) \to (X, x)$, there is an isomorphism between the fundamental group and the group of deck transformations:*

$$\pi_1(X, x) \to \mathrm{Aut}(p).$$

Moreover, there is a correspondance between subgroups of $\pi_1(X, x)$ and coverings of (X, x). The normal subgroups correspond to Galois covers. Equivalently, Galois coverings with group G correspond to surjective homomorphisms $\pi(X, x) \to G$.

Example 4. In Example 1 we have $\pi_1(S^1) = \mathbb{Z}$. There are three types of subgroup of $\mathbb{Z}$. First of all, there is the whole group, this corresponds to the trivial cover with the identity map. Secondly, there are the groups generated by an integer $n \neq 0, \pm 1$, which corresponds to the covers with cyclic group $\mathbb{Z}/n$. Finally, there is the 0-group, which corresponds to the universal cover $\mathbb{R} \to S^1$.

Example 5. The fundamental group of a topological space X not always easy to compute. A more tractable object is usually its abelianization, which is isomorphic to the first homology group of X. Since by the above theorem Galois coverings of a topological space with abelian group of deck transformations G corresponds to a surjection $\pi_1(X) \to G$, it must necessarily factor through $\pi_1(X) \to \pi_1(X)^{ab} = H_1(X) \to G$.

3 Analytical covers of varieties

In this section we will consider covers with structure additional to being continuous. Part of the reason for introducing this is that many interesting covers actually come equipped with this or that additional structure.
A map between complex analytic varieties is said to be a holomorphic cover if in the definition of cover we replace continuous with holomorphic. It is not hard to see that if X is a complex analytic space, and $Y \to X$ is a topological covering, there is a unique complex analytic structure, using that the two spaces locally look the same, on Y such that Y is complex analytic and $Y \to X$ is a complex analytic covering.

Example 6. Suppose $\Omega, \Omega' \subseteq \mathbb{C}$ are open subsets. An analytical covering $f : \Omega' \to \Omega$ is a holomorphic function f such that $f'(z) \neq 0$ for all $z \in \Omega'$. This follows from the inverse function theorem, which states that under this assumption, f admits a local holomorphic inverse around z. The same example works in higher dimensions, given that we instead use the condition that the determinant of the Jacobian is non-zero $\det f \neq 0$. The map $z \mapsto z^n$ is thus a analytic covering outside of $z = 0$.

Typical complex varieties are the affine varieties. These are realized as the zero-set of a set of polynomials $f_i \in \mathbb{C}[z_1, \ldots, z_n]$. Most of the time one is interested in certain compactifications of these. Recall the complex projective space $\mathbb{P}^n_{\mathbb{C}}$, a certain natural compactification of $\mathbb{C}^n$. It is the space of $n+1$-tuples $(z_0, \ldots, z_n) \neq 0$, modulo the identification $(z_0, \ldots, z_n) \simeq \lambda(z_0, \ldots, z_n)$ for any $\lambda \in \mathbb{C}^\times$. The coordinates are written $[Z_0 : Z_1 : \ldots : Z_n]$, and $\mathbb{C}^n$ is naturally a subset via the inclusion $(z_1, \ldots, z_n) \mapsto [1 : z_1 : \ldots : z_n]$. A polynomial $F \in \mathbb{C}[x_0, \ldots, x_n]$ is homogenous of degree d if $F(\lambda x) = \lambda^d F(x)$ for any $x \in \mathbb{C}^{n+1}, \lambda \in \mathbb{C}$. A projective variety is a variety defined as the set of zeros of a set of homogenous polynomials. An algebraic map of projective varieties is a holomorphic map which is describable by polynomials.

Example 7. The projective line $\mathbb{P}^1$ is homeomorphic to the sphere, S^2, and is the one-point compactification of $\mathbb{C}$. It is the unique complex and algebraic structure on S^2, up to biholomorphism. An algebraic map from $\mathbb{P}^1$ to $\mathbb{P}^1$ is given by $[X : Y] \mapsto [F_0(X,Y) : F_1(X,Y)]$, where F_i are homogenous polynomials of the same degree without common zeros. Equivalently, it is given by a rational function f_0/f_1 where f_0 and f_1 are polynomials. Since the fundamental group of the sphere is trivial, these can never be topological covers, unless it is a trivial cover.

Example 8. It is not true that two analytical or algebraic structures on the same topological object give rise are always equivalent. Consider the plane curve given by the polynomial $ZY^2 = X^3 + aZ^2X + Z^3b$, such that $4a^3 + 27b^2 \neq 0$. These are all homeomorphic to a donut, and it is a fact that two such curves are algebraically equivalent, or biholomorphic, if and only if they have the same j-invariant $j(a,b)$, where

$$j(a,b) = \frac{a^3}{4a^3 + 27b^2}$$

A beautiful theorem of Serre, usually known as part of the GAGA-principle (Géométrie Analytique et Géométrie Algèbrique, [6]), states that if X is a complex projective manifold, then any analytical covering $F : Y \to X$ with finite fibers is algebraic. This means, that Y itself is a projective variety, and that $F : Y \to X$ can be described by polynomial equations. This has the following consequence. If X is a complex projective variety, and $Y \to X$ is a finite topological cover, then Y can be equipped with an analytical cover and then by GAGA an algebraic structure, and $Y \to X$ must necessarily be algebraic. This statement is false if we instead suppose that the covering is infinte. For example, the Fermat curve

$$X^d + Y^d = Z^d$$

is, for $d > 1$, a projective variety of (complex) dimension one, with universal covering space being the upper half plane, which is not algebraic.

4 Algebraic Curves and Riemann Surfaces

The most basic examples of varieties are the compact varieties of complex dimension one. Since they have real dimension two, they have also been called Riemann surfaces. In what follows, I use the term "algebraic curve" to emphasize the algebraic nature, and the term "Riemann surface" to emphasize the analytical structure. They were first introduced by Bernhard Riemann in the second half of the 19th century, and are still a topic of study. One of their main features is that they tie together topology, complex analysis and algebraic geometry in a fascinating way. They also provide testing grounds for higher dimensional questions, but also contain open problems, in topology, complex analysis, algebraic geometry and number theory. The topological classification is however simple: they are all homeomorphic to a donut with a number of holes. This number is the genus of the curves. They can however have various different complex structures which distinguish them.
The simplest Riemann surface is the (unique) Riemann surface on genus 0, $\mathbb{P}^1$, homeomorphic to the 2-sphere, S^2. It is realized as the complex plane $\mathbb{C}$ with a single "infinite" point, so that $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. The genus one Riemann surfaces are all tori, and can be viewed as parallelograms in the complex plane, where the opposite sides are identified. Setting one side to be the line between 0 and 1, the parallelogram is determined by a single τ in the upper half plane, and in general different τ define different Riemann surfaces of genus one. Also see Example 8 in the previous section.

Example 9. The planar curves of degree d are described by the set of zeros of homogenous polynomials in three variables

$$F = \sum_{i+j+k=d} a_{ijk} X^i Y^j Z^k.$$

If their partial deriviatives do not simultaneously vanish at a point $\mathbb{C}^3 \setminus 0$, these define compact Riemann surfaces of genus $g = (d-1)(d-2)/2$.

It is a fact that actually all compact Riemann surfaces are algebraic, i.e. given by a set of equations in projective space. The field of functions of a Riemann surface C,

$\mathbb{C}(C)$ is the field determined by all meromorphic functions $C \to \mathbb{C}$. The algebraicity of compact Riemann surfaces implies that a map $f : C' \to C$ is equivalent to a map $f^* : \mathbb{C}(C) \to \mathbb{C}(C')$, where the obvious implication associates to a meromorphic function $g : C \to \mathbb{C}$ the meromorphic function $f^*g = g \circ f : C' \to C \to \mathbb{C}$. Also, since they are algebraic curves, $\mathbb{C}(C)$ must be a finite extension of $\mathbb{C}(z) = \mathbb{C}(\mathbb{P}^1)$. These are just given by irreducible polynomials of $\mathbb{C}(x)[y]$, i.e. polynomials $f(y)$ with coefficients in $\mathbb{C}(z)$. It then also makes sense to talk about Riemann surfaces defined over a field $K \subseteq \mathbb{C}$. These can either be characterized as finite field extensions of $K(z)$, or as projective varieties defined by polynomials with coefficients in K, these lead to the same concept.

The last technical ingredient we shall need is a stronger form of the GAGA-principle, known as Riemann's existence theorem. For this, suppose that C is a Riemann surface, Δ a finite set of points, and $\widetilde{C} \to C \setminus \Delta$ a topological cover. Then there a Riemann surface C' and an algebraic (and thus holomorphic) map $f : C' \to C$ which restricts to $\widetilde{C} \to C \setminus \Delta$, which then corresponds to a field extension $K(C) \to K(C')$.
This provides a very strong link between algebra and topology, which we illustrate with the relation with the fundamental group. Let $\Delta \subseteq \mathbb{P}^1$ be a finite set of points of cardinality r say. Then a d-sheeted cover of $\mathbb{P}^1 \backslash \Delta$ corresponds to, by Galois theory, a homomorphism from $\pi_1(\mathbb{P}^1 \backslash \Delta)$ to the symmetric group on d elements, such that the image acts transitively on $\{1, \ldots, d\}$. The group $\pi_1(\mathbb{P}^1 \backslash \Delta)$ is generated by clockwise loops $\ell_1, \ell_2, \ldots, \ell_r$ around each point in Δ, with the relationship $\ell_1 \cdot \ldots \cdot \ell_r = 1$, so this gives us a wealth of examples of algebraic curves together with maps $C \to \mathbb{P}^1$ ramified outside of Δ.

Example 10. More generally, suppose that C is a g-holed donut, $g \geq 1$. Then the fundamental group $\pi_1(C)$ has the description as the group generated by generators $a_1, \ldots, a_g, b_1, \ldots, b_g$, with the single relation

$$a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1} = 1.$$

In particular the first homology group, being the abelianization of the fundamental group, is just $\mathbb{Z}^{2g}$, whose quotients then correspond to Galois covers with abelian fundamental group, of at most $2g$ generators.
If we remove a set Δ of r points from C, then the fundamental group $\pi_1(C \setminus \Delta)$ is the group generated by $a_1, \ldots, a_g, b_1, \ldots, b_g, \ell_1, \ldots, \ell_r$, with the relation

$$a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1} \ell_1 \ldots \ell_r = 1.$$

The same computation as the one with the Riemann sphere above can then be made. The upshot for the Riemann sphere is that it is the unique complex structure on the 2-sphere, while a genus g Riemann surface have infinitely many different structures.

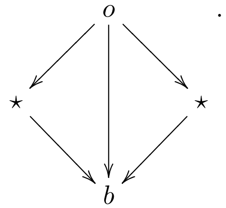
5 Dessins d'enfants

In this section we will focus on the case of coverings ramified in at most three points. By a transformation, we can suppose the points are $0, 1$ and ∞ (in projective coordiantes: $[0:1], [1:1], [1:0]$ respectively). While all compact Riemann surfaces can be realized as ramified covers of $\mathbb{P}^1$, three points adds extra restrictions. An important observation is that in this case, then the corresponding curve can be represented by polynomials with coefficients in a number field, i.e. a finite field extension of $\mathbb{Q}$. This is rather astonishing, as the curves which can be defined over a number field are very special (in the same sense the algebraic numbers are special in the set of complex numbers). From the point of view of algebraic geometry, it is however essentially a standard usage of Weil's descent theory. In the 70's, Alexander Grothendieck wondered if it could possibly be true that the converse is also true, i.e. if the curve is defined over a number field, can it be realized as a cover over $\mathbb{P}^1$ ramified at three points? This seemed as a very optimistic assertion, even though there was no counter example. To his astonishment, during the International Congress of Mathematicians in Helsinki '78, the russian mathematician G. V. Belyi announced this precise statement (see [1]). Moreover, the proof was completely elementary, and only used general properties of polynomials. The complete argument fit easily on two pages.

More precisely, the above discussion says that any curve C, together with a holomorphic map $\beta : C \to \mathbb{P}^1$ only ramified at three points, is necessarily defined over a number field. The pair (C, β) is called a Belyi pair.

We will now describe a topological receipi which describes such pairs. On the sphere, color the point 0 white, and the point 1 black, and draw the line $[0, 1]$ between them as in the picture below. If we are given a Belyi pair (C, β), we can then associate $\beta^{-1}([0, 1])$. This is a graph traced on C, whose nodes are the inverse images of 0 and 1, and it is bipartite, that is every edge has exactly one white and one black node.

Conversely, given a bipartite graph Σ on a g-hole donut S, put a star in each open cell $S \setminus \Sigma$, and draw from each star to all white and black nodes in that cell. Then every original edge with black and white node has lines going out from it to form a butterfly (to use the terminology from [7]).



Flapping the wings of the butterfly identifies it with the sphere, and doing it for every edge constructs a map $S \to S^2$, which is a covering of the sphere ramified at the white, black and star node, which we can identify with 0, 1 and ∞. By Riemann's existence theorem this corresponds to a Belyi pair (C, β), i.e. a complex structure C on S.

To summarize, a bipartite graph on a g-holed donut corresponds to a Belyi pair of a curve defined over a number field. This is rather incredible. The bipartite

graph together with an embedding into a g-holed torus is what is known as a "dessin d'enfant" (French: child's drawing), baptised as such by Grothendieck. In *Esquisse d'un programme* [5], Grothendieck writes (translation from french by Leila Schneps):
"*This discovery, which is technically so simple, made a very strong impression on me, and it represents a decisive turning point in the course of my reflections, a shift in particular of my center of interest in mathematics, which suddenly found itself strongly focused. I do not believe that a mathematical fact has ever struck me quite so strongly as this one, nor had a comparable psychological impact. This is surely because of the very familiar, non-technical nature of the objects considered, of which any child's drawing scrawled on a bit of paper (at least if the drawing is made without lifting the pencil) gives a perfectly explicit example. To such a dessin we find associated subtle arithmetic invariants, which are completely turned topsy-turvy as soon as we add one more stroke.*"
That basically any drawing by a child corresponds to a "dessin d'enfant" is an old theorem, which says that any finite graph can be embedded into a g-holed donut. It can always be colored white-black, by adding a color in the middle of two if necessary.
Dessins d'enfants and their relations to covers of the sphere were already used in work by Felix Klein in 1978/79 ([3], [4], without Belyi's theorem). There, he called them *Linienzüge* (German: plural of "line-track").
As we described in the previous section, a ramified covering of the sphere corresponds to a field extension of $\mathbb{C}(z)$, i.e. by a polynomial $f(y)$ with coefficients in $\mathbb{C}(z)$. Below we list a couple of examples of the correspondance between polynomials and dessins d'enfants. **some well-chosen dessins d'enfants, on $\mathbb{P}^1$ and on tori**

6 Galois actions on dessins

Since any dessin corresponds to a curve defined over a number field, this allows us to let the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ act on the dessins. A very direct way to do so is by applying $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ to the Belyi morphism $\beta : C \to \mathbb{P}^1$. Since β is represented by a polynomial with coefficients in $\overline{\mathbb{Q}}$, $\sigma \cdot \beta$ can be understood as the action of the Galois groups in these coefficients. This gives rise to a new Belyi pair $(C^\sigma, \sigma \cdot \beta)$, which in general is not the same as (C, β). In fact, C and C^σ need not even be biholomorphic algebraic curves. If we understand C as given by the set of zeros of homogenous polynomials in $\overline{\mathbb{Q}}$, C^σ is just given by letting σ act on the coefficients.

Example 11. Consider again Example 8, and suppose that $a, b \in \overline{\mathbb{Q}}$. The proof of Belyi's theorem associates to the function $(x, y) \mapsto x$ from $E(a, b) : y^2 = x^3 + ax + b$ to $\mathbb{P}^1$ a Belyi pair $(E(a, b), \beta)$. Then $E(a, b)^\sigma = E(\sigma \cdot a, \sigma \cdot b)$, and this is biholomorphic to $E(a, b)$ if and only if their j-invariants are equal, i.e. if $j(a, b) = \frac{a^3}{4a^3+27b^2}$ is fixed by σ.

The above then defines an action on every finite quotient $\pi_1(\mathbb{P}^1 \setminus 0, 1, \infty)/N$, where for our purposes we will suppose N is a normal subgroup. For two such groups $N' \subseteq N$, the action of the Galois group will respect the natural map

$$\pi_1(\mathbb{P}^1 \setminus 0, 1, \infty)/N' \to \pi_1(\mathbb{P}^1 \setminus 0, 1, \infty)/N$$

and defines an action on the whole inverse system of such subgroups. In more technical terms, this defines a representation $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}(\widehat{F_2})$, where $\widehat{F_2}$ is the profinite completion of $\pi_1(\mathbb{P}^1 \setminus 0, 1, \infty) = F_2$, the free group on two elements. Example 11 above moreover proves the following important corollary to Belyi's theorem:

Corollary 6.1. *The representation*

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}(\widehat{F_2})$$

is faithful, i.e. if $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts trivially on all dessins, it is necessarily the trivial element. In other words, any element of the Galois group of $\mathbb{Q}$ can be understood as set of (compatible) automorphisms of dessins or as a certain element of $\mathrm{Aut}(\widehat{F_2})$.

It is an open question how to determine exactly what elements of $\mathrm{Aut}(\widehat{F_2})$ come from the Galois group of $\mathbb{Q}$. In [2], Drinfeld defines a much smaller group $\widehat{GT}$, the Grothendieck-Teichmüller group, which still contains the Galois group. It is not known whether they are equal or not.
There are two main questions in the field of dessins. The first question is basic: Given a dessin, how can we describe the associated meromorphic function? Above we have a list of examples. This is Some discrete invariants one can associate to a dessin include the number of black and white dots, these must necessarily be preserved under conjugation. Another is the degree sequence: This lists the number of edges going out of the white respectively the black dots. It is however known that these discrete invariants do not characterize conjugate graphs.

For the interested reader further references, from which also some of the above material is taken, can be found in [7] (for an early account of the theory), [8], [9] and of course the original *Esquisse d'un Programme* in [5].

References

[1] G. Belyi, *On Galois extensions of a maximal cyclotomic field* Izv. Akad. Nauk SSSR, Ser. Mat. 43:2 (1979), 269-276 (in Russian), [English transl.: Math. USSR Izv. 14 (1979), 247-256].

[2] V. Drinfel'd *On quasitriangular quasi-Hopf algebras and a group closely connected to* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, Leningrad Math. J., 2 (1991), 829–860.

[3] F. Klein, *Über die Transformation siebenter Ordnung der elliptischen Funktionen* Math. Annalen 14: collected as pp. 90–135 (in Oeuvres, Tome 3).

[4] F. Klein, *Ueber die Transformation elfter Ordnung der elliptischen Functionen* Mathematische Annalen 15 (3–4): 533–555, collected as pp. 140–165 (in Oeuvres, Tome 3).

[5] A. Grothendick , *Esquisse d'un Programme* (1984 manuscript)

[6] J.-P. Serre, *Géométrie algébrique et géométrie analytique*, Annales de l'Institut Fourier 6 (1956), 1–42.

[7] G.B. Shabat, V. A. Voevodsky, *Drawing curves over number fields*, The Grothendieck Festschrift Volume III, Birkhäuser (1984), 199–227.

[8] F. Herrlich, G. Schmithüsen, *Dessins d'enfants and origami curves* in Handbook of Teichmüller Theory, Volume II, IRMA Lectures in Mathematics and Theoretical Physics Vol. 13, 2009.

[9] L. Schneps, *Dessins d'enfants on the Riemann sphere* in The Grothendieck Theory of Dessins d'Enfant, London Math. Soc. Lecture Notes 200, Cambridge Univ. Press, 1994

[10] ,

[11] ,

[12] ,

[13] ,